Navier-Stokes Equation

(2)

lec # 13

INTRODUCTION

Navier-Stokes equation in fluid mechanics a partial differential equation that describes the flow of incompressible fluids. The equation is a generalization of the equation devised by swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and frictionless fluids. The Navier-Stokes equations are the broadly applied mathematical model to examine and changes on those properties (Velocity, Pressure, temperature, density and viscosity) during dynamic and thermal interactions.

Brief History of Navier Stokes equation

The knowledge of fluid flow cannot be called a discipline of science until **Isaac Newton** published his famous work Philosophiae Naturalis Principa Mathematica. In **1687,** Newton started to state in his principa that for straight parallel and uniform flow, the shear stress between layers is propotional to the velocity gradient in the direction perpendicular to the layers. With the development of calculus, many problems were solved in the frame of ideal fluid or inviscid fluid In **1738, Bernoulli** proved that the gradient of pressure is proportional to the acceleration. Later the famous differential equations were derived by Euler in **1752**, **d'Alembert** used inviscid theory in the form of potential solution of the incompressible Euler equations, to prove that the drag of the body of any shape moving through an inviscid fluid is zero, which is known as **"d'Alembert's paradox."** The result was obviously in contradiction to an abundance of evidence of the real world, hence mathematical fluid mechanics and engineering hydrodynamics were developed into separated branches.

Many researchers tried to add a friction term into **Euler's differential equation.** **Navier (1822), Cauchy (1828), Poisson (1829) and Saint-venant (1843)** suggested the function concerning the molecular mechanism. **Stokes (1845)** first used the absolute viscosity and assume that

1. The fluid is continuous and the

The stress tensor has a linear relation with the strain tensor.

2. The fluid is isotropic (the properties are independent of the direction and the frames concerned.

3. The form of the static-fluid pressure can be obtained as the strain rate is zero. The Navier-Stokes equation of motion is finally obtained.

• Description OF Navier Stokre equation:.

The Navier-Stokre equations developed by Claude-Louis Navier and George Gabriel Stokre in 1822 are equation which can be used to determine the velocity vector field that applies to a fluid, given some initial conditions. They arise from the application of Newton's second law in combination with a fluid stress (due to viscosity) and a pressure term. For almost all real situations they result in a system of non-linear partial differentlal equations; however with certain simplification (such as 1-dimensional) they can sometime be reduced to linear differential equations. Usually, however they remain non-linear which makes them difficult or impossible to solve; that is what causes the turbulence and unpredictability in their results.

The equations can perfectly describe how fluid flow behaves in a pipe or they can dictate how fluids behave in video games.

These equations are also used in weather prediction as they can predict air currents

- **Derivation**

The Navier-Stokes equations can be derived from the basic conservation and continuity equation applied to proptires of fluids. In order to derive the equations of fluid motions, we must derive the continuity equation of mass and momentum and finally combine the conservation equations with physical understanding of what a fluid is.

The Stresses as defined in the preceding section can be substituted into the differential equations of motion and simplified by using continuity equation for incompressible flow.

For rectangular coordinates the results are be derived as follows:

we already know the Differential Equations of Motions:-

$$\text{(A)} \leftarrow \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\text{(B)} \leftarrow \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\text{(C)} \leftarrow \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Also by the Stress-Deformation Relationships

For incompressible Newtonian fluids it is known that the stress are linear related to the rates of deformation and can be expressed in cartesian coordinates

(For normal stresses)

$$\sigma_{xx} = -P + 2\mu \frac{\partial u}{\partial x} \quad -(1)$$

$$\sigma_{yy} = -P + 2\mu \frac{\partial v}{\partial y} \quad -(2)$$

$$\sigma_{zz} = -P + 2\mu \frac{\partial w}{\partial z} \quad -(3)$$

(For shear stresses)

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad -(4)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \quad -(5)$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \quad -(6)$$

Put equation (1), (4) and (6) in (A)

This implies that

$$\rho g_x + \frac{\partial}{\partial x}\left(-P + 2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_x - \frac{\partial P}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \mu \frac{\partial}{\partial x}\frac{\partial w}{\partial z} + \mu \frac{\partial^2 u}{\partial z^2} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho g_x - \frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \mu \frac{\partial}{\partial x}\frac{\partial w}{\partial z} + \mu \frac{\partial^2 u}{\partial z^2} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Since $\frac{\partial}{\partial x}$ is common we want to make continuity equation then

$$-\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Since we know the equation

of continuity for incompressible fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Then this implies that

$$-\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}(0)$$

$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\boxed{-\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)} \quad (*)$$

The equation (*) is the x-component of Navier-Stoke equation which describes the fluid flow along x-direction.

Y-Component of Navier-Stoke equation:.

By eq (B)

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

Put equation (2) (4) and (5) in (B)

$$\rho g_y + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(-P + 2\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right)$$

$$= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho g_y + \mu\frac{\partial}{\partial y}\frac{\partial u}{\partial x} + \mu\frac{\partial^2 v}{\partial x^2} - \frac{\partial P}{\partial y} + \mu\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2}{\partial z^2}$$

$$+ \mu\frac{\partial}{\partial y}\frac{\partial w}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Since we know that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Then this implies that

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu \frac{\partial}{\partial y}(0) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\boxed{\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right)}$$

The final result shows that The Y-component of Navier stoke equation.

Z-component of Navier-Stoke equation:-

By eq (c) we have

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Put equation (3), (5) and (6) in (c). we have

$$\rho g_z + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + \frac{\partial}{\partial z}\left(-P + 2\mu\frac{\partial w}{\partial z}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho g_z + \mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial z}\frac{\partial u}{\partial x}\mu + \mu\frac{\partial}{\partial z}\frac{\partial v}{\partial y} + \mu\frac{\partial^2 w}{\partial y^2} - \frac{\partial P}{\partial z} + \mu\frac{\partial^2 w}{\partial z^2} + \mu\frac{\partial^2 w}{\partial z^2} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{\partial}{\partial z}(0) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\boxed{\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)}$$

Vectors Differntial Form OF Navier Stoke eq.

$$-\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V} = \rho \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right)$$

$$\Rightarrow \boxed{-\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V} = \rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right)}$$

Physical Meaning OF Navier Stoke Eq.

Term vise

$$-\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V} = \rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right)$$

$$\underbrace{-\nabla P}_{\text{Pressure Gradient}} + \underbrace{\rho \vec{g}}_{\text{Body Force Term}} + \underbrace{\mu \nabla^2 \vec{V}}_{\text{Viscous term}} = \underbrace{\rho \frac{\partial \vec{V}}{\partial t}}_{\text{local Acceleration}} + \underbrace{\rho (\vec{V} \cdot \nabla) \vec{V}}_{\text{Convective Acceleration}}$$

$-\nabla P$ = Pressure Gradient = Fluid flows in the direction of largest change in pressure.

$\rho \vec{g}$ = Body force term = external forces that act on the fluid (such as gravity, electromagnetic, etc).

$\mu \nabla^2 \vec{V}$ = viscous term = viscosity controlled velocity diffusion term.

$\rho \frac{\partial \vec{V}}{\partial t}$ = local acceleration = change of velocity with time

$\rho (\vec{V} \cdot \nabla) \vec{V}$ = Convective Acceleration .

• Dimensionless Navier-Stokes Equation

It is convenient to make the Navier-Stokes equation dimensionless. we define the dimensionless Euler variable as:

- $\bar{r} = \frac{r}{L}$

we may write as

$\bar{x} = \frac{x}{L} \Rightarrow x = L\vec{x}$

$\bar{y} = \frac{y}{L} \Rightarrow y = L\vec{y}$

$\bar{z} = \frac{z}{L} \Rightarrow z = L\vec{z}$

where **'L' is a characterstic length.**

we also define a dimenionsless velocity

- $\bar{V} = \frac{V}{U}$

$\Rightarrow \bar{u} = \frac{u}{U} \Rightarrow u = \bar{u} U$

$\Rightarrow \bar{v} = \frac{v}{U} \Rightarrow v = \bar{v} U$

$\Rightarrow \bar{w} = \frac{w}{U} \Rightarrow w = \bar{w} U$

where **'U' is a characterstic velocity.**

The dimensionless pressure and nabla

operator are defined

- $\bar{P} = \frac{P}{\rho U^2} \Rightarrow P = \rho U^2 \bar{P}$

- $\bar{\nabla} = L\nabla \Rightarrow \nabla = \frac{\bar{\nabla}}{L}$

- $T = \frac{Ut}{L} \Rightarrow t = \frac{TL}{U}$

Now From x-component of Navier-Stokes equation we have

$$-\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Now by Putting the above values of 'P', '∇', 'T', u, x and dividing ρ both sides we have

$$\frac{\partial(\bar{u}U)}{\partial(\frac{TL}{U})} + \frac{\bar{u}U\partial(\bar{u}U)}{\partial(L\bar{x})} + \frac{\bar{v}U\partial(\bar{u}U)}{\partial(L\bar{y})} + \frac{\bar{w}U\partial(\bar{u}U)}{\partial(L\bar{z})}$$

$$= \frac{-1}{\rho}\frac{\partial(\rho U^2\bar{P})}{\partial(L\bar{x})} + g_x + \frac{\mu}{\rho}\left(\frac{\partial^2(\bar{u}U)}{\partial(L\bar{x})^2} + \frac{\partial^2(\bar{u}U)}{\partial(L\bar{y})^2} + \frac{\partial^2(\bar{u}U)}{\partial(L\bar{z})^2}\right]$$

$$\frac{U^2}{L}\left(\frac{\partial\bar{u}}{\partial T} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right) = -\frac{U^2}{L}\frac{\partial\bar{P}}{\partial\bar{x}} + g_x + \frac{\nu U}{L^2}\left(\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{y}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right)$$

By dividing $\frac{U^2}{L}$ both sides we have

$$\frac{\partial\bar{u}}{\partial T} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}} = -\frac{\partial\bar{P}}{\partial\bar{x}} + \frac{L}{U^2}g_x + \frac{\nu}{LU}\left(\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{y}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right)$$

$$\boxed{\frac{\partial \bar{u}}{\partial \bar{T}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} = \frac{-\partial \bar{P}}{\partial \bar{x}} - \frac{K}{Fr^2} + \frac{1}{Re_L}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right)}$$

where $\boxed{Re_L = \frac{UL}{\nu}}$ and $\boxed{Fr = \frac{U}{\sqrt{gL}}}$

$$\Rightarrow Fr^2 = \frac{U^2}{gL}$$

- Similarly in Y-direction
  Dimensionless form of Navier Stoke eq.

$$\boxed{\frac{\partial \bar{v}}{\partial \bar{T}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{z}} = \frac{-\partial \bar{P}}{\partial \bar{y}} - \frac{K}{Fr^2} + \frac{1}{Re_L}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right)}$$

- In Z-direction

$$\boxed{\frac{\partial \bar{w}}{\partial \bar{T}} + \bar{u}\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} = \frac{-\partial \bar{P}}{\partial \bar{z}} - \frac{K}{Fr^2} + \frac{1}{Re_L}\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right)}$$

Vector Form of dimensionless Navier Stoke Equation:-

$$\frac{\partial \bar{\mathbf{V}}}{\partial \bar{T}} + (\bar{\mathbf{V}} \cdot \bar{\nabla})\bar{\mathbf{V}} = -\bar{\nabla}\bar{P} - \frac{K}{Fr^2} + \frac{1}{R_L}\bar{\nabla}^2\bar{\mathbf{V}} \rightarrow (J)$$

Important Note:-

* we can deduce the dimensionless Euler equation by

taking high value of Reynolds number.

To shown let $R_L \rightarrow \infty$ for fixed Euler variables we will generate the result from the dimensionless Navier-Stoke equation for incompressible flow

$$\frac{1}{R_L} = \frac{1}{\infty} = 0$$

This implies from equation (J)

Thus we obtain the result

$$\boxed{\frac{\partial \bar{V}}{\partial T} + (\bar{V} \cdot \bar{\nabla})\bar{V} = -\bar{\nabla}\bar{P} - \frac{K}{Fr^2}}$$

$\downarrow$

Dimensionless Euler Equation.

APPlication OF Navier Stoke Equation:-

- An exact solution of the 3D Navier-Stoke equation is simplified to 2D on the surface of a globe.
- An initial velocity along the equator

produces two symmetric flow towards the poles

- The velocity flows are arrived at without the use of numerical iterations speeding up the simulation.
- Air flow in a duck
- water flow in an open channel
- Combustion in cylinder
- Evaporation of water/condensation

Examples

Example # 1:-

Plane Coutte Flow:

Consider the flow of a viscous Newtonian fluid between two parallel plates located at $y=0$ and $y=h$. The upper plate is moving with velocity U. Calculate flow field?

Sol:

Assume the following

* steady flow: $\frac{\partial}{\partial t} = 0$
* Two dimensional flow: $v = 0$, $\frac{\partial u_i}{\partial x} = 0$ → Parallel fully developed flow
* No pressure gradient: $\frac{\partial P}{\partial x_i} = 0$

For two dimensional flow

$$w = 0, \quad \frac{\partial}{\partial z} = 0$$

The streamise

Navier Stoke equation

$$-\cancel{\frac{\partial P}{\partial x}}^{0} + \cancel{\rho g_x}^{0} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{0}\right) = \rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}}\right)$$

we get $\mu \frac{\partial^2 u}{\partial y^2} = 0$

On Integrating

$$\frac{\partial u}{\partial y} = A$$

Again Integrating

$$\Rightarrow \boxed{u = Ay + B} \quad — (1)$$

By the Boundary Conditions

$$u(0) = 0 \Rightarrow B = 0$$

$$u(h) = U$$

$$\Rightarrow U = Ah + 0$$

$$\boxed{A = \frac{U}{h}}$$

Putting the value in (1)

$$u(y) = \frac{Uy}{h} + 0$$

$$\boxed{u(y) = \frac{Uy}{h}} \quad \text{Ans}$$

This shows that the velocity varies linearly between the plate for $P = 0$.

Example #2

For a two dimensional incompressible flow in the x-y plane show that the z-component of the vorticity ζ varies in accordance with the equation

$$\frac{D\zeta_z}{Dt} = \nu \nabla^2 \zeta_z$$

What is the Physical interpretation of this equation for a non-viscous flow?

Solution:

For the two dimensional flow with $w = 0$.

Navier-Stokes equation for x-component reduce to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

For the y-component

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

Differentiating equation (1) with respect to y and equation (2) with respect to x and by subtracting we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\mu}{\rho}\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] \quad (3)$$

By definition

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

rewritting eq (3)

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\mu}{\rho}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \quad (4)$$

since each term in Paranthesis in ζ is 'ζ' it follows that equation (4) is

$$\frac{\partial \zeta_z}{\partial t} + u\frac{\partial \zeta_z}{\partial x} + v\frac{\partial \zeta_z}{\partial y} = \frac{\mu}{\rho}\left(\frac{\partial^2 \zeta_z}{\partial x^2} + \frac{\partial^2 \zeta_z}{\partial y^2}\right) \quad -(5)$$

the left hand side of equation (5) can be expressed as $\frac{D\zeta_z}{Dt}$ where The operator $\frac{D}{Dt}$ is The material derivative. The right hand side of eq (5) can be written as

$$\upsilon \nabla^2 \zeta_z$$

where $\upsilon = \frac{\mu}{\rho}$ so equation (5) can be written as

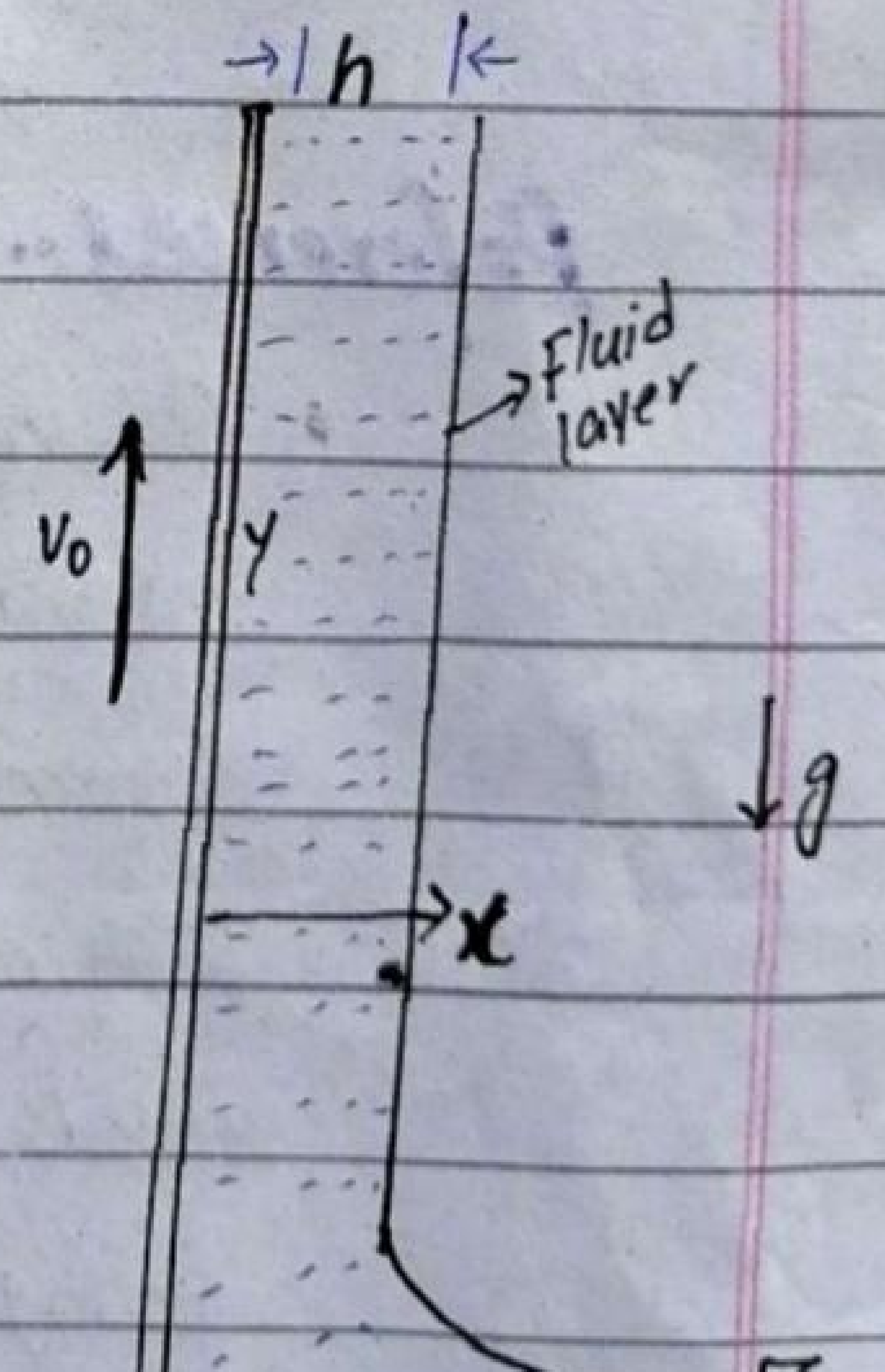
$$\frac{D\zeta_z}{Dt} = \upsilon \nabla^2 \zeta_z$$

For a nonviscous flow $\upsilon = 0$ and in this case $\frac{D\zeta_z}{Dt} = 0$

Thus for a two dimensional flow of an incompressible nonviscous fluid the change in the velocity of a fluid particle ϕ as it moves through the flow field is zero.

Example #3:- A wide moving belt passes through a container of a viscous liquid The belt moves vertically upward with a constant velocity V_0 Because of viscous forces the belt pick up a flim of fluid of thickness h. Gravity tends to made the fluid drain down the belt Assume that the flow is laminar, steady and fully developed.

Find :- Use Navier-Stokes equations to determine the average velocity of the fluid film as it is dragged up the belt.



Sol:-

Since the flow is assumed to be fully developed the only velocity component in the y-direction $u=0=w=0$ it follows from continuity equation $\frac{\partial v}{\partial y}=0$ So the Navier-Stoke equations for the x-direction y and z-direction simply reduced to

$$\frac{\partial P}{\partial x}=0 \quad \frac{\partial P}{\partial z}=0$$

This result indicates that the The vary over horizontal plane

Since the pressure on the surface of the film is atmospheric the pressure throughout the film must be atomospheric. The equation of motion in y direction reduce to

$$0 = -\rho g + \mu \frac{d^2v}{dx^2}$$

$$\frac{d^2v}{dx^2} = \frac{\gamma}{\mu}$$

Integrating w.r.t x above

$$\frac{dv}{dx} = \frac{\gamma}{\mu}x + C_1 \quad -(1)$$

on the film surface $x = h$ the drag of the air on film negligible so $\tau_{xy} = \mu\left(\frac{dv}{dx}\right)$

if $\tau_{xy} = 0$ at $x = h$

$\Rightarrow \quad \frac{dv}{dx} = 0 \quad$ at $x = h$

$\Rightarrow$ From (1) $\quad 0 = \frac{\gamma h}{\mu} + C_1$

$$C_1 = \frac{-\gamma h}{\mu}$$

$$\frac{dV}{dx} = \frac{\gamma}{\mu}x - \frac{\gamma h}{\mu}$$

Again integrating

$$V = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + C_2$$

At $x = 0 \quad v = v_0$

$$v_0 = C_2$$

$$\boxed{V = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + v_0}$$

the flow rate

$$q = \int_0^h v\,dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + v_0\right)dx \Rightarrow q = v_0 h - \frac{\gamma h^3}{3\mu}$$

The average velocity is $q = Vh \Rightarrow \boxed{V = v_0 - \frac{\gamma h^2}{3\mu}}$

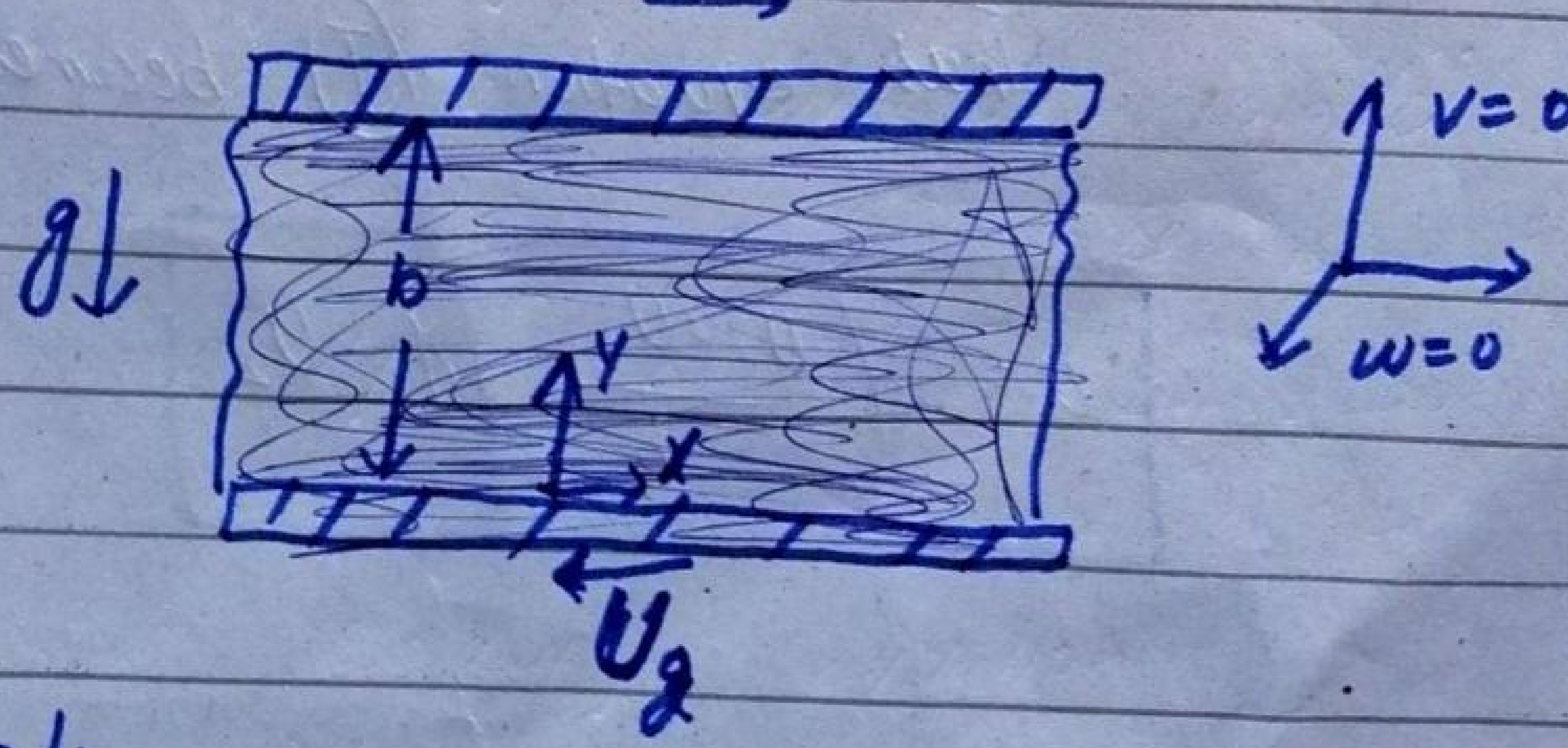
Average velocity

Excercise Problem :-

Question 1:-

An incompressible viscous fluid is placed between horizontal, infinite parallel plates as shown in figure. The two plates move in opposite direction with constant velocities U_1 and U_2. The pressure gradient in the x-direction is zero and the only body force is due to the fluid weight. Use Navies-Stokes equations to dervie an expression for the velocity distribution between the plates. Assume laminar flow.

Fig



Sol:-

For the specified conditions

$v=0$, $w=0$ $\frac{\partial p}{\partial x}=0$ and $g_x=0$,

So that the x-component of The Navier Stoke equations reduced to $\frac{\partial^2 u}{\partial y^2}=0$

$$\Rightarrow \frac{d^2u}{dy^2}=0 \quad -(I)$$

Integrating (I) yields

$$u = c_1 y + c_2 \quad -(J)$$

For $y=0$, $u=-U_2$ and therefore from fig (2)

$$y=0 \quad u=-U_2$$

$$\Rightarrow -U_2 = 0 + c_2$$

$$\boxed{c_2 = -U_2}$$

For $y=b$, $u=U$ so that

$$U_1 = c_1 b - U_2$$

$$\Rightarrow c_1 = \frac{U_1+U_2}{b}$$

Thus equation (J) becomes

$$\boxed{u = \left(\frac{U_1+U_2}{b}\right)y - U_2}$$

Summary:-

The Navier-Stokes equations are useful because they describe the Physics of many phenomena of scientific and engineering interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. These balance equations arise from applying Issac Newton's second law to fluid motion together with the assumption that the stress in the fluid is the sum of diffusing viscous term (propotional to the gradient velocity) and a pressure term hence describing viscous flow.

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