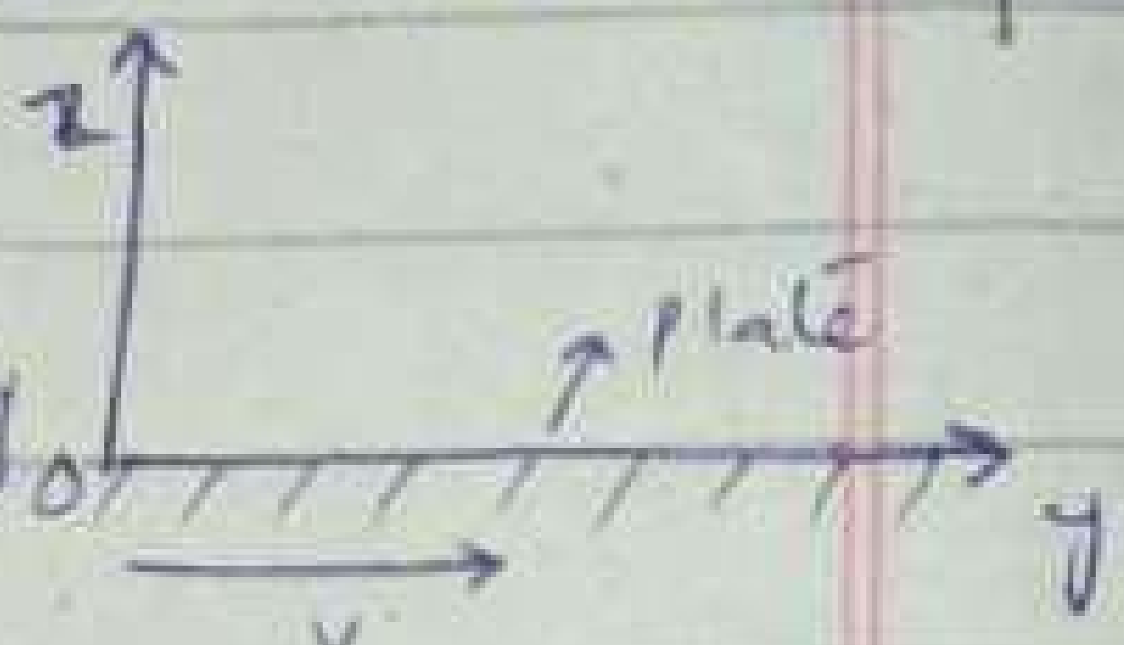
Lecture 9 Suddenly accelerated Flat Plate

Consider a semi-infinite flat plate immersed in a viscous fluid. The inertial co-ordinate system is shown as. At time $t<0$, the plate is static



At $t=0$, the plate is suddenly given a velocity in the positive y-direction. We seek the resultant velocity and pressure field in the fluid as a result of sudden acceleration of the flat plate. As from the figure, the plate induces a velocity only in the y-direction, since the plate is assumed to have no appreciable thickness. Hence the velocity components $u = w = 0$ and $v = v(z,t)$. Such a flow is seen to satisfy the (D.F) continuity equation. The Navier-stokes equations reduce to the simple form

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} \quad \text{—(1)}$$

$$\frac{dP}{dz} = -\rho g \quad \text{—(2)}$$

our pressure is the hydrostatic pressure

The boundary conditions for our problem are

$$u = V \quad \text{at } z=0 \quad \text{for } t>0$$

$$v \to 0 \quad \text{as } z \to \infty \quad \text{for all } t$$

eq ① is known as diffusion equation.

We suspect that the velocity component v will be some function of a combination of location z and real time t. Thus

$$v = V f(\eta) \quad — ①$$

$$\eta = \frac{1}{2} \frac{z}{\sqrt{\nu t}}$$

$$\frac{\partial v}{\partial t} = V \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = V f' \left(\frac{-\eta}{2t} \right)$$

$$\frac{\partial v}{\partial t} = \frac{-V}{2t} \eta f'$$

$$\frac{\partial v}{\partial z} = V \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} = V f' \left(\frac{1}{2\sqrt{\nu t}} \right)$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{V}{2} f'' \cdot \frac{1}{\sqrt{\nu t}} \cdot \frac{1}{2\sqrt{\nu t}} = \frac{V}{4\nu t} f''$$

So eq ① become

$$\frac{-\cancel{V}}{\cancel{2t}} \eta f' = \cancel{\nu} \frac{\cancel{V}}{{}_2\cancel{4} \cancel{\nu t}} f''$$

$$\Rightarrow f'' + 2\eta f' = 0$$

observe that we have transformed the partial differential equation by using the single independent similarity variable η, and then our differential equation is of a form which we can now solve

$f = 1$ at $\eta = 0$ $\qquad v = v f(\eta)$

$f \to 0$ as $\eta \to \infty$. $\qquad v = v f(\eta)$

$f'' + 2\eta f' = 0$ — (3) $\qquad f(\eta) = 1$

$f = 1$ at $\eta = 0$ $\qquad v \to 0$ as $\eta \to \infty$

$f \to 0$ as $\eta \to \infty$ $\qquad f \to 0$

to solve the differential equation (3), we get

$$g(\eta) = \frac{df}{d\eta}$$

to obtain a reduce first order differential equation:

$$\frac{dg}{d\eta} + 2\eta g = 0 \quad \text{— (4)}$$

The equation is homogeneous and separable, integration yeilds

$$g = C_1 e^{-\eta^2} = \frac{df}{d\eta}$$

$$\frac{df}{d\eta} = C_1 e^{-\eta^2}$$

$$f(\eta) = C_1 \int_0^{\eta} e^{-\eta^2} d\eta + C_2$$

we evaluate the constants of integration C_1 and C_2 using the B.Cs as

$f = 1$ at $\eta = 0$, $f \to 0$ as $\eta \to 0$

$$c_2 = 1 \quad , \quad c_1 = \frac{-1}{\int_0^\infty e^{-\eta^2} d\eta}$$

Hence the solution for similarity function $f(\eta)$ is

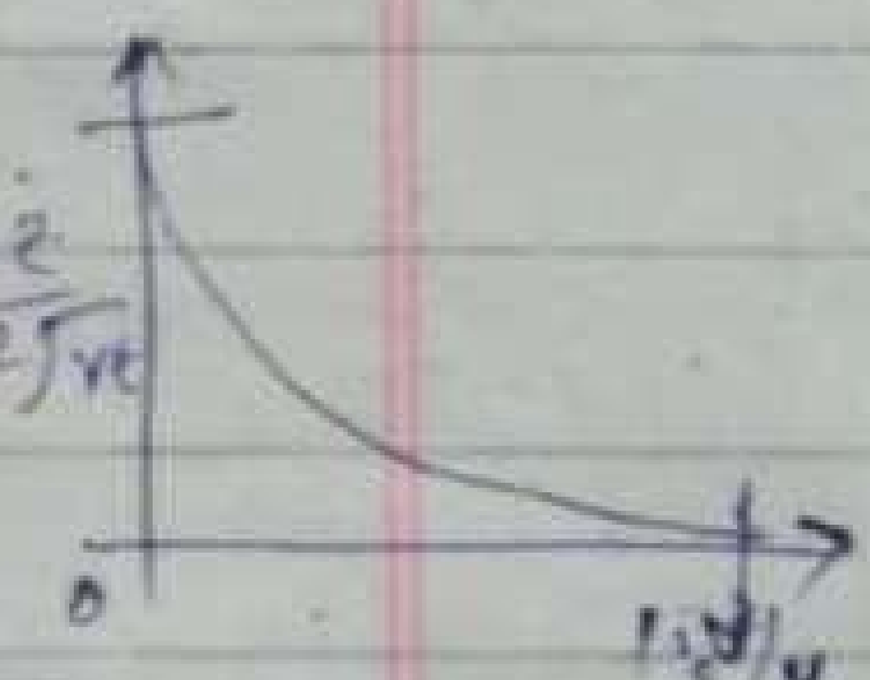
$$f(\eta) = 1 - \frac{\int_0^\eta e^{-\eta^2} d\eta}{\int_0^\infty e^{-\eta^2} d\eta}$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta$$

$$\operatorname{erf} \eta = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta$$ → found in probability theory.

(A) becomes

$$V = V\left(1 - \operatorname{erf} \frac{z}{\sqrt{4vt}}\right)$$



by taking $\eta = 1$,

$$\eta(\delta, t) = \frac{\delta}{2\sqrt{vt}}$$

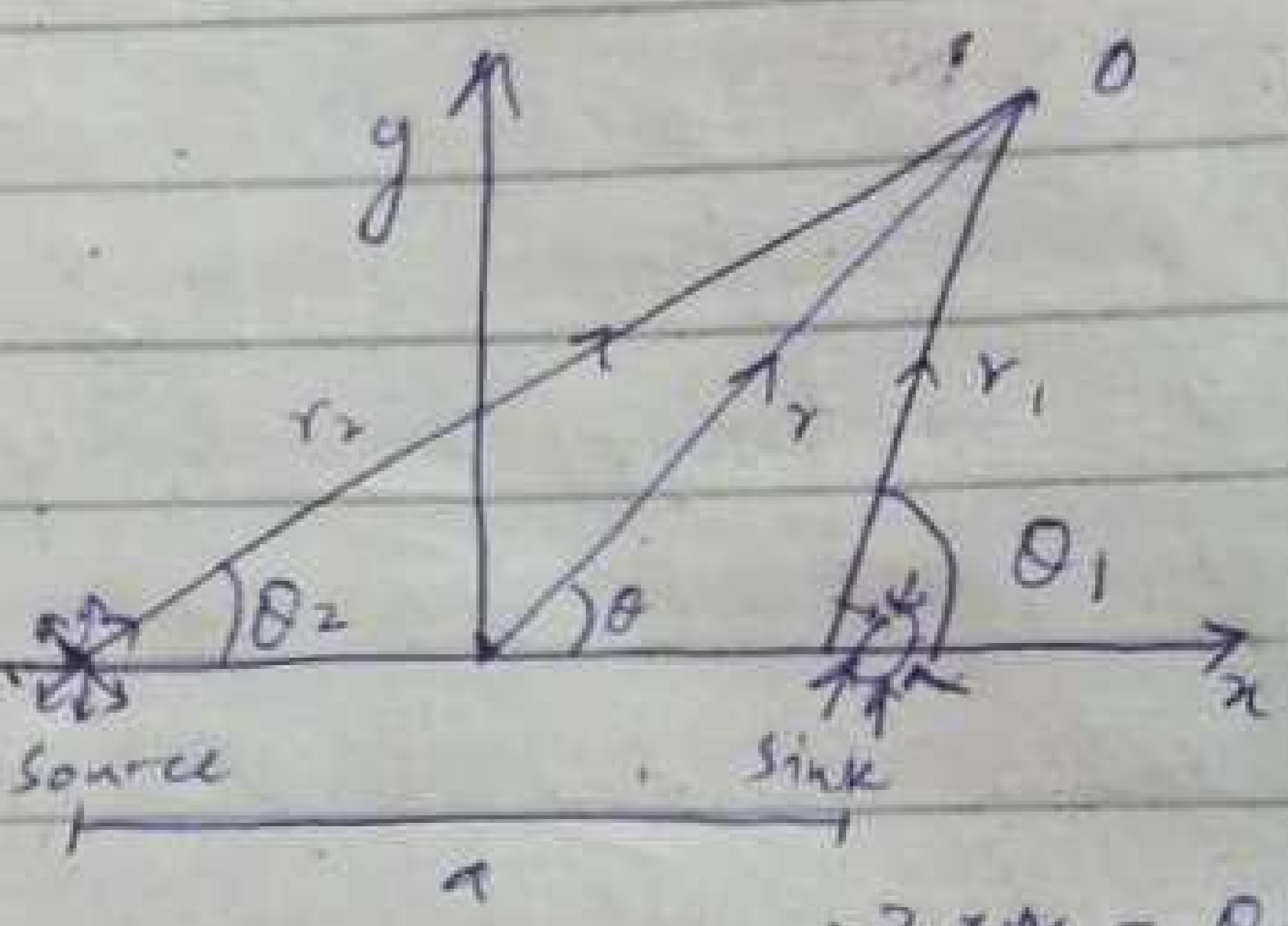
$$1 = \frac{\delta}{2\sqrt{vt}} \Rightarrow \delta = 2\sqrt{vt}$$

$$t = \frac{L}{V} \quad , \quad \delta = 4L\sqrt{\frac{\nu}{VL}}$$

$$\delta = \frac{4L}{\sqrt{R_L}}$$

Doublet: The find basic potential flow to be considered is one that is formed by combining a source and sink in a spacial way.
The combined stream function for the pair is

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$



$$\frac{-2\pi\psi}{m} = \theta_1 - \theta_2$$

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2)$$

From fig. $= \dfrac{\tan\theta_1 - \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$

$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$$

$$\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$$

By substituting;

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$$

$$\psi = \frac{-m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

typical streamlines for this flow for small values of the distance a

$$\psi = -\frac{m}{2\pi} \cdot \frac{2a\,r\sin\theta}{r^2-a^2} = -\frac{m a r \sin\theta}{\pi(r^2-a^2)}$$

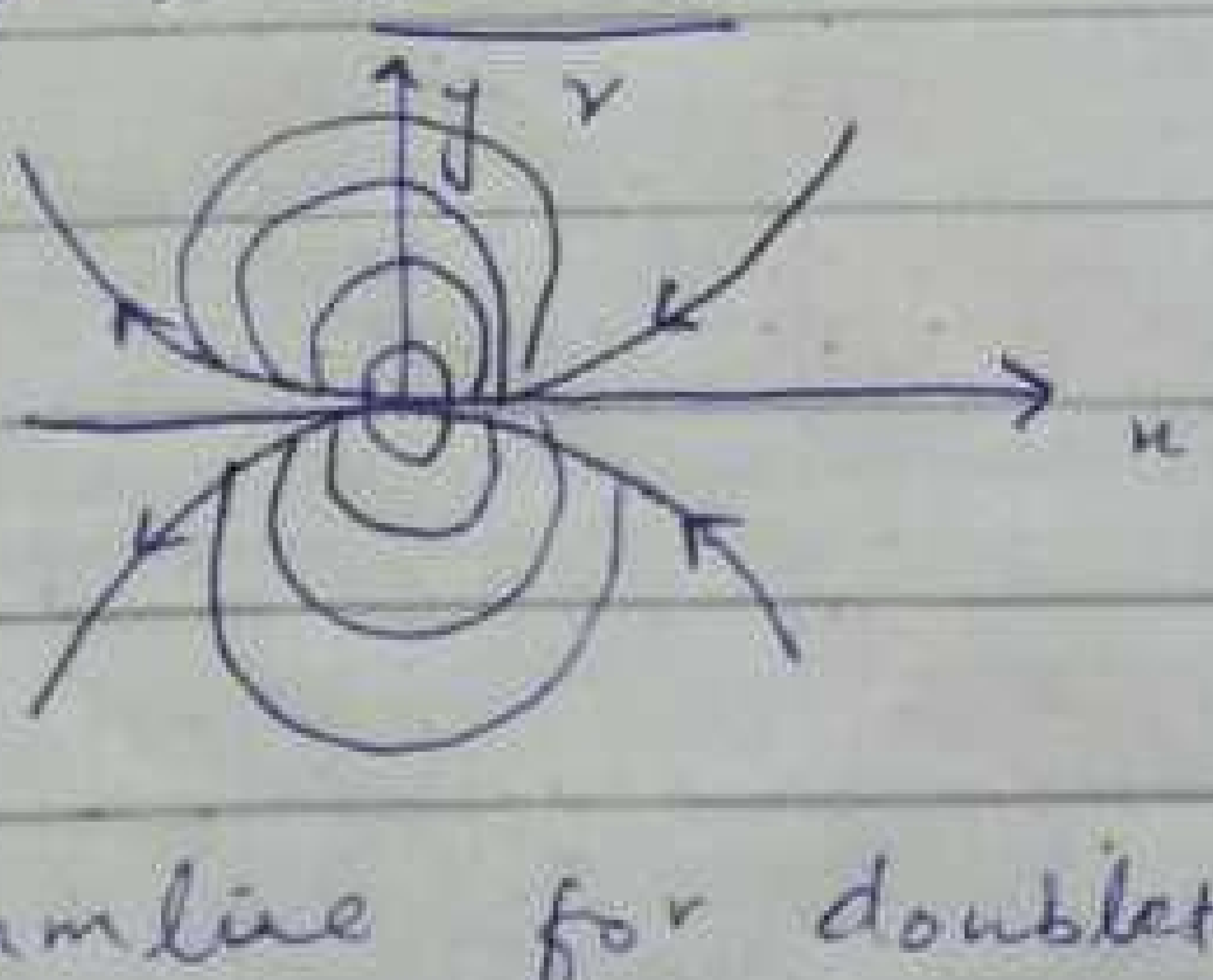
Since the tangent of an angle approaches the value of the angle for small angles. The so called doublet is formed by letting the source and sink approaches one another $(a \to 0)$, while increasing the strength m $(m \to \infty)$, So that the product $\frac{ma}{\pi}$ remains constant. In this case, since

$$\frac{r}{r^2-a^2} \to \frac{1}{r}$$

$$\psi = -K\frac{\sin\theta}{r}$$

where K is a constant equal to $\frac{ma}{\pi}$ is called the strength at the doublet. The corresponding velocity potential for the doublet is

$$\phi = \frac{K\cos\theta}{r}$$



Streamline for doublet

plots of lines of constant ψ reveal that the stream lines for a doublet are circles through the origin tangent to the x-axis.

Just as Sources and Sink are not physical realistic entities ~~are~~ neither are doublet. However the doublet when combined with other basic potential flows provides a useful representation of some fields of practical interest.