27-3-20 Lecture 24

Flow Past a wedge

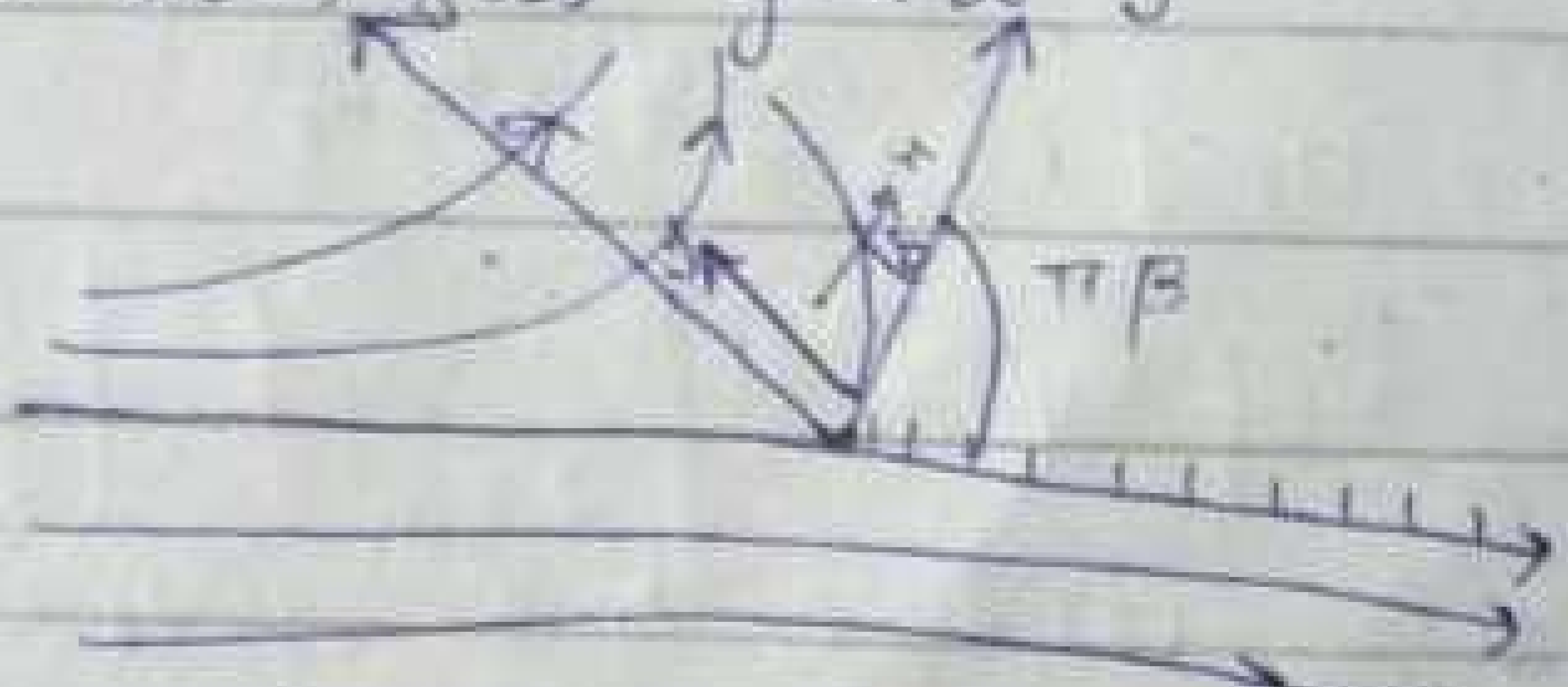
In the case of two dimensional motion, the boundary layer equation and their boundary conditions are given by equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad -(2)$$

The boundary conditions are

$$\left.\begin{array}{l} u = 0 \quad v = 0 \text{ at } y = 0 \\ u \to U(x) \text{ as } y \to \infty \end{array}\right\} (3)$$



Flow past a wedge

In most cases, it is convenient to integrate the eq. of continuity by the introduction of a stream function $\psi(x,y)$ so that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{— (4)}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Group of Transformation:

$$\left.\begin{aligned} \psi(x, y) &= \sqrt{\frac{2}{m+1}} \sqrt{\nu U_1}\, x^{\frac{m+1}{2}} f(\eta) \\ \eta &= y \sqrt{\frac{m+1}{2} \frac{U_1}{\nu}}\, x^{\frac{m-1}{2}} \\ U(x) &= U_1 x^m \end{aligned}\right\} \quad (5)$$

Thus the velocity components become

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{\frac{2}{m+1}} \sqrt{\nu U_1}\, x^{\frac{m+1}{2}} f(\eta)\right)$$

$$= \sqrt{\frac{2}{m+1}} \sqrt{\nu U_1}\, x^{\frac{m+1}{2}} \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\because \quad \frac{\partial \eta}{\partial y} = \sqrt{\frac{m+1}{2} \frac{U_1}{\nu}}\, x^{\frac{m-1}{2}}$$

$$\Rightarrow u = \sqrt{\frac{\cancel{2}}{\cancel{m+1}}} \sqrt{\cancel{\nu} U_1} \cdot x^{\frac{m+1}{2}} f'(\eta) \cdot \sqrt{\frac{\cancel{m+1}}{\cancel{2}} \frac{U_1}{\cancel{\nu}}}\, x^{\frac{m-1}{2}}$$

$$= U_1 f'(\eta)\, x^{\frac{m+1}{2} + \frac{m-1}{2}}$$

$$= U_1 f'(\eta) x^m$$

where $U(x) = U_1 x^m$, $U_1 = \frac{U(x)}{x^m}$

$$\Rightarrow u = \frac{U}{\cancel{x^m}} \cdot f'(\eta)\, \cancel{x^m}$$

$$\boxed{u = U f'(\eta)} \quad \text{— (6)}$$

$\frac{m-1}{2} - 1$ $\frac{m-1-2}{2}$

Similarly; $v = -\frac{\partial \psi}{\partial x}$

$$v = -\frac{\partial}{\partial x}\left(\sqrt{\frac{2}{m+1}}\sqrt{\nu U_1}\; x^{\frac{m+1}{2}} f(\eta)\right)$$

$$= -\sqrt{\frac{2}{m+1}}\sqrt{\nu U_1}\left[x^{\frac{m+1}{2}} \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{m+1}{2x} x^{\frac{m+1}{2}} f\right]$$

$$\because \frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{m+1}{2}\frac{U_1}{\nu}}\; x^{\frac{m-1}{2}}\right)$$

$$= y\sqrt{\frac{m+1}{2}\frac{U_1}{\nu}} \cdot \frac{m-1}{2} x^{\frac{m-3}{2}}$$

$$= \eta \cdot \frac{m-1}{2x}$$

$$v = -\sqrt{\frac{2}{m+1}}\sqrt{\nu U_1}\; \frac{x^{\frac{m+1}{2}}}{2x}\left[(m-1)\eta f' + (m+1) f\right]$$

$$= -\sqrt{\frac{2}{m+1}}\sqrt{\nu U_1}\; \frac{x^{\frac{m+1}{2}-1}}{2}(m+1)\left[f + \frac{m-1}{m+1}\eta f'\right]$$

$$= -\frac{\cancel{\sqrt{2}}}{\sqrt{\cancel{m+1}}}\sqrt{\nu U_1}\; \frac{x^{\frac{m-1}{2}}}{\cancel{\sqrt{2}}\sqrt{2}}\sqrt{\cancel{m+1}}\sqrt{m+1}\left[f + \frac{m-1}{m+1}\eta f'\right]$$

$$= -\sqrt{\frac{m+1}{2}\nu U_1}\cdot x^{\frac{m-1}{2}}\left[f + \frac{m-1}{m+1}\eta f'\right] \quad (7)$$

$$U = U_1 x^m \Rightarrow \frac{dU}{dx} = m U_1 x^{m-1}$$

$$U\frac{dU}{dx} = U_1 m x^{m-1} \cdot U_1 x^m = m U_1^2 x^{2m-1}$$

$$m = \frac{\beta}{2-\beta} \Rightarrow m(2-\beta) = \beta$$

$$2m - m\beta = \beta \Rightarrow \beta = \frac{2m}{m+1}$$

$u = U f'$, $V = -\sqrt{\frac{m+1}{2} \nu U_1 \; x^{m-1}} \left[f + \frac{m-1}{m+1} \eta f' \right]$

$\beta = \frac{2m}{m+1}$, $U = U_1 x^m$

$$\frac{\partial u}{\partial x} = f' \frac{\partial U}{\partial x} + U \frac{\partial f'}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$= f' U_1 m x^{m-1} + U f'' (m-1) \frac{\eta}{2x}$$

$$= f' U_1 m x^{m-1} + U_1 x^m f'' \frac{(m-1)}{2x}$$

$$\boxed{\frac{\partial u}{\partial x} = U_1 m x^{m-1} f' + \frac{U_1 x^{m-1}}{2} f'' (m-1)}$$

$$\frac{\partial u}{\partial y} = U \frac{\partial f'}{\partial \eta} \frac{\partial \eta}{\partial y} + f' \cancel{\frac{\partial U}{\partial y}}^{0}$$

$$= U_1 x^m f'' \sqrt{\frac{m+1}{2} \frac{U_1}{\nu}} \; x^{\frac{m-1}{2}}$$

$\frac{m-1}{2} + 1$

$\frac{m-1+2}{2}$

$m-$

$$\boxed{\frac{\partial u}{\partial y} = U_1 f'' \sqrt{\frac{m+1}{2} \frac{U_1}{\nu}} \; x^{\frac{3m-1}{2}}}$$

$$\frac{\partial^2 u}{\partial y^2} = x^{\frac{3m-1}{2}} U_1 \sqrt{\frac{m+1}{2\nu} U_1} \; f''' \cdot \frac{\partial \eta}{\partial y}$$

$$= x^{\frac{3m-1}{2}} U_1 \sqrt{\frac{m+1}{2\nu} U_1} \; f''' \left[\sqrt{\frac{m+1}{2} \frac{U_1}{\nu}} \; x^{\frac{m-1}{2}} \right]$$

$$= x^{\frac{3m-1}{2}} U_1^2 \left(\frac{m+1}{2\nu} \right) f''' \cdot x^{\frac{m-1}{2}}$$

$\frac{3m-1+m-1}{2}$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = x^{2m-1} \; U_1^2 \left(\frac{m+1}{2\nu} \right) f'''}$$

Put in $\frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} = \frac{u\,du}{dx} + \frac{\nu\partial^2 u}{\partial y^2}$

$$\Rightarrow uf'\left[u_1 m x^{m-1} f' + u_1 \frac{x^{m-1}}{2} f''(m-1)\eta\right] + \left[-\sqrt{\frac{m+1}{2}\nu u_1 x^{m-1}}\left[f + \frac{m-1}{m+1}\eta f'\right]\right] \times$$

$$\left[u_1 f''\sqrt{\frac{m+1}{2\nu}u_1}\cdot x^{\frac{3m-1}{2}}\right] = m u_1^2 x^{2m-1} + \cancel{\nu} x^{2m-1} u_1^2\left(\frac{m+1}{2\cancel{\nu}}\right)f'''$$

$$\Rightarrow u_1 x^m f'\left[u_1 m x^{m-1} f' + \frac{u_1 x^{m-1}}{2} f''(m-1)\eta\right] - \left[u_1 f''\sqrt{\frac{m+1}{2}\cancel{\nu} u_1 x^{m-1}}\left(\frac{m+1}{2\cancel{\nu}}u_1\right) x^{\frac{3m-1}{2}}\right] \times$$

$$\left(f + \frac{m-1}{m+1}\eta f'\right) = m u_1^2 x^{2m-1} + \frac{x^{2m-1}}{2} u_1^2 (m+1) f'''$$

$$\Rightarrow m u_1^2 x^{2m-1} f'^2 + u_1^2 \frac{x^{2m-1}}{2} f' f''(m-1)\eta - u_1^2 f''\left(\frac{m+1}{2}\right) x^{2m-1}\left(f + \frac{m-1}{m+1}\eta f'\right)$$

$$= m u_1^2 x^{2m-1} + \frac{x^{2m-1}}{2} u_1^2 (m+1) f'''$$

$$\Rightarrow (u_1^2 \cancel{x^{2m-1}})\left[m f'^2 + f'f''\frac{(m-1)}{2}\eta - \left(\frac{m+1}{2}\right)ff'' - \left(\frac{m-1}{2}\right)f'f''\eta\right]$$

$$= u_1^2 \cancel{x^{2m-1}}\left(m + \frac{m+1}{2}f'''\right)$$

$$\Rightarrow \left(\frac{m+1}{2}\right)f''' + m - mf'^2 - f'f''\frac{(m-1)}{2}\eta + \left(\frac{m+1}{2}\right)(ff'') - \left(\frac{m-1}{2}\right)f'f''\eta = 0$$

multiplying by $\frac{2}{m+1}$

$$f''' + \frac{2m}{m+1} - \frac{2m}{m+1}f'^2 - f'f''\frac{m-1}{m+1}\eta + \left(\frac{m+1}{2}\right)\left(\frac{2}{m+1}\right)ff'' - \frac{m-1}{m+1}f'f''\eta = 0$$

$$f''' + \frac{2m}{m+1} - \frac{2m}{m+1}f'^2 + ff'' = 0$$

$$f''' + ff'' + \frac{2m}{m+1}\left[1 - f'^2\right] = 0$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

B.Cs: $f = f' = 0$ at $\eta = 0$, $f' \rightarrow 1$ as $\eta \rightarrow \infty$.