Properties of inverse Laplace Transform :-

formulas from Laplace Transform

① $L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

② $L[e^{at}] = \frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

③ $L[\sin at] = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

④ $L[\cos at] = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

⑤ $L[\sinh at] = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

⑥ $L[\cosh at] = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

⑦ $L[1] = \frac{1}{s} \Rightarrow L^{-1}\left[\frac{1}{s}\right] = 1$

⑧ $L[t] = \frac{1}{s^2} \Rightarrow L^{-1}\left[\frac{1}{s^2}\right] = t$

Examples:-

① $L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$

② $L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$

③ $L^{-1}\left[\frac{1}{s^6}\right] = L^{-1}\left[\frac{1}{s^{5+1}}\right] = \frac{t^5}{5!} = \frac{t^5}{120}$

$\xrightarrow{\text{mul \& div by 2}} = L^{-1}\left[\frac{2}{s^2+(2)^2}\right]\cdot\frac{1}{2}$

④ $L^{-1}\left[\frac{1}{s^2+4}\right] = L^{-1}\left[\frac{1}{s^2+(2)^2}\right] = \frac{1}{2}\sin 2t$

⑤ $L^{-1}\left[\frac{s}{s^2+9}\right] = L^{-1}\left[\frac{s}{s^2+(3)^2}\right] = \cos 3t$

$2\cdot\frac{1}{3}L^{-1}\left[\frac{3}{s^2-(3)^2}\right]$

⑥ $L^{-1}\left[\frac{2}{s^2-9}\right] = L^{-1}\left[\frac{2}{s^2-(3)^2}\right] = 2L^{-1}\left[\frac{1}{s^2-(3)^2}\right] = 2\cdot\frac{1}{3}\sinh 3t$

⑦ $L^{-1}\left[\frac{1}{s^{5/2}}\right] = L^{-1}\left[\frac{1}{s^{3/2+1}}\right] \longrightarrow L^{-1}\left[\frac{1}{s^{n+1}}\right]$

$= \frac{t^{3/2}}{3/2\,!} = \frac{t^{3/2}}{3/2\cdot\frac{1}{2}\cdot 1!}$ $\quad \frac{3}{2}$.

$= \frac{4t^{3/2}}{3\cdot 1!}$

Properties of laplace Inverse:-

① Shifting Property:-

If: $L[f(t)] = F(S)$ then $L[e^{at} f(t)] = F(S-a)$

then:

If: $L^{-1}[F(S)] = f(t)$ then: $L^{-1}[F(S-a)] = e^{at} f(t)$

e.g.:- $L^{-1}\left[\frac{1}{(S+2)^3}\right] = ? = e^{-2t} L^{-1}\left[\frac{1}{S^3}\right]$ $\Rightarrow S^{\overset{n+1}{2+1}}$

$= e^{-2t} \cdot \frac{t^2}{2!}$

$$L^{-1}\left[\frac{1}{(S+2)^3}\right] = \boxed{\frac{e^{-2t} \cdot t^2}{2}}$$

e.g.:- $L^{-1}\left[\frac{S}{S^2+2S+2}\right] = ? = L^{-1}\left[\frac{S}{(S+1)^2+1}\right]$ add & sub 1.

$= L^{-1}\left[\frac{S}{(S+1)^2+1}\right]$ or $= L^{-1}\left[\frac{S+1-1}{(S+1)^2+1}\right]$

$= L^{-1}\left[\frac{S+1}{(S+1)^2+1}\right] - L^{-1}\left[\frac{1}{(S+1)^2+1}\right]$

$= e^{-t} \cdot L^{-1}\left[\frac{S}{S^2+1}\right] - e^{-t} L^{-1}\left[\frac{1}{S^2+1}\right]$

$L^{-1}\left[\frac{S}{S^2+2S+2}\right] = e^{-t} \cdot \cos t - e^{-t} \sin t$

$= \boxed{e^{-t}(\cos t - \sin t)}$

② **Multiply by t property:- (Derivative in L^{-1})**

If: $L[t\,f(t)] = -\dfrac{d}{ds} F(S)$

Then: $L^{-1}[F(S)] = f(t)$ & $L^{-1}\left[\dfrac{d}{ds} F(S)\right] = -t.f(t)$

③ **Divide by t property:- (integral in L^{-1}).**

If: $L\left[\dfrac{f(t)}{t}\right] = \int_s^{\infty} F(S)\,ds$

Then: $L^{-1}\left[\int_s^{\infty} F(S)\,ds\right] = \dfrac{f(t)}{t}$.

④ **Derivative property:- (multiply by s in L^{-1})**

If: $L\left[\dfrac{d}{dt} f(t)\right] = sf(s) - f(0)$.

Then: $L^{-1}[sF(S)] = \dfrac{d}{dt} f(t)$ if $f(0) = 0$

$\because$ as: $L^{-1}(\text{const.}) =$ not exist.

⑤ **Integral Property:- (Divide by s in L^{-1}**

If: $L\left[\int_0^t f(t)\,dt\right] = \dfrac{F(S)}{s}$

Then: $L^{-1}\left[\dfrac{F(S)}{s}\right] = \int_0^t f(t)\,dt$.

⑥ **Shifting property:**

If: $L[e^{at} f(t)] = F(S-a)$

Then: $L^{-1}[F(S-a)] = e^{at} f(t)$

① Partial fraction method:-

Examples:-

① Evaluate:

$$L^{-1}\left[\frac{1}{(s+2)(s-3)}\right] = ?$$

using partial fraction:-

$$\frac{1}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)} \quad — ①$$

$$1 = A(s-3) + B(s+2) \quad — ②$$

put $s=3$ & $s=-2$ into ② one by one:-

at $s=3$:- $1 = B(3+2)$

$$\boxed{B = 1/5}$$

at $s=-2$:- $1 = A(-2-3)$

$$\boxed{A = -1/5}$$

putting A & B in ①:-

$$\frac{1}{(s+2)(s-3)} = \frac{-1}{5(s+2)} + \frac{1}{5(s-3)}$$

Taking laplace Inverse both side:-

$$L^{-1}\left[\frac{1}{(s+2)(s-3)}\right] = L^{-1}\left[\frac{-1}{5(s+2)}\right] + L^{-1}\left[\frac{1}{5(s-3)}\right]$$

$$= \frac{-1}{5} L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{5} L^{-1}\left[\frac{1}{s-3}\right]$$

$$\boxed{L^{-1}\left[\frac{1}{(s+2)(s-3)}\right] = -\frac{1}{5} e^{-2t} + \frac{1}{5} e^{3t}}$$

Example:-

(2) Evaluate: $L^{-1}\left[\log \frac{s+1}{s-1}\right]$

(3) Evaluate:

$$L^{-1}\left[\log \frac{s^2+1}{s(s+1)}\right]$$

using: $L^{-1}\left[\frac{d}{ds} F(s)\right] = -t f(t)$

solutions:-

(2) let $F(s) = \log \frac{(s+1)}{(s-1)} = \log(s+1) - \log(s-1)$

diff. both side w.r.t 's':-

$$\frac{d}{ds} F(s) = \frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

Taking laplace inverse both sides:-

$$L^{-1}\left[\frac{d}{ds} F(s)\right] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$-t f(t) = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s-1}\right]$$

$$-t f(t) = e^{-t} - e^{t}$$

or: $f(t) = \frac{e^{-t} - e^{t}}{-t}$

$$L^{-1}[F(s)] = \frac{e^{+t} - e^{-t}}{+t}$$

so,

$$\boxed{L^{-1}\left[\log \frac{s+1}{s-1}\right] = \frac{e^{t} - e^{-t}}{t}}$$

Evaluate $L^{-1}\left[\log \frac{s^2+1}{s(s+1)}\right]$

(3) Let $F(s) = \log \frac{(s^2+1)}{s(s+1)}$ $\qquad \log a(b) = \log a + \log b$

or:

$$F(s) = \log(s^2+1) - \log s(s+1)$$
$$= \log(s^2+1) - [\log s + \log(s+1)]$$
$$= \log(s^2+1) - \log s - \log(s+1)$$

diff. both side w.r.t s:-

$$\frac{d}{ds} F(s) = \frac{d}{ds}\left[\log(s^2+1) - \log s - \log(s+1)\right]$$

$$= \frac{1}{s^2+1}(2s) - \frac{1}{s} - \frac{1}{s+1}$$

Taking Laplace inverse both sides:-

$$L^{-1}\left[\frac{d}{ds} F(s)\right] = L^{-1}\left[\frac{2s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$-t f(t) = 2\cos t - 1 - e^{-t}$$

$$f(t) = \frac{2\cos t - 1 - e^{-t}}{-t}$$

or: $L^{-1}[F(s)] = \frac{2\cos t - 1 - e^{-t}}{-t}$

or: $$\boxed{L^{-1}\left[\log \frac{s^2+1}{s(s+1)}\right] = \frac{1 + e^{-t} - 2\cos t}{t}}$$

Scanned with CamScanner

Properties of Laplace Transform:-

① Time shifting:-

$$L[e^{\pm at} f(t)] \xrightarrow{L} F(s \mp a)$$

② Multiply by t:-

$$L[t f(t)] \longrightarrow -\frac{d}{ds} F(s)$$

③ Divide by t:-

$$L\left[\frac{f(t)}{t}\right] \longrightarrow \int_s^{\infty} F(s)$$

④ L.T of derivatives:-

$$L\left[\frac{d^n}{dt^n} f(t)\right] \longrightarrow s^n F(s) - s^{n-1} f^{n-n}(0) \ldots - s^{n-n} f^{n-1}(0)$$

⑤ L.T of integrals:-

$$L\left[\int_0^t f(t)\,dt\right] \Longrightarrow \frac{F(s)}{s}$$

⑥ Conjugation:-

$$L[f^*(t)] \longrightarrow F^*(s^*)$$

⑦ Time reversal:-

$$L[f(-t)] \longrightarrow F(-s)$$

⑧ Time scaling:-

$$L[f(at)] \longrightarrow \frac{1}{|a|} F(s/a)$$

(9) Time shifting:-

$$L[f(t \pm t_0)] \Rightarrow F(S)e^{\pm st_0}$$

(10) Freq. shifting:-

$$L[e^{\pm s_0 t} f(t)] \longrightarrow F(S \mp S_0)$$

(11) Convolution in time:-

If: $L[f_1(t)] \longrightarrow F_1(S)$

$L[f_2(t)] \longrightarrow F_2(S)$

Then: $L[f_1(t) * f_2(t)] \longrightarrow F_1(S) \cdot F_2(S)$