Let us now suppose that the given system of coplanar forces reduces to a single force **R** at the point $O'$ (case 3(b)). This requires that $G'=0$, *i.e.*,

$$G-xY+yX=0. \qquad \ldots (2{\cdot}13)$$

It follows that if the system reduces to a single force, any point $(x,y)$ on its line of action satisfies equation (2·13). Hence equation (2·13) is the equation of the line of action of the resultant.

## 2.15 Reduction of a System of Parallel Forces

Among the systems of coplanar forces, the system of parallel forces is of special importance in Mechanics. In the following we shall consider the composition of such forces. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be a system of forces all parallel to a vector **a**. Then we can write

$$\mathbf{P}_1=k_1\mathbf{a},\ \mathbf{P}_2=k_2\mathbf{a},\ \ldots,\ \mathbf{P}_n=k_n\mathbf{a}$$

or simply $\mathbf{P}_i=k_i\,\mathbf{a},\ i=1,2,\ldots,n,$

where $k_i$ is a +ve or −ve constant according as $\mathbf{P}_i$ has the same sense as **a** or it has the opposite sense. The sum, **R**, of these forces is given by

$$\begin{aligned}\mathbf{R}&=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n\\&=k_1\mathbf{a}+k_2\mathbf{a}+\ldots+k_n\,\mathbf{a}\\&=(k_1+k_2+\ldots+k_n)\mathbf{a}\\&=k\mathbf{a},\end{aligned} \qquad \ldots (2{\cdot}15)$$

$$\text{where } k=k_1+k_2+\ldots+k_n=\sum_{i=1}^{n} k_i. \qquad \ldots (2{\cdot}16)$$

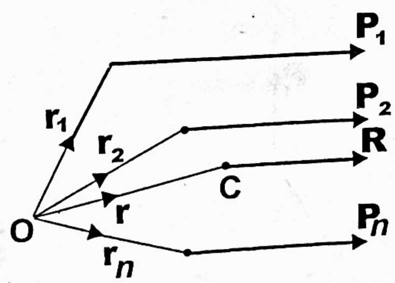

Fig. 2.26

Let the position vectors of the points of application of the forces $\mathbf{P}_i$ relative to an origin $O$ be $\mathbf{r}_i$ as shown in Fig. 2·26. Then the total moment **G** of these forces about $O$ is

$$\begin{aligned}\mathbf{G}&=\mathbf{r}_1\times\mathbf{P}_1+\mathbf{r}_2\times\mathbf{P}_2+\ldots+\mathbf{r}_n\times\mathbf{P}_n\\&=\mathbf{r}_1\times(k_1\mathbf{a})+\mathbf{r}_2\times(k_2\mathbf{a})+\ldots+\mathbf{r}_n\times(k_n\mathbf{a})\\&=(k_1\mathbf{r}_1+k_2\,\mathbf{r}_2+\ldots+k_n\,\mathbf{r}_n)\times\mathbf{a}\\&=\frac{k_1\,\mathbf{r}_1+k_2\,\mathbf{r}_2+\ldots+k_n\,\mathbf{r}_n}{k}\times k\mathbf{a}\end{aligned} \qquad \ldots (2{\cdot}17)$$

$$=\mathbf{r}\times\mathbf{R},$$

$$\text{where} \qquad \mathbf{r}=\frac{k_1\,\mathbf{r}_1+k_2\,\mathbf{r}_2+\ldots+k_n\,\mathbf{r}_n}{k} \qquad \ldots (2{\cdot}18)$$

Thus, in general, (unless $k=0$) the system of paral[lel] forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ is equivalent to a force $\mathbf{R}$ parallel to th[e] forces. It is clear from equation (2·17) that the force $\mathbf{R}$ ac[ts] at a point $C$ whose position vector $\mathbf{r}$ is given by equatio[n] (2·18). It may be noticed that if $k=0$, the force $\mathbf{R}$ vanish[es] and the point $\mathbf{r}$ recedes to infinity. In such a case, the expres[sion] (2·17) becomes indeterminate and $\mathbf{G}$ may have a finit[e] value. If $\mathbf{G} \neq 0$, the forces reduce to a couple of moment $\mathbf{G}$.

It is interesting to notice that the position vector $\mathbf{r}$ depends only on $k_i$ and $\mathbf{r}_i$. Thus if the forces are turned about their points of application through the same angle so that they remain parallel, the vector $\mathbf{r}$ remains the same and thus the position of $C$ is not altered. The point $C$ is known as the *centre of the parallel forces.*

*N.B.* If the given system consists of a couple $(\mathbf{P}_1, -\mathbf{P}_1)$ and a force $F_2$ parallel to the forces of the couple (*cf:* Art. 2·12), the resultant is a single force given by

$$\mathbf{R}=\mathbf{P}_1-\mathbf{P}_1+\mathbf{P}_2$$
$$=\mathbf{P}_2$$

*i.e.* the system is a single force equal to $F_2$, acting through the centre of the given parallel forces.

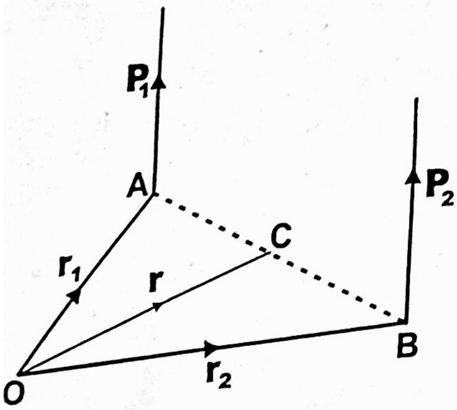


Fig. 2.27

**Two Parallel Forces :** Let us now consider a simple system of two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ each parallel to a vector $\mathbf{a}$. Let $A$ and $B$ be their points of application. We can write

$$\mathbf{P}_1=k_1\mathbf{a} \text{ and } \mathbf{P}_2=k_2\mathbf{a}.$$

If $k_1$ and $k_2$ have the same sign the forces, as in Fig. 2·27, are said to be *like parallel forces.* If $k_1$ and $k_2$ have opposite signs, then the forces, as in Fig. 2·28, are said to be *unlike parallel forces.* For definiteness let $k_1 > 0$.

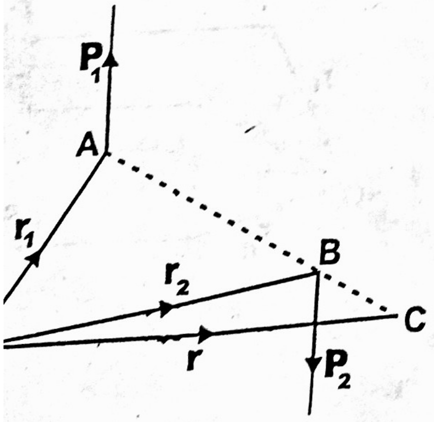


Fig. 2.28

These forces reduce to a single force

$$\mathbf{R}=(k_1+k_2)\mathbf{a}. \qquad \ldots (2·19)$$

The moment of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ about a point $O$ is given by

$$\mathbf{G}=\mathbf{r}_1 \times \mathbf{P}_1+\mathbf{r}_2 \times \mathbf{P}_2$$
$$=(k_1\mathbf{r}_1+k_2\mathbf{r}_2) \times \mathbf{a}$$
$$=\frac{k_1\mathbf{r}_1+k_2\mathbf{r}_2}{k_1+k_2} \times (k_1+k_2)\mathbf{a}$$
$$=\mathbf{r} \times \mathbf{R}, \qquad \ldots (2·20)$$

where

$$\mathbf{r}=\frac{k_1\mathbf{r}_1+k_2\mathbf{r}_2}{k_1+k_2}, \qquad \ldots (2·21)$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of $A$ and $B$ relative to $O$.

If $k_2 > 0$ then the system reduces to a force

$$\mathbf{R}=(k_1+k_2)\mathbf{a}=\mathbf{P}_1+\mathbf{P}_2,$$

acting at a point $C$ (*see* equation (2·20)) whose position vector is $\mathbf{r}$. It is clear from equation (2·21) that $C$ divided the directed segment $\overline{AB}$ in the ratio $k_2 : k_1$.

If $k_2 < 0$ but $k_1+k_2 \neq 0$, then the system reduces to a force

$$\mathbf{R}=\mathbf{P}_1+\mathbf{P}_2$$

acting at a point $C$ which divides the extension of the directed segment $\overline{AB}$ in the ratio $k_2 : k_1$.

If $k_2 < 0$ and $k_1+k_2=0$, *i.e.*, $k_1=-k_2=k$, then

$$\mathbf{R}=O$$

and
$$\mathbf{G}=(k_1\mathbf{r}_1+k_2\mathbf{r}_2)\times\mathbf{a}=k_1(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{a}$$
$$=(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{P}_1 \qquad \ldots (2{\cdot}22)$$

is the moment of $\mathbf{P}_1$ about $B$. Thus in this case the system reduces to a couple.

## 2.16 Conditions of Equilibrium of a Coplanar Force System

We have seen that any system of coplanar forces can be reduced to a single force $\mathbf{R}$ acting at an arbitrary point of the plane of the forces together with a couple G. A system of coplanar forces is said to be *in equilibrium* if

$$\mathbf{R}=O \text{ and } G=O \qquad \ldots (2{\cdot}23)$$

If the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ through the points $(x_1, y_1)$, $(x_2, y_2), \ldots, (x_n, y_n)$ have resolved parts $(X_1, Y_1)$, $(X_2, Y_2), \ldots, (X_n, Y_n)$ respectively, and

$$X=\Sigma X_i \text{ and } Y=\Sigma Y_i$$

then
$$\mathbf{R}=X\mathbf{i}+Y\mathbf{j}$$

and
$$G=\Sigma (x_i Y_i - y_i X_i).$$

Thus the equilibrium conditions (2·14) imply that

$$X=\Sigma X_i=0$$
$$Y=\Sigma Y_i=0 \qquad \ldots (2{\cdot}24)$$
and
$$G=\Sigma (x_i Y_i - y_i X_i)=0$$

Equations (2·24) are known as the basic equations of equilibrium and can be interpreted as follows :

*A system of coplanar forces is in equilibrium if the algebraic sum of the resolved parts of the forces in any two directions* is zero and the algebraic sum of the moments of all the forces about any arbitrary point is zero.*

This is, however, not the only set of conditions which ensure the equilibrium of a force system. In the following we consider other sets of conditions which also guarantee equilibrium.

## Other Forms of Conditions of Equilibrium

(*i*) *A system of coplanar forces is in equilibrium if the algebraic sum of moments of all the forces about two different points O and A is zero and the algebraic sum of the resolved parts in some one direction, not perpendicular to OA, is zero.*

Let us take $O$ as the origin and the direction in question as the $x$-axis. Let $A$ have the rectangular coordinates $(x, y)$. Since $OA$ is not perpendicular to the $x$-axis, $x \neq 0$. Let $X$, $Y$ be the algebraic sums of the resolved parts of the forces and $G$ and $G'$ be the algebraic sum of their moments about $O$ and $A$ respectively. Then

$$G' = G - x\,Y + y\,X.$$

Now we are given that

$$G = 0$$

$$G' = G - x\,Y + y\,X = 0$$

and $\quad X = 0. \qquad (2{\cdot}25)$

These equations give

$$X = 0,\ Y = 0 \text{ and } G = 0,$$

so that the forces are in equilibrium.

(*ii*) *A system of coplanar forces is in equilibrium if the algebraic sum of the moments of all the forces about three different points O, A, B, not in the same straight line, are separately zero.*

Let us take $O$ as the origin and $OA$ as the $x$-axis. Let $A$ and $B$ have the rectangular coordinates $(x, 0)$ and $(x', y')$

*The two directions need not be at right angles to each other but they must not be parallel.

respectively. Since $B$ does not lie on the $x$-axis, $y' \neq 0$. Let $X$, $Y$ be the sums of the resolved parts and $G$, $G'$ and $G''$ be the moments of the forces about $O$, $A$ and $B$ respectively. Then

$$G' = G - xY$$

$$G'' = G - x'Y + y'X.$$

The conditions $G=0$, $G'=0$ and $G''=0$ give $X=0$, $Y=0$ and $G=0$ which guarantee the equilibrium of the forces.

## 2.17 Examples

**Example 1.** Forces of magnitude $P$, $2P$, $3P$, $4P$ act respectively along the sides $AB$, $BC$, $CD$, $DA$ of a square $ABCD$, of side $a$, and forces each of magnitude $(8\sqrt{2})P$ act along the diagonals $BD$, $AC$. Find the magnitude of the resultant force and the distance of its line of action from $A$.

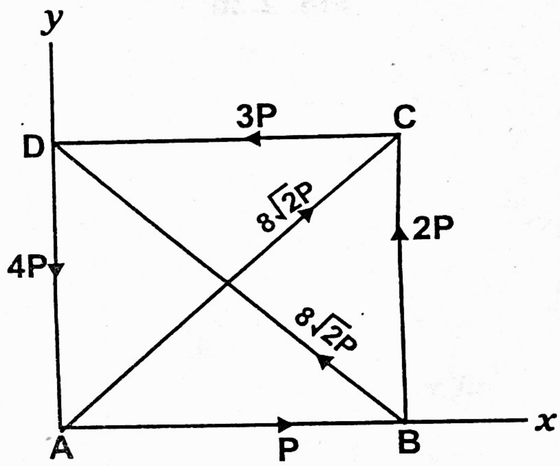


Fig. 2.29

**Sol.** Take $AB$, $AD$ respectively as axes of $x$ and $y$ and reduce the system of forces at $A$ (Fig. 2·29). If $X$, $Y$ are the resolved parts of the single force and $G$ is the moment of the couple, we have

$$X = P - 3P + 8\sqrt{2}P.\frac{1}{\sqrt{2}} - 8\sqrt{2}P.\frac{1}{\sqrt{2}} = -2P,$$

$$Y = 2P - 4P + 8\sqrt{2}P.\frac{1}{\sqrt{2}} + 8\sqrt{2}P.\frac{1}{\sqrt{2}} = 14P,$$

and

$G$ = sum of the moments of the forces about $A$

$$= 8\sqrt{2}P.\frac{a}{\sqrt{2}} + 2Pa + 3Pa = 13Pa.$$

Hence the resultant force is of magnitude $R$, where

$$R^2 = (2P)^2 + (14P)^2 = 200P^2,$$

so that $\quad R = (10\sqrt{2})\,P.$

The equation of the line of action of the resultant is

$$G - xY + yX = 0,$$

or $\quad 13a - 14x - 2y = 0,$

whose perpendicular distance from $A$ is

$$\frac{13a}{\sqrt{(14^2+2^2)}} = \frac{13a}{10\sqrt{2}}.$$

**Example 2.** Forces $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ act along the sides of a regular hexagon taken in order. Show that they will be in equilibrium if

$$\Sigma P = 0 \text{ and } P_1 - P_4 = P_3 - P_6 = P_5 - P_2.$$