4.4.9 The Navier-Stokes Equations

The Navier-Stokes equations are used to solve many fluid flow problems. These equations represent the differential form of the conservation of linear momentum and are applicable in describing the motion of a fluid particle at an arbitrary location in the flow field at any instant of time.

We start with Cauchy's equation of motion:

$$(\mathbf{g} - \mathbf{a})\rho + \nabla \cdot \mathbf{P} = 0 \tag{4.98}$$

The stress dyadic $\mathbf{P}$ is expressed in terms of the rate of strain dyadic $\dot{\mathbf{S}}$ and pressure p through Eqs. (4.59) or (4.60). The rate of strain tensor $\dot{\mathbf{S}}$ is related to the velocity gradients through Eq. (4.39). Substituting the stresses of Eqs. (4.61)–(4.66) into Eqs. (4.99)–(4.101) results in the governing equations of motion in Cartesian coordinates for *compressible* fluid flow.

1. x-component

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho} \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial D}{\partial x}\right) \tag{4.106}$$

2. y-component

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho} \cdot \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial D}{\partial y}\right) \tag{4.107}$$

3. z-component

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho} \cdot \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial D}{\partial z}\right) \tag{4.108}$$

We can easily express the above compressible equations in *incompressible form* by setting the dilation D equal to zero. One of the more convenient incompressible forms is the vector equation

$$\boxed{\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V}} \quad M = 0 \tag{4.109}$$

(a) (b) (c) (d) (e)

The dimensions of each term are L/T^2; i.e., each term is an acceleration.

The term (a), $\partial\mathbf{V}/\partial t$, represents the *local acceleration* of the fluid particle at a fixed point in space. For steady flow, this term is zero.

The term (b), $(\mathbf{V}\cdot\nabla)\mathbf{V}$, is the *convective acceleration* of the fluid particle, and it predicts how the flow differs from one space location to the next at the same instant of time. Uniform flow has no convective acceleration, of course, since "uniform" means "of the same value."

The term (c), $\mathbf{g}$, represents the acceleration due to gravity.

The term (d), $-(1/\rho)\nabla p$, is the *pressure acceleration* due to the "pumping" action of the flow.

The last term, (e), $\nu\nabla^2\mathbf{V}$, is the *viscous deceleration* due to the fluid's frictional resistance to objects moving through it.

The physical significance of the various terms of Eq. (4.109) can be illustrated to some extent by considering a few examples of fluid flow.

Consider the motion of an inviscid fluid. Such fluids are considered to be *ideal*, so that the Navier-Stokes equation for inviscid fluid flows would then include terms (a)–(d) of Eq. (4.109):

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \mathbf{g} - \frac{1}{\rho}\nabla p \tag{4.110}$$

Popularly called *Euler's equation*, this is a first-order nonlinear partial differential equation and has some rather interesting solutions. Ideal fluid flows will be treated in succeeding chapters in much greater detail.

Another example of the Navier-Stokes equation is the case of very slow fluid motion. Such fluid flow problems are mathematically defined as those where the total acceleration $D\mathbf{V}/Dt$ [terms (a) and (b) of Eq. (4.109)] and the acceleration due to gravity, term c , are both zero. The Navier-Stokes equation for very slow motion is then

$$\nabla p = \mu \nabla^2 \mathbf{V} \tag{4.111}$$

and is popularly called *Stokes flow* [4.3], or creep flow. It applies to the analysis of fluid behavior in lubrication mechanics, capillary flows, and certain molten metals.

Example 4.12. Very Slow Motion

Consider the two-dimensional motion of a cover plate moving with velocity V_∞ in the y-direction located a fixed distance h above a fixed flat plate. Let there be a fluid of density ρ and dynamic viscosity μ between the two plates. Assume no x- and z-component of velocity. Also assume the flow is steady and that $g_z = 0$. Calculate the pressure, velocity, and shear stress distributions (Fig. E4.12).

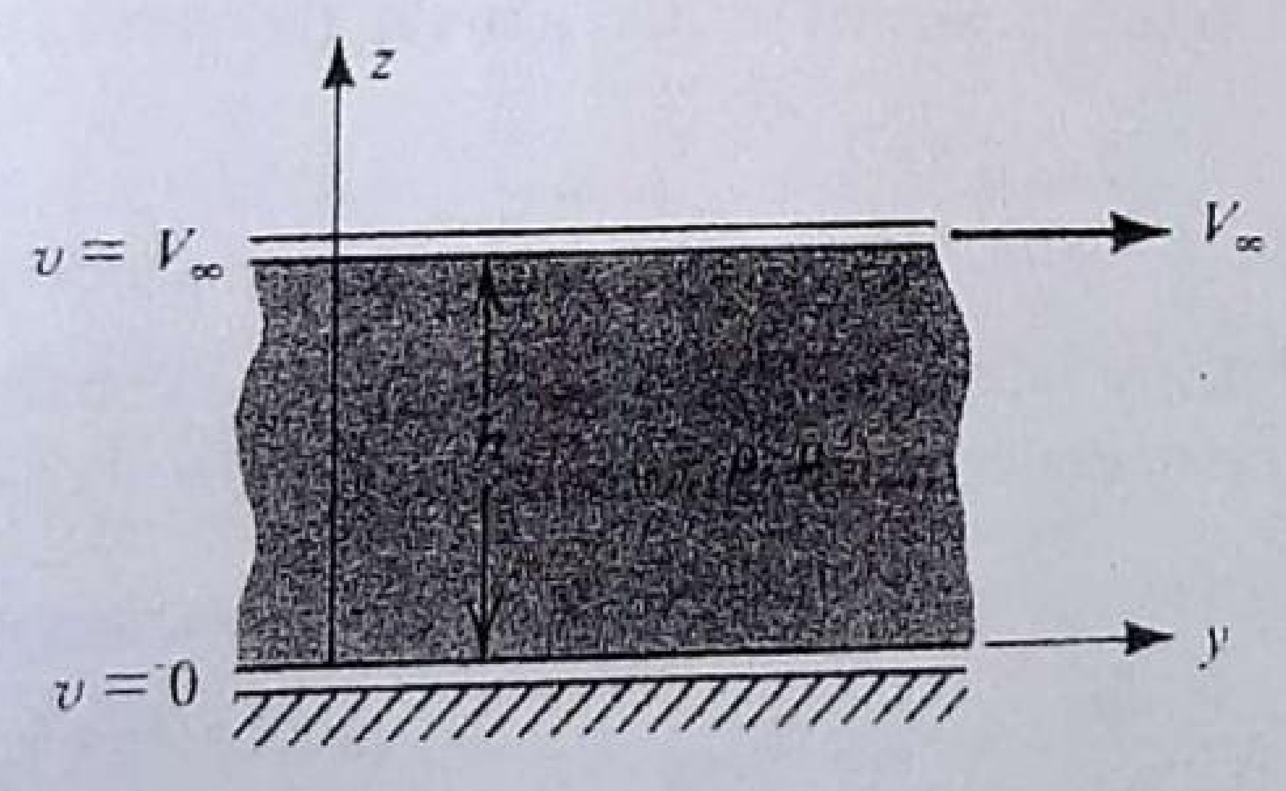

Figure E4.12

Solution:

Step 1.

Identify the characteristics of the fluid and flow field.

Assume the fluid is incompressible and real. The flow is steady and two-dimensional.

Example 4.12 *(Con't.)*

Step 2.
Write the appropriate form of the governing flow equations.

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.24}$$

and the Navier-Stokes equation is

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V} \tag{4.109}$$

Using the assumptions given on the velocity field $u = w = 0$, we obtain from the continuity equation

$$\frac{\partial v}{\partial y} = 0 \tag{i}$$

The x-component scalar forms of the Navier-Stokes equations yields

$$\frac{\partial p}{\partial x} = 0 \tag{ii}$$

so that for this example

$$p = p(y, z) \tag{iii}$$

The y-component scalar form of the Navier-Stokes equation yields

$$\frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{iv}$$

Substituting the results obtained from Eqs. (i) and (4.108) into Eq. (iv) results in

$$\frac{\partial p}{\partial y} = \mu\frac{d^2 v}{dz^2} \tag{v}$$

Since $\nu \neq \nu(x,y,t)$, Eq. (v) is readily integrable. Treating dp/dy as a constant, we have a simple second degree linear ordinary differential equation with the solution

$$v = \frac{1}{2\mu}\frac{dp}{dy}z^2 + c_1 z + c_2 \tag{vi}$$

From the given boundary conditions of the example

$$v = V_\infty \quad \text{at} \quad z = h \tag{vii}$$

and

$$v = 0 \quad \text{at} \quad z = 0 \tag{viii}$$

the coefficients c_1 and c_2 are evaluated with the result that the fluid flow velocity

Example 4.12 *(Con't.)*

$$v = \frac{1}{2\mu}\frac{dp}{dy}(z^2 - zh) + \frac{V_\infty}{h}z \qquad \text{(ix)}$$

As shown by Eq. (4.108), the pressure p varies only in the y- and z-direction or

$$p = \mu\left(\frac{d^2v}{dz^2}\right)y + gz + \text{const.} \qquad \text{(x)}$$

where d^2v/dz^2, we recall, is a constant.

The shear stress $p_{xz} = p_{xy} = 0$, since the velocities $v = v(z)$ and $u = w = 0$. The only nonzero shear stress is

$$p_{zy} = \mu\frac{\partial v}{\partial z}$$

$$= \frac{1}{2}\frac{dp}{dy}(2z - h) + \frac{\mu V_\infty}{h} \qquad \text{(xi)}$$

Note that the velocity field v is parabolic, and the stress field is linear. Note also that the stress p_{zy} is not zero at $h/2$ as we might suspect. From Eq. (x) we see that the shear stress is zero at the midpoint only if the upper plate is stationary (i.e., $V_\infty = 0$). This type of problem will reappear as one of the major topics in Chap. 12 on pipe flow.

This completes the solution.

Parallel flow constitutes that class of motion where all velocity components save one are zero. If that flow were in the x-direction, then terms ⓑ and ⓒ of Eq. (4.109) would be zero. The Navier-Stokes equation then becomes

$$\boxed{\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)} \qquad (4.112)$$

The above equation is a partial differential linear equation. One way to solve it is to try to transform it into an ordinary differential form. That will, of course, depend upon the particular flow problem.

The Navier-Stokes equation can also be expressed in other coordinate forms. Expressing the operator ∇ and D/Dt in cylindrical coordinates, Eq. (4.109) becomes

1. r-component

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + w\frac{\partial v_r}{\partial z}$$

$$= g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}\right.$$

$$\left. + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) \qquad (4.113)$$