Chapter No. 16

Application of the Laplace Transform.

Significance of Laplace Transform in ENA

" The Laplace Transform is Significant for a number of reasons:.

1st ⇒ It can be applied to a wider variety of inputs than Phasor Analysis.

2nd ⇒ It Provides an easy way to solve circuit Problem, involving initial conditions, because it allow us to work with algebric equations instead of differential equations.

3rd ⇒ The Laplace Transform is capable of Providing us, in one single operation, the total response of the circuit consisting of both the natural & forced responses.

Definition of the Laplace Transform

" The Laplace Transform is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$

$$\underset{\text{Time Domain}}{f(t)} \xleftrightarrow[\text{ILa}]{\mathcal{L}} \underset{\text{S-Domain}}{F(S)}$$

(Complex Freq Domain)"

Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or $\mathcal{L}[f(t)]$ is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

where s is complex variable given by

$$s = \sigma + j\omega$$

Real ↙ ↘ Imaginary Part

$\bar{0}$ ↓ Lower Limit

Laplace Transform Pairs

~~Single~~	Time Domain $f(t)$ (Table 15.2)	S-Domain / Frequency domain $F(s)$
Impulse ~~unit~~ signal	$\delta(t)$	1
Step	$u(t)$	$1/s$
	e^{-at}	$\frac{1}{s+a}$
	t	$1/s^2$
	t^n	$\frac{n!}{s^{n+1}}$
	$t\, e^{-at}$	$\frac{1}{(s+a)^2}$
	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} F(s)\, e^{st}\, ds$$

ILT

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

Def: System "A system is a mathematical model of a physical process relating the input to the output".

→ It is entirely appropriate to consider circuit as system" (modeling of circuit in the S-domain)

16.2 Circuit Element Models

Steps in Applying the Laplace Transform:

1.) Transform the circuit from the time domain to the S-domain

$$f(t) \xrightarrow{\mathcal{L}} F(S)$$

Time Domain ⟶ S-Domain (Freq. Domain)

2). Solve the circuit using Nodal Analysis, Mesh Analysis, Source Transformation, Superposition, or any circuit analysis technique

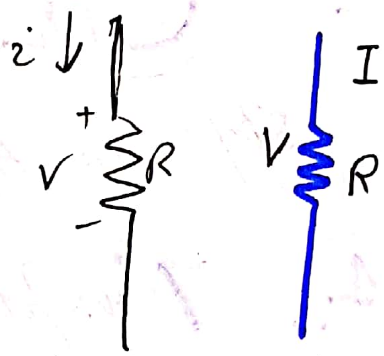
3). Take the inverse transform of the solution & thus obtain the solution in the time domain

$$\mathcal{L}^{-1}[F(S)] \longrightarrow f(t)$$

S-Domain Time Domain

Re-call / Revise chapter No. 9, As, we we transform a circuit in the time domain to the frequency domain. ($j\omega$)

Here In Ch. No. 19, we also transform a circuit in the time domain to the frequency or S-domain by Laplace transforming each term in the circuit.



(SAME)

(1) For **Resistor**

The voltage - C

Time Domain		S-Domain
$V = iR$ or $V(t) = i(t)\cdot R$	$\xRightarrow{\text{taking Laplace}}$	$V(s) = I(s)R$

(2) **Inductor**

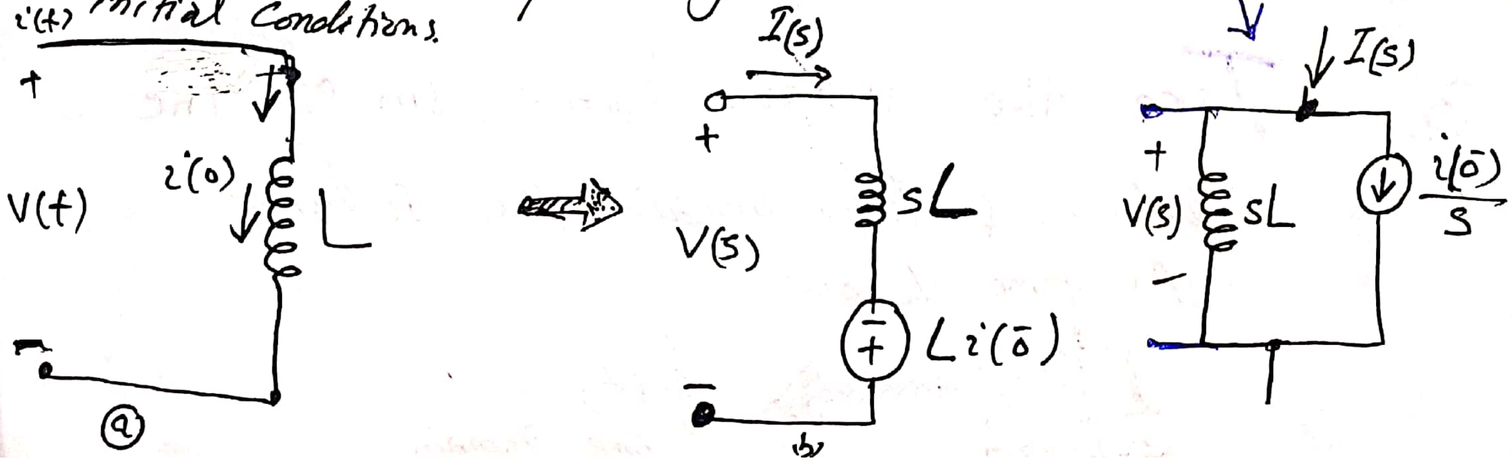
$$V(t) = L\frac{di(t)}{dt}$$

$$V(s) = L\,[sI(s) - i(\bar{0})]$$
$$V(s) = sLI(s) - Li(\bar{0})$$

or

$$i(t) = \frac{1}{L}\int v(t)\,dt + i(\bar{0}) \Rightarrow I(s) = \frac{1}{sL}V(s) + \frac{i(\bar{0})}{s}$$

Note Polarity of Source Representing initial Conditions.



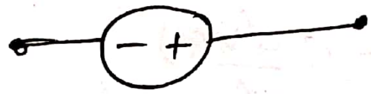
3.

Time Domain

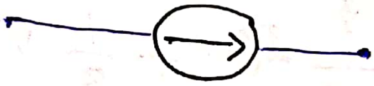
S-Domain

"Laplace"

$V(t)$



$V(S)$



$i(t)$

$I(S)$

$+\ V(t)\ -$

R $i(t)$

$$V(t) = i(t).R$$
$$V(t) = R.i(t)$$

"Voltage Across Resistor."

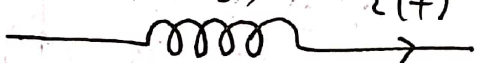
$+\ V(S)\ -$

R $I(S)$

$$V(S) = RI(S)$$

(SAME WORKING in S-Domain)

$+\ V(t)\ -$ $i(t)$



L

Voltage Across Inductor $V(t) = L\dfrac{d(i)}{dt}$ $\rightarrow$ (1)

$V(S)\ -$ $I(S)$

sL $Li(0^-)$

$$V(S) = SLI(S) - Li(0^-)$$

($V = iR$ ohm's law)

⇒ taking the Laplace of Eq No. (1). on both sides, the voltage across the Inductor transform $V(t) \longrightarrow V(S)$, Now, we take the laplace transform of $L\frac{d(i(t))}{dt}$, we get term $S \times L$ (times), where S come from derivative of $L\frac{d}{dt}i(t) = SLI(S)$, $V(S) = SL\,I(S)$, $V \propto I$, R's Prop. Constant Here (SL) s ~

⇒ $Li(0^-)$, we notice the polarity here is opposite to the polarity we discuss here, & that represents the initial voltage due to current $i(0^-)$ here i Initial current

$Z_s = SL$ —ooo—

$Z_s = 1/sc$ —||—

S-domain & Phasor impedance agree for $\boxed{S = j\omega}$

$S = j\omega$

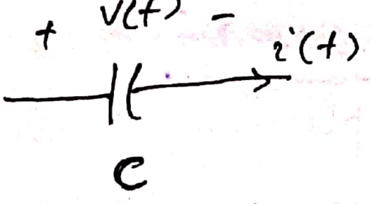
in ch. No. 9

ch. No. 16

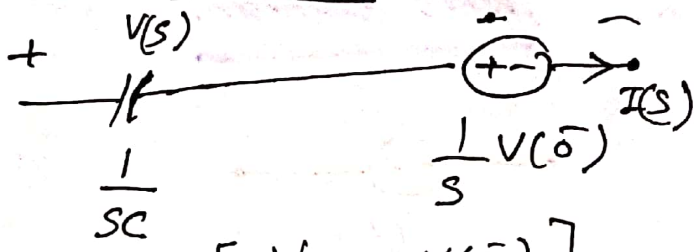
• Resistor	$= V_R = iR$	$V = (R)I$	$V = RI(S)$
• Inductor	$= V_L = L\frac{di}{dt}$	$V_L = (j\omega L)I$	$V_L(s) = SLI(S)$
• Capacitor	$= V_c = \frac{1}{C}\int i\,dt$	$V_c = \left(\frac{1}{j\omega c}\right)\cdot I$	$V_c(s) = \frac{1}{Sc}\cdot I$

4)

Time



S- Domain



$$i(t) = C\frac{dv(t)}{dt}$$

& Voltage Across capacitor

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

"taking Laplace on both sides"

$$I(S) = C\left[S\,V(S) - V(0^-)\right]$$

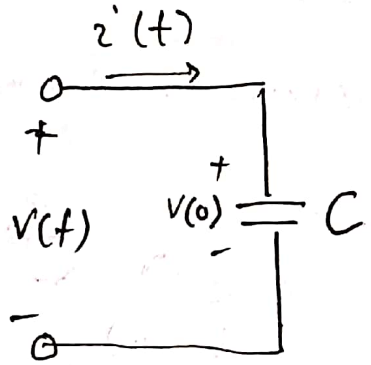
$$I(S) = SCV(S) - CV(0^-)$$

$$V(S) = \frac{1}{SC}I(S) + \frac{1}{S}V(0^-)$$

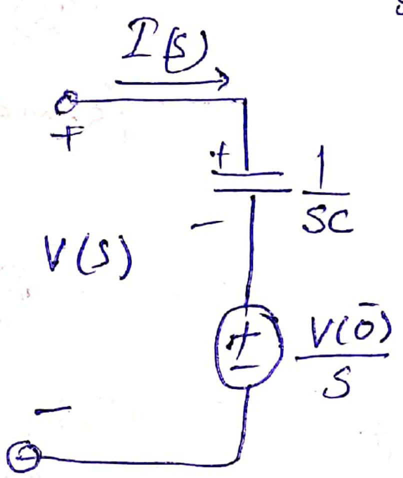
$Z = 1/SC$ ⇒ Impedance of Capacitor.

$$\frac{1}{C}\int_{-\infty}^{0^-} i(t)\,dt.$$

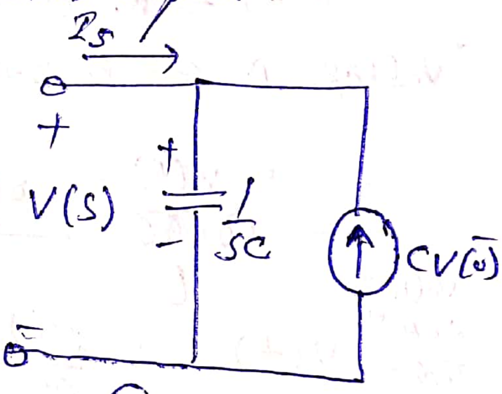
$\frac{1}{S}V(0^-)$ is voltage Source due to initial Voltage Across capacitor.



(a)
time - Domain



(b)
S-Domain Equivalent



(c)
S-Domain

"Note"

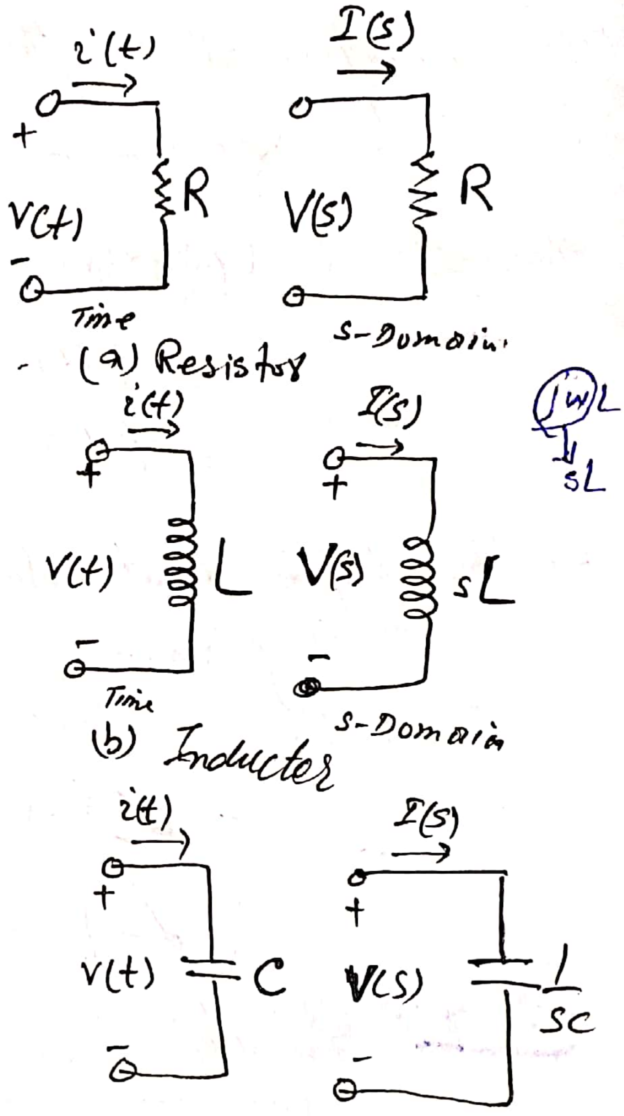
— The elegance of using the Laplace transform in circuit Analysis lies in the automatic inclusion of the initial conditions in the transformation Process, Thus Providing a Complete [Transient & steady State) Solution.

If ~~we~~ Assume $\boxed{\text{Zero Initial Condition}}$ for the inductor & capacitor, the above ~~for~~ Equations on Page #6, reduce to:

Resistor: $V(s) = RI(s)$

Inductor: $V(s) = sLI(s)$

Capacitor: $V(s) = \frac{1}{sC} I(s)$



(a) Resistor

(b) Inductor

(c) Capacitor.

Impedance Z(s)

"We define the impedance in the s-domain as the ratio of the voltage transform to the current transform under zero initial condition. i.e

$$Z(s) = \frac{V(s)}{I(s)}$$

So, the Impedance of the three (R, L & C) circuit are

Resistor: $Z(s) = R$ $\left(V(s) = R\,I(s) \quad \frac{V(s)}{I(s)} = R = Z \right)$

Inductor: $Z(s) = sL$ $\left(V(s) = sL\,I(s) \quad \frac{V(s)}{I(s)} = sL \right)$

Capacitor: $Z(s) = \frac{1}{sC}$ $\left(\frac{V(s)}{I(s)} = \frac{1}{sC} \right)$ From Above Eq.

Admittance : $Y(s)$

The admittance in the s-domain is the Reciprocal of the impedance . i.e

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

⇒ Re-call all Properties of Laplace

$$f(t) \longrightarrow F(s)$$

$$a\,f(t) \longrightarrow a\,F(s) \qquad \text{The linearity Property.}$$

⇒ The dependent source can have only two controlling values, a constant time either voltage or a current source. Thus

$$a f(t) \longrightarrow a F(s)$$

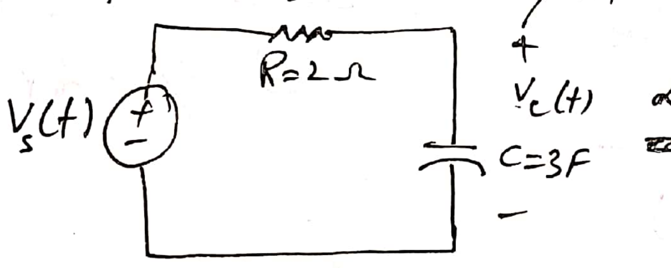
$$\mathcal{L}[a v(t)] = a V(s)$$

$$\mathcal{L}[a i(t)] = a I(s)$$

Example for understanding

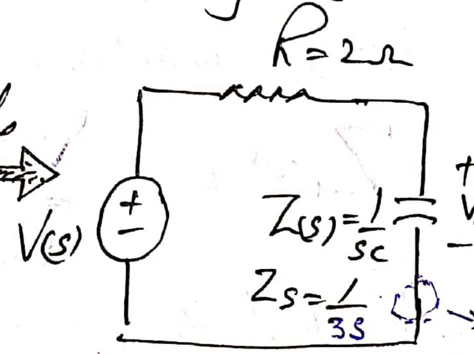
Find Voltage Across Capacitor

Assuming $V(0^-) = 0$ Initial Voltage



Time Domain

$\mathcal{L}$ ⇒



→ No Voltage Source due to Zero Initial voltage

To Find the voltage Across Capacitor, we use of V.D.R.

$$V_c(s) = \frac{1/3s}{2 + \frac{1}{3s}} \times V_s(s) \quad \text{⇒ applied Voltage.}$$

(1)

Example 16.1

Find $V_o(t)$ in the circuit of Figure 16.4, assuming zero initial conditions.

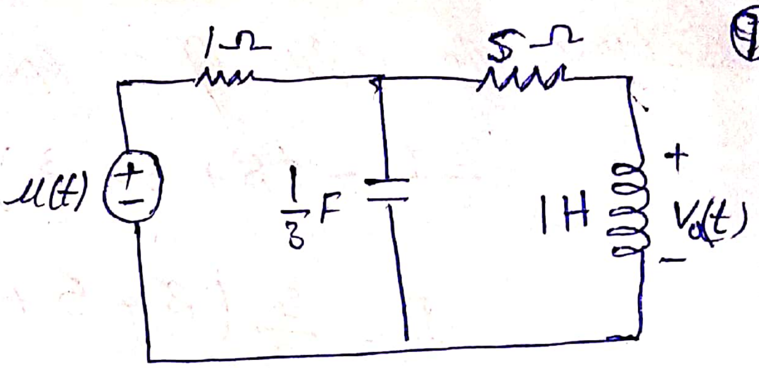


Fig. 16.4 (Time Domain)

Solutions:

We first transform the circuit from the time domain to the s-domain.

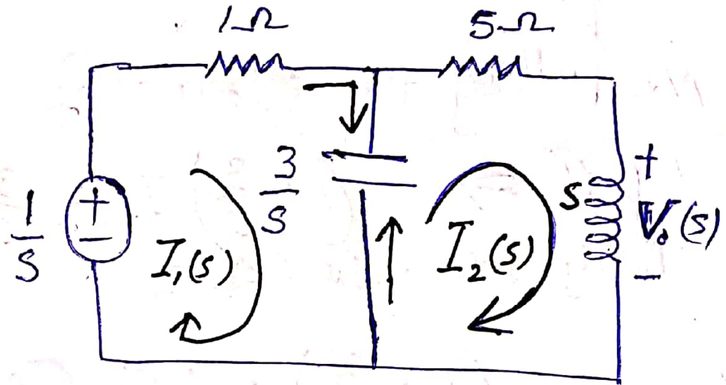


Fig. 16.5 (S-Domain)

$$u(t) \Rightarrow \frac{1}{s}$$

$$1H \Rightarrow sL = s \times 1 = s$$

$$\frac{1}{3}F \Rightarrow \frac{1}{sC} = \frac{1}{s \cdot \frac{1}{3}} = \frac{3}{s}$$

Now, we apply Mesh Analysis for Mesh #1

Loop #1

$$-\frac{1}{s} + 1I_1 + \frac{3}{s}(I_1 - I_2) = 0$$

$$\frac{1}{s} = 1I_1 + \frac{3}{s}I_1 - \frac{3}{s}I_2$$

taking I_1 common

$$\left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2 = 1/s \longrightarrow (1)$$

For Mesh #2

$$5I_2 + sI_2 + \frac{3}{s}(I_2 - I_1) = 0$$

$$5I_2 + sI_2 + \frac{3}{s}I_2 - \frac{3}{s}I_1 = 0$$

Rearranging

$$-\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2 = 0 \longrightarrow (2)$$

Scanned with CamScanner

Deriving the value of I_1 from Eq ②

$$\frac{3}{s} I_1 = \left(s + 5 + \frac{3}{s}\right) I_2$$

$$I_1 = \frac{s}{3}\left(s + 5 + \frac{3}{s}\right) I_2$$

$$I_1 = \frac{1}{3}\left(s^2 + 5s + 3\right) I_2 \longrightarrow ③$$

Now Putting the Value of I_1 in Eq ①

Eq ① $\Rightarrow$ $\left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2 = \frac{1}{s} \longrightarrow ①$

Putting the value of I_1

$$\left(1 + \frac{3}{s}\right)\frac{1}{3}\left(s^2 + 5s + 3\right) I_2 - \frac{3}{s} I_2 = \frac{1}{s}$$

To simplify the equation, multiplying through by 3s on both sides gives:

LCM

$$\left[\left(\frac{s+3}{s}\right)\left(\frac{s^2 + 5s + 3}{3}\right)\right] I_2 - \frac{3}{s} I_2 = \frac{1}{s}$$

$$\left[\frac{(s+3)(s^2 + 5s + 3)}{3s}\right] I_2 - \frac{3}{s} I_2 = \frac{1}{s}$$

$$\left[\frac{s^3 + 5s^2 + 3s + 3s^2 + 15s + 9}{3s}\right] I_2 - \frac{3}{s} I_2 = \frac{1}{s}$$

$$\left[\frac{s^3 + 8s^2 + 18s + 9}{3s}\right] I_2 - \frac{3}{s} I_2 = \frac{1}{s}$$

Now we can multiply 3s on both sides.

$$\cancel{3s}\, I_2 \frac{\left(s^3 + 8s^2 + 18s + 9\right)}{\cancel{3s}} - \frac{3 \times 3\cancel{s}}{\cancel{s}} I_2 = \frac{3\cancel{s}}{\cancel{s}}$$

$$\left[s^3+8s^2+18s+\cancel{9}-\cancel{9}\right]I_2 = 3$$

$$(s^3+8s^2+18s)I_2 = 3$$

$$\Rightarrow \quad I_2 = \frac{3}{s^3+8s^2+18s} \longrightarrow (4)$$

Now, Our main focus on, Voltage across Inductor.

$\because V_L(s) = sLI_2$, $L=1$

$V \propto I$

Ohm's La.

$V = R \cdot I$

$V_L = sLI$

$V_o = s(1)I_2 = s \cdot I_2$

$$V_o(s) = sI_2$$

$$V_o(s) = \cancel{s}\left(\frac{3}{\cancel{s}(s^2+8s+18)}\right)$$

$$V_o(s) = \frac{3}{s^2+8s+18}$$

simplifying for making function.

$$V_o(s) = \frac{3}{s^2+2(4)s+16+2}$$

$$V_o(s) = \frac{3}{(s+4)^2+(\sqrt{2})^2}$$

$$\boxed{\begin{aligned}&(s+4)^2+(\sqrt{2})^2\\&s^2+16+2(4)s+2\\&s^2+18+8s\\&s^2+8s+18\end{aligned}}$$

$$V_o(s) = \frac{\sqrt{2}}{\sqrt{2}} \times \frac{3}{(s+4)^2+(\sqrt{2})^2}$$

$$V_o(s) = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{(s+4)^2+(\sqrt{2})^2}$$

Re-call.

$$\because \frac{\omega}{(s+a)^2+\omega^2} = e^{-at}\sin\omega t$$

$\omega = \sqrt{2}$

$a = 4$

$$\frac{\omega}{(s+a)^2+\omega^2} = e^{-4t}\sin\sqrt{2}t$$

Taking the Inverse Laplace yield

$$V_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\cdot\sin\sqrt{2}t \ V$$

$t \geq 0$.

Practice Problem 16.1

Determine $v_o(t)$ in the circuit of Fig. 16.6, assuming Zero initial conditions.

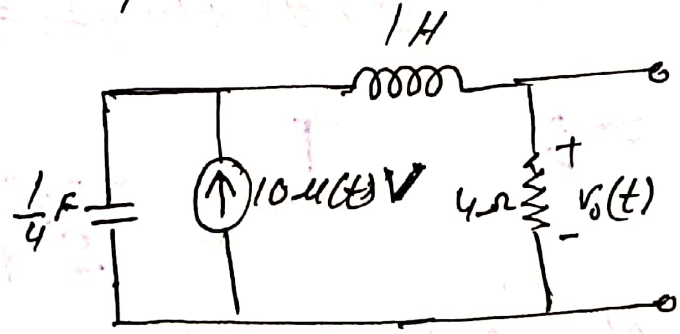
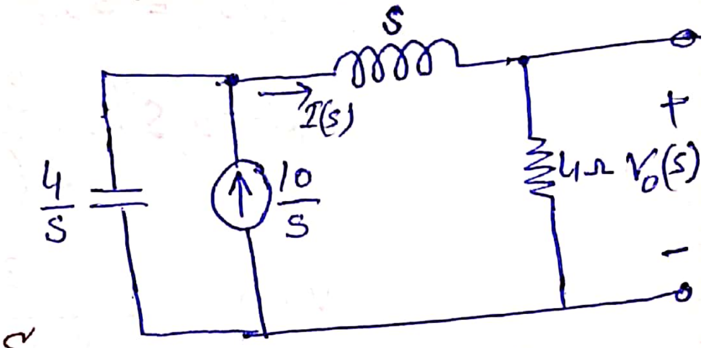


Fig. 16.6

Solution

Transform the circuit from Time Domain to Frequency (S) domain

Use the Transformed values to obtain S-Domain circuit



(S-Domain Ckt)
Fig (2)

(i) Current Source

$$10u(t) \Longrightarrow \frac{10}{s}$$

(ii) Inductance

$$L = 1H \Longrightarrow sL = s$$

(iii) Capacitance

$$C = \frac{1}{4}F \Longrightarrow \frac{1}{sC} = \frac{1}{s \cdot \frac{1}{4}} = \frac{4}{s}$$

The current in any branch is calculated using (CDR) Current Division Rule.
Calculate the current $I(s)$ in the circuit figure 2

$$I(s) = \frac{4/s}{\frac{4}{s} + s + 4} \times \frac{10}{s}$$

$$I(s) = \left(\frac{10}{s}\right)\left(\frac{4}{4 + s^2 + 4s}\right)$$

$$I(s) = \frac{40}{s(s^2 + 4s + 4)}$$

The output voltage $V_0(s)$ (Voltage Across 4Ω) is Calculated by multiplying the current $I(s)$ with the Resistance 4Ω.

$V = IR$

$\therefore I(s) = \frac{40}{s(s^2+4s+4)}$

$$V_0(s) = 4I(s)$$

$$V_0(s) = 4\left(\frac{40}{s(s^2+4s+4)}\right)$$

$$V_0(s) = \frac{160}{s(s^2+4s+4)}$$

$\therefore (s+2)^2 = s^2+4+4s$ ($2\cdot 2(s)$)

$(a+b)^2 = a^2+b^2+2ab$

$$V_0(s) = \frac{160}{s(s+2)^2} \longrightarrow (1)$$

Take Partial fractions to the equation (1).

$$\frac{160}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \longrightarrow (2)$$

multiplying with $s(s+2)^2$ on both sides, we get

$$160 = A(s+2)^2 + Bs(s+2) + Cs \longrightarrow (3)$$

$$160 = A(s^2+4+2(2s)) + B(s^2+2s) + Cs$$

$$160 = As^2 + 4A + 4As + Bs^2 + 2Bs + Cs \longrightarrow (3)$$

$$160 = s^2(A+B) + s(4A+2B+C) + 4A \longrightarrow (4)$$

Compare constant terms in Eq. (4)

$$160 = 4A$$

$$A = \frac{160}{4} = 40 \Rightarrow \boxed{A = 40}$$

Substitute $s+2=0 \Rightarrow \boxed{s=-2}$ in eq No. (3)

$$160 = A(\cancel{-2}+\cancel{2})^2 + Bs(\cancel{-2}+\cancel{2}) + C(-2)$$

$$160 = C(-2)$$

$$C = -\frac{160}{2} = -80 \Rightarrow \boxed{C = -80}$$

Compare s^2 term on both sides in equation (3)

$$0 = A + B \qquad As \Rightarrow \boxed{A = 40}$$

$$A = -B$$

$$40 = -B \Rightarrow \boxed{B = -40}$$

Substitute $\boxed{A = 40}$, $\boxed{B = -40}$ & $\boxed{C = -80}$ in eq (1)

$$\frac{160}{s(s+2)^2} = \frac{40}{s} - \frac{40}{s+2} - \frac{80}{(s+2)^2}$$

$$V_o(s) = \frac{40}{s} - \frac{40}{s+2} - \frac{80}{(s+2)^2} \longrightarrow (5)$$

Applying Inverse Laplace transform Eq (5)

$$\mathcal{L}^{-1} V_o(s) = \mathcal{L}^{-1}\left[\frac{40}{s} - \frac{40}{s+2} - \frac{80}{(s+2)^2}\right]$$

$$V_o(t) = 40\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 40\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 80\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$V_o(t) = \left(40u(t) - 40e^{-2t}u(t) - 80te^{-2t}u(t)\right) V$$

$$\boxed{V_o(t) = 40\left[1 - e^{-2t} - 2te^{-2t}\right]u(t)\ V}$$

Example: 16.2 find $V_o(t)$ in the circuit of Fig. 16.7, Assume $V_o(0^-) = 5V$ ⇒ Initial Condition

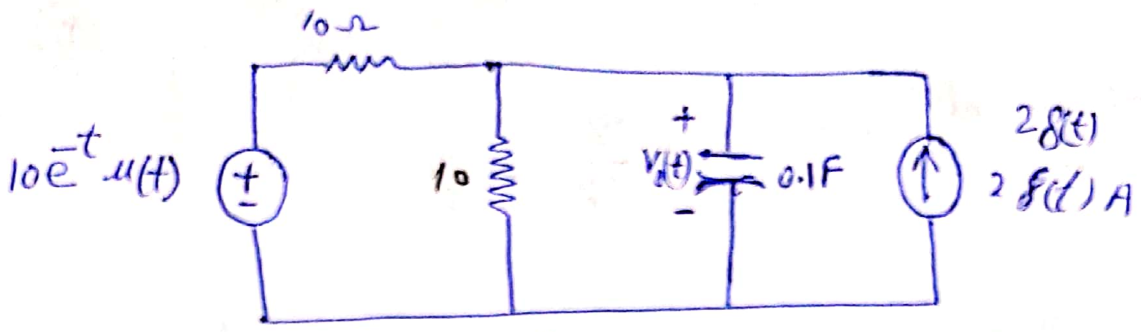


Fig 16.7 (Time Domain)

Solution

We transform the circuit to the S-Domain

$$10e^{-t}u(t) \Rightarrow \frac{10}{s+a}$$

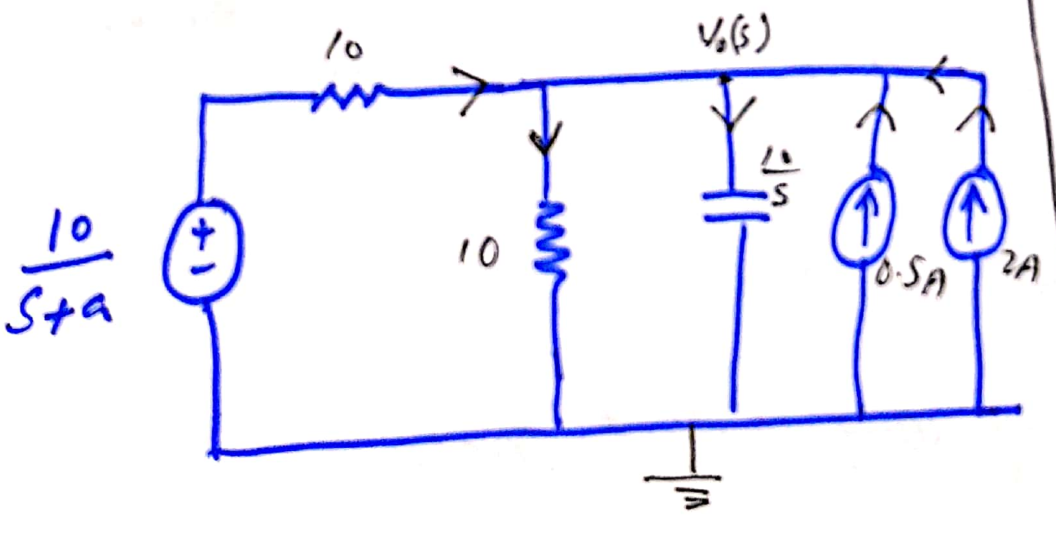
$$0.1F \Rightarrow Z = \frac{1}{sC} \text{ \& } CV(0^-) \text{ Initial Condition}$$

$$Z = \frac{1}{s(0.1)} = \frac{10}{s} \qquad V(0^-) = 5v$$

$$CV(0^-) = 0.1 \times 5 = 0.5$$

$$2\delta(t) = \mathcal{L}[2\delta(t)] = 2(1) = 2$$

Re-Draw the circuit in Fig 16.7 in S-Domain



$e^{-at} = \frac{1}{s+a}$

$e^{-t} = \frac{1}{s+1}$ $\boxed{a=1}$

Ex. 15.1 for Reference

$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}$

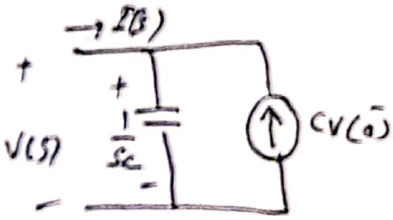
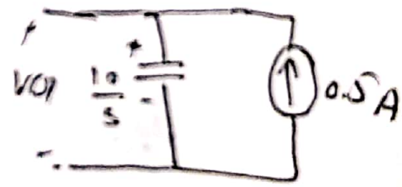


Fig 16.2

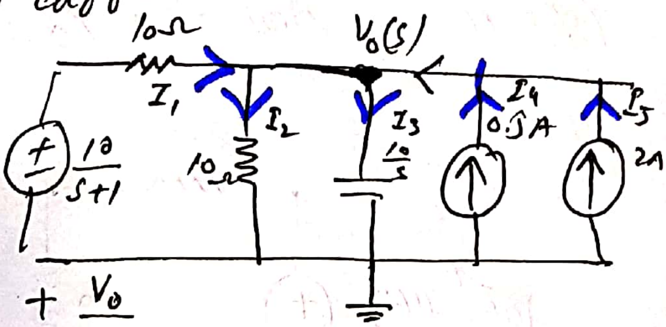


$\mathcal{L}[\delta(t)] = 1$

we apply Nodal Analysis at the Top node,

In coming current = out going current

$$I_1 + I_4 + I_5 = I_2 + I_3$$



$$\frac{10/s+1 - V_0}{10} + 0.5 + 2 = \frac{V_0}{10} + \frac{V_0}{10/s}$$

$$\frac{\cancel{10}}{(s+1)\cancel{10}} - \frac{V_0}{10} + 2.5 = \frac{V_0}{10} + \frac{sV_0}{10}$$

$$\frac{1}{s+1} + 2.5 = \frac{V_0}{10} + \frac{V_0}{10} + \frac{sV_0}{10}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_0}{10} + \frac{sV_0}{10}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_0 + sV_0}{10}$$

$$\frac{1}{s+1} + 2.5 = \frac{1}{10} V_0 (2+s)$$

Re-arrange

$$\frac{1}{s+1} + 2.5 = \frac{1}{10} V_0 (s+2)$$

For Easyness

Multiplying through by 10 on both Side

$$\frac{10}{s+1} + 25 = V_0 (s+2)$$

Now solving by using Partial fraction

$$\frac{10}{S+1} + 25 = V_0\,(S+2)$$

$$V_0 = \frac{1}{S+2}\left[\frac{10}{S+1} + \frac{25}{1}\right]$$

$$V_0 = \frac{1}{S+2}\left[\frac{10+25(S+1)}{S+1}\right] \quad \text{LCM}$$

$$V_0 = \frac{1}{S+2}\left[\frac{10+25S+25}{S+1}\right] \quad \text{Simplifying}$$

$$V_0 = \frac{25S+35}{(S+1)(S+2)}$$

Using Partial Fraction

$$V_0 = \frac{25S+35}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}$$

Multiplying with $(S+1)(S+2)$ on both Sides

$$25S+35 = A(S+2) + B(S+1) \longrightarrow (1)$$

Putting $S+1=0 \Rightarrow \boxed{S=-1}$ in Eq (1)

$$-25+35 = A(-1+2) + B(\cancel{-1+1})$$

$$10 = A(1) + 0$$

$$\boxed{A=10}$$

Now $S+2=0 \Rightarrow \boxed{S=-2}$ in Eq (1)

$$25(-2)+35 = A(\cancel{-2+2}) + B(-2+1)$$

$$-50+35 = 0 + B(-1)$$

$$\cancel{-}15 = \cancel{-}B$$

$$\boxed{B=15}$$

Rearranging

$A = 10$
$B = 15$ Putting the Values

$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

$$V_o(s) = 10\left(\frac{1}{s+1}\right) + 15\left(\frac{1}{s+2}\right)$$

Taking the Inverse Laplace Transform.

$$v_o(t) = \left(10e^{-t} + 15e^{-2t}\right)u(t)\ V$$

$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$

$\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$

Example 16.1, 16.2, 16.3, 16.4, 16.5, 16.6, ~~16.~~ 16.7 16.8, ~~16.10~~

P.P. 16.1 ⟶ 16.8, 16.10 & Ex. Prob. 16.1, 16.2 (16.3), 16.5, 16.16

(i) Application 2-(i) Network Stability (Poles & Zeros)

(ii) Network Synthesis

Summary

		Freq ($j\omega$)	Freq S-Domain.
→ Resistor	$v_R = Ri$ →	$V_R = RI$ →	$V_R = RI$
→ Inductor	$v_L = L\frac{di}{dt}$ →	$V_L = j\omega LI$ →	$V_L = sLI - Li(0^-)$ initial Cond.
→ Capacitor	$v_c = \frac{1}{C}\int i\,dt$ →	$V_c = \frac{I_c}{j\omega C}$ →	$V_c = \frac{I_c}{sC} - \frac{v(0^-)}{s}$