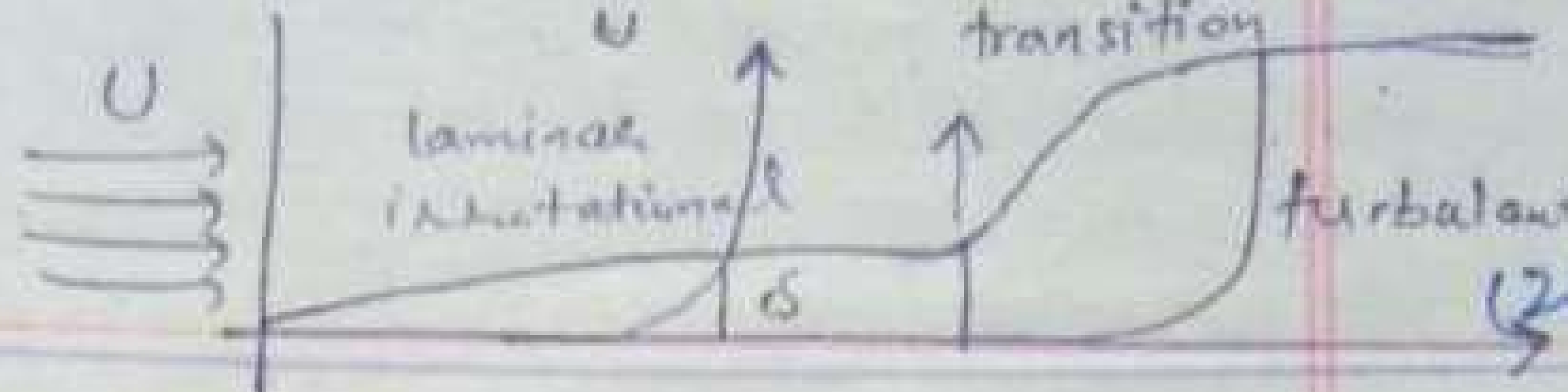
7/4/20 Lecture 6

Boundary layer flows:

The effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies where the fluid ~~in a region called the boundary layer, in which~~ adheres to the boundary. Thus close to a body in a region called the boundary layer, is ~~which~~ where shear stresses exert an increasingly large effect on the fluid as we move toward the solid boundary because of the increased velocity gradient $\frac{\partial u}{\partial y}$ as $y \to 0$



(26)

The boundary layer thickness of each region is different, and so one of the first task before us to estimate the thickness of the laminar region. The thickness of boundary layer is denoted by δ, and in the y- direction. Since we are concerned solely with the boundary layer of the flow, then the order of magnitude of y is δ. The largest value of the velocity component u is the free- stream velocity U, The inertial force per unit volume is $\rho u \left(\frac{\partial u}{\partial x}\right)$ and for laminar flow the viscous ~~flow~~ force per unit volume is

$$\frac{\partial \tau_{xy}}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} \qquad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

if the flow is parallel, making the following esti- -mation on order of magnitude

$$u \sim U \quad , y \sim \delta$$

$$\frac{\partial u}{\partial x} \sim \frac{u}{x} \quad , \quad \frac{\partial u}{\partial y} \sim \frac{u}{\delta}$$

$$\frac{\partial^2 u}{\partial y^2} \sim \frac{u}{\delta^2}$$

and setting from Navier- Stokes equation.

$$\rho u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$\rho \cancel{U} \frac{U}{x} \sim \mu \frac{\cancel{U}}{\delta^2}$$

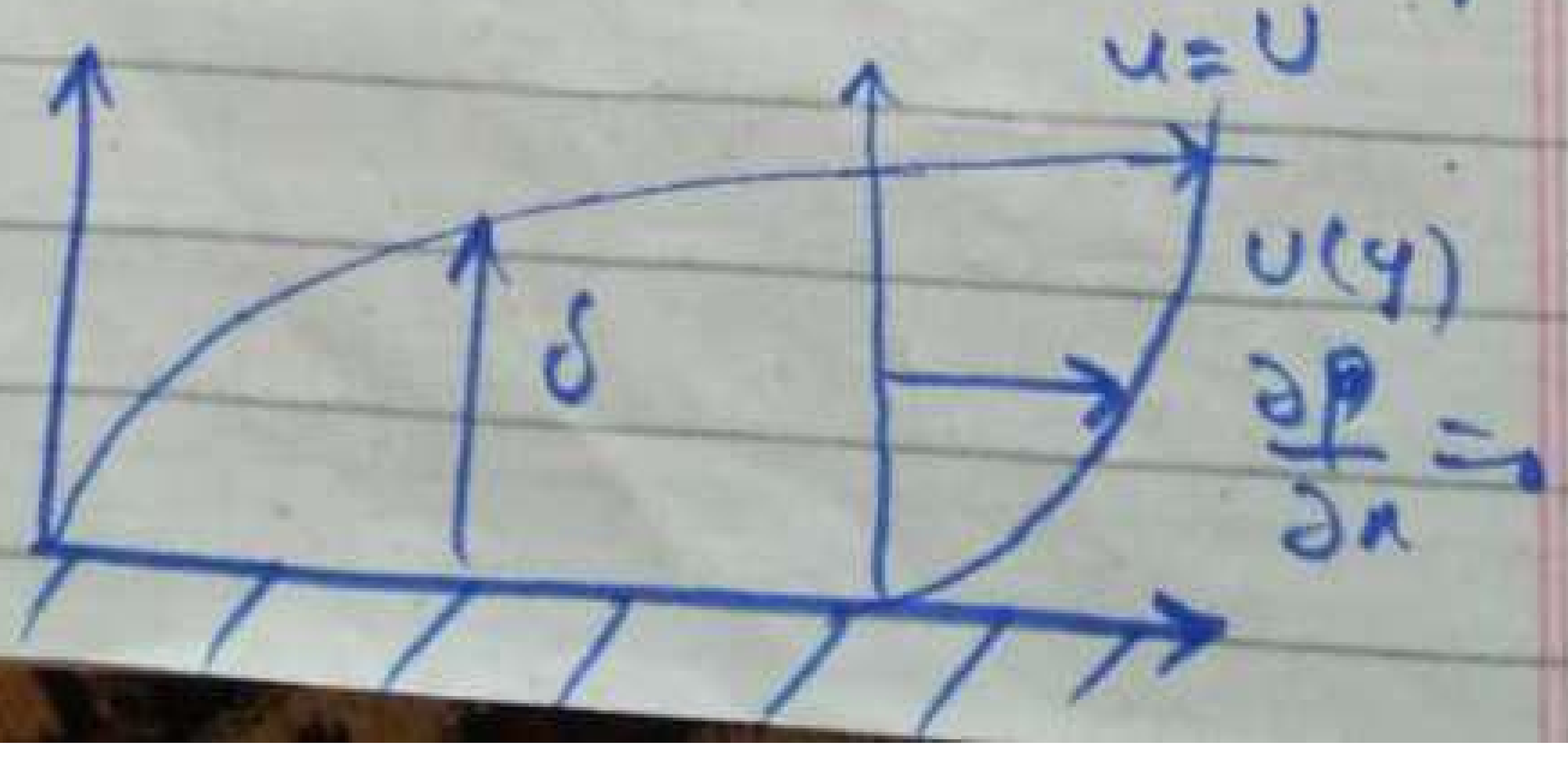
$$\delta^2 \sim \frac{\mu}{\rho U x} \Rightarrow \delta \sim \sqrt{\frac{\mu}{\rho U x}} \sim \frac{1}{\sqrt{Re_x}}$$

This shows that the boundary layer thickness for laminar flow over a flat plate is inversely proportional to the square root of density and free stream velocity, and directly proportional to the square root of the dynamic viscosity, and the distance from the leading edge of the plate.

EX Using the prandtl boundary layer eq.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{--- (1)}$$

Show that the velocity profile for a laminar flow past a flat plate has an infinite radius of curvature on the surface of the plate



Sol :- The radius of the curvature 'ρ' of the distribution of velocity u(y) is that used in the elementary calculus.

$$\rho = \frac{\left[1 + \left(\frac{du}{dy}\right)^2\right]^{3/2}}{d^2u/dy^2} \Rightarrow ②$$

The boundary conditions at the surface of the plate are $u = v = 0$ at $y = 0$ — ③

Substituting the boundary conditions given in ③ into ①

$$\cancel{u}^{0}\frac{\partial u}{\partial x} + \cancel{v}^{0}\frac{\partial u}{\partial y} = -\frac{1}{S}\cancel{\frac{\partial p}{\partial x}}^{0} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{d^2u}{dy^2} = 0 \quad \text{at } y = 0 \qquad \tau_w = \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

Substituting the gradient of the shear stress into eq ② ; $\rho = \infty$ (which means that very closed to the surface of the plate, the velocity is linear and the shear stress is constant.

Ex 2 :- Reduce the prandtl boundary layer eq. to a simpler form that that given by equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \overset{\Rightarrow -1/S\,\partial p/\partial x}{u\frac{du}{dx}} + \nu\frac{\partial^2 u}{\partial y^2} \quad — ①$$

$$\frac{\partial p}{\partial y} = 0 \; — ② \; ; \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \; — ③$$

for a) flow over a flat plate
b) the case $\tau_{yx} = C_1$
& c) The case where $v \propto \nu$
d) Solve the prandtl boundary layer eq. for the special case $v = \nu$ and where the pressure gradient $\frac{\partial P}{\partial x}$ is zero.

Sol:- a) For flow past a flate plate, $\frac{\partial P}{\partial x} = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad -(i)$$

b) for constant shear stress case the eq (1) reduces to

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)$$

$$C_1 = \mu\left(\frac{\partial u}{\partial y}\right)$$

$$u\frac{\partial u}{\partial x} + \frac{C_1 v}{\mu} = -\frac{1}{\rho}\cdot\frac{\partial P}{\partial x} \qquad \Rightarrow \frac{\partial u}{\partial y} = \frac{C_1}{\mu}$$

which can be altered to yeild

$$P + \frac{1}{2}\rho u^2 = -\frac{C_1}{\nu}\int v\,dx$$

thus total pressure can be determined if we know, how the y-component of velocity v vanishes in the flow.

c) For the case $v \propto \nu$, the prandtl boundary layer equations reduce to

$$\frac{d^2u}{dy^2} - a\frac{du}{dy} = -\frac{1}{\mu}\frac{dP}{dx}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial u}{\partial x} + \frac{\mu}{\rho}\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\rho u\frac{\partial u}{\partial x} + \mu\frac{\partial u}{\partial y} = -\frac{\rho}{\rho}\frac{\partial p}{\partial x} \quad ?$$

$$\rho u\frac{\partial u}{\partial x} = \mu\frac{d^2u}{dy^2}$$

$$\mu\frac{d^2v}{dy^2} + \mu\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{d^2u}{dy^2} + \frac{du}{dy} = -\frac{1}{\mu}\frac{dp}{dx}}$$

we note that the left hand side is a function of y and the right hand side is the function of x, which satisfies that the left and right hand sides are constant.

Setting $\frac{dp}{dx}$ to zero, reduces to

$$\frac{d^2u}{dy^2} - a\frac{du}{dy} = 0. \quad \text{where } a = -1$$

$$\frac{d(du/dy)}{du/dy} = a\,dy$$

integrating and taking antilogs on b-sides yeilds

$$\frac{du}{dy} = c_1 e^{ay} \Rightarrow u = \frac{c_1}{a} e^{ay} + c_2$$

we need two boundary conditions to evaluate the constants of integration. These would stem from known conditions of the velocity and stress field on specific boundaries that define the flow.

Assignment:- Express prandtl's boundary layer eq. for laminar flow over a flat surface if v has the same form as u i.e. $v \propto u$.

Express $u = u(x, y)$