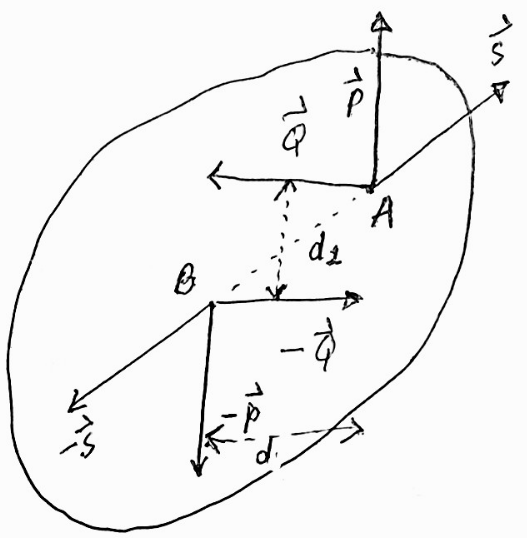
Mar. 26, 2020

EQUIVALENT COUPLES

Theorem The effect of couple upon a rigid body is unaltered if it is replaced by any other couple of the same moment lying in the same plane.

PROOF



Let a couple $(P, -P)$ with arc d_1 is acting on a R-B. Let A and B be any two points on the lines of action of $\vec{P}$ and $-\vec{P}$.

Let G be the magnitude of the moment of this couple, then

$$G = Pd \longrightarrow (1)$$

Resolving the forces $\vec{P}$ and $-\vec{P}$ into component forces $\vec{S}$ and $\vec{Q}$ and $-\vec{S}$ and $-\vec{Q}$. Now $\vec{S}$ through A and $-\vec{S}$ through B are equal in magnitude and opposite in direction and their line of action is same, so they cancel the effect of each other. Now we are left with two forces $\vec{Q}$ through A and $-\vec{Q}$ through B. These two forces are equal in magnitude and opposite in direction and their line of action is not same so they form a couple $(\vec{Q}, -\vec{Q})$

Let d_1 be the arm of this couple

then

$$G_1 = Qd_1 \longrightarrow (2)$$

Thus the couple of forces $\vec{P}, -\vec{P}$ and arm d is replaced by another couple of forces $\vec{Q}, -\vec{Q}$ and arm d_1.

Now we will prove that

$$G = G_1$$

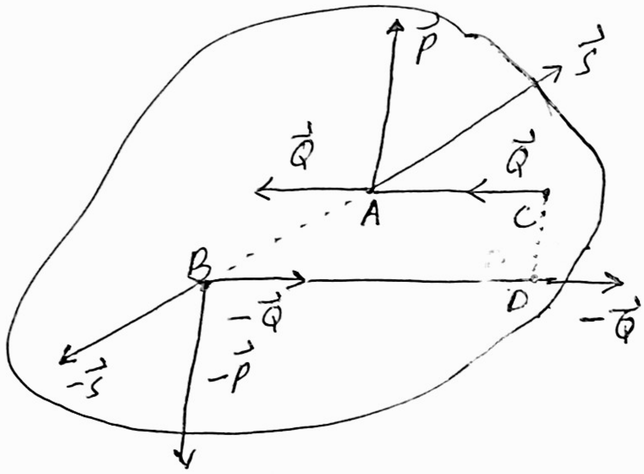
Taking moment of forces about point B. According to varignon's theorem

$$Pd = S(0) + Qd_1$$

$$G = G_1 \quad (\text{By } (1) \ \& \ (2))$$

Moment of couple $(\vec{P}, -\vec{P})$ = Moment of couple $(\vec{Q}, -\vec{Q})$.

Note that if points of application of $\vec{Q}$ and $-\vec{Q}$ are transferred from A and B to any points C and D in their lines of action



Since the location of points A and B and the directions of AC and BD are arbitrary, the location of the couple of forces $\vec{Q}$ and $-\vec{Q}$ in the plane is also arbitrary.

So we conclude that

(1) A couple can be transferred anywhere in its plane

(2) It is possible to change the magnitudes of the forces of a couple and its arm arbitrarily without changing the moment of the couple.

COMPOSITION OF COUPLES

Conditions for the equilibrium of Couples

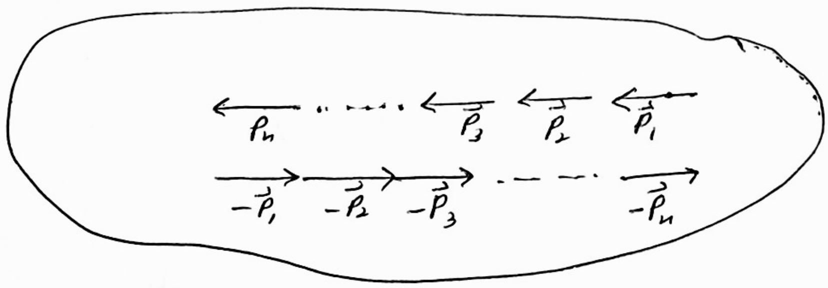
Theorem

Coplanar couples of moments $G_1, G_2, \dots G_n$ are equivalent to a single couple lying in the same plane, whose moment G is given by

$$G = G_1 + G_2 + \dots + G_n$$

Proof

Consider a system of coplanar couples $(\vec{P_1}, -\vec{P_1}), (\vec{P_2}, -\vec{P_2}), (\vec{P_3}, -\vec{P_3}), \dots (\vec{P_n}, -\vec{P_n})$ with common arm 'd' acting on a R.B.



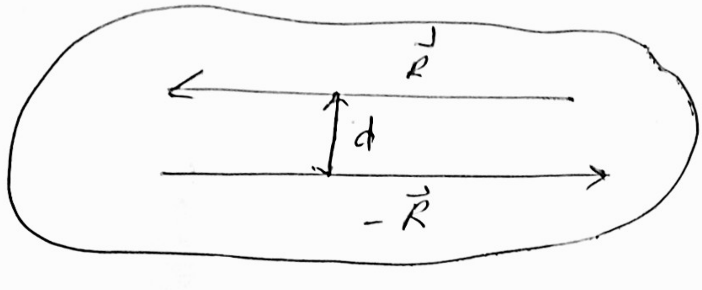
Let $G_1, G_2, G_3 \dots G_n$ be the magnitudes of the moments of these couples respectively.

Then
$$G_1 = P_1 d$$
$$G_2 = P_2 d$$
$$G_3 = P_3 d$$
$$\dots$$
$$G_n = P_n d$$

Let $\vec{R} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \dots + \vec{P_n}$

$$-\vec{R} = (-\vec{P_1}) + (-\vec{P_2}) + (-\vec{P_3}) + \dots + (-\vec{P_n})$$

Now the new resultant couple is $(\vec{R}, -\vec{R})$ with arm d. Let G be the magnitude of moment of this couple



then

$$G = Rd$$
$$= (P_1 + P_2 + P_3 + \cdots + P_n)\,d \quad \text{by (2)}$$
$$= P_1 d + P_2 d + P_3 d + \cdots + P_n d$$
$$G = G_1 + G_2 + G_3 + \cdots + G_n \quad \text{by (1)}.$$

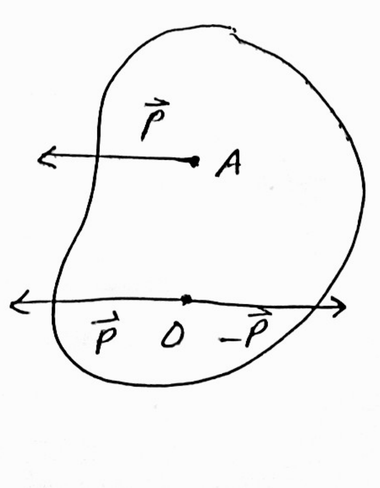
Necessary and sufficient condition for equilibrium of a system of coplanar couples is that $G_1 + G_2 + G_3 + \cdots + G_n = 0$

A FORCE AND A COUPLE

Theorem A force $\vec{P}$ acting on a R.B can be moved to any point O of the R.B, provided a couple is added, whose moment is equal to the moment of $\vec{P}$ about O.

Proof

Let a force $\vec{P}$ act at a point A of the R.B.
We introduce two forces $\vec{P}$ and $-\vec{P}$ without altering the effect of force $\vec{P}$ on body.



The force $\vec{P}$ at A and $-\vec{P}$ at O form a couple $(\vec{P}, -\vec{P})$.

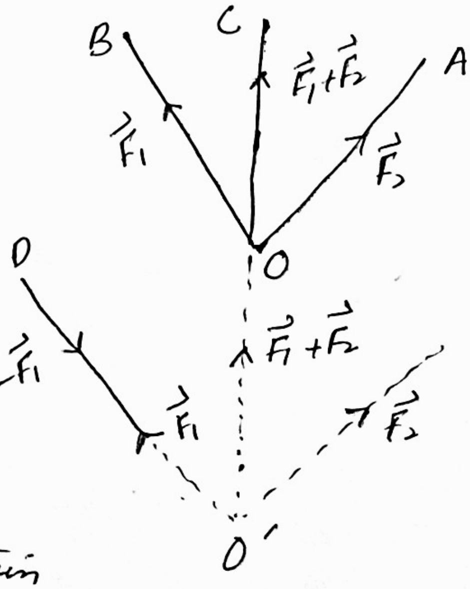
Hence the given ~~force~~ force $\vec{P}$ at A is equivalent to a force $\vec{P}$ at O together with a couple $(\vec{P}, -\vec{P})$ whose moment is equal to the moment about O of the force $\vec{P}$ acting at A.

<u>CONVESELY</u>: A single force and a couple acting in the same plane upon a R.B are equivalent to the single force acting in a direction parallel to its original direction.

<u>Proof</u>

Let the given system consist of the couple $(\vec{F_1}, -\vec{F_1})$ and the force $\vec{F_2}$ (acting through O')

Let $\vec{F_2}$ is not parallel to $\vec{F_1}$. Let $\vec{F_2}$ meet $\vec{F_1}$ in O. Then $\vec{F_1}$, $\vec{F_2}$ are equivalent to a force $\vec{F_1}+\vec{F_2}$ acting at O along a line OC.



Let OC meet the line of action of $-\vec{F_1}$ in O'.
The point of application of $\vec{F_1}+\vec{F_2}$ may be shifted to O'

The force $\vec{F_1} + \vec{F_2}$ acting at O' can be resolved into two forces

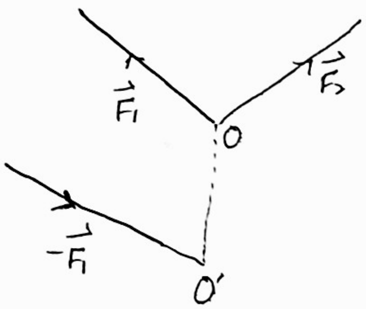
(i) $\vec{F_2}$, parallel to the original force $\vec{F_2}$

(ii) $\vec{F_1}$, acting along the line of action of $-\vec{F_1}$.

The forces $\vec{F_1}, -\vec{F_1}$ acting at O' balance each other. Therefore given system is equivalent to a single force $\vec{F_2}$ acting at O'.

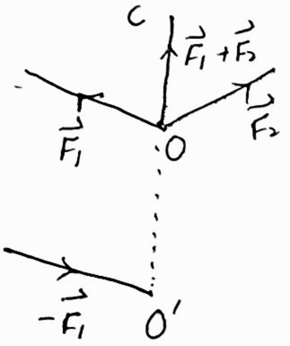
Aside

Step 1



A couple $(\vec{F_1}, -\vec{F_1})$ and a force $\vec{F_2}$

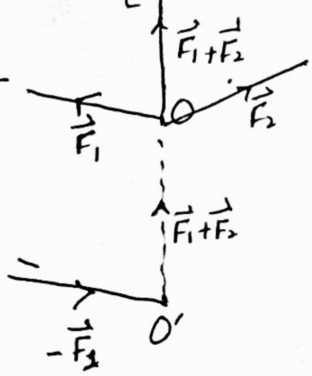
Step 1



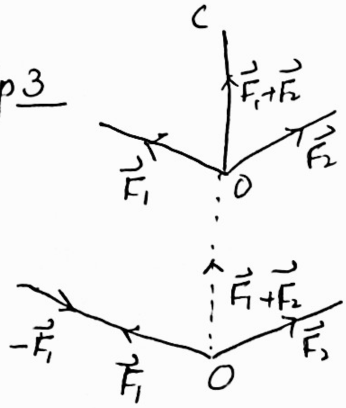
Resultant of $\vec{F_1}, \vec{F_2}$

$\vec{F_1} + \vec{F_2}$ (By parallelogram law)

Step 2



Step 3



Point of application of $\vec{F_1}+\vec{F_2}$ shifted to O'

$\vec{F_1} + \vec{F_2}$ resolved into two forces $\vec{F_2}$ & $\vec{F_1}$