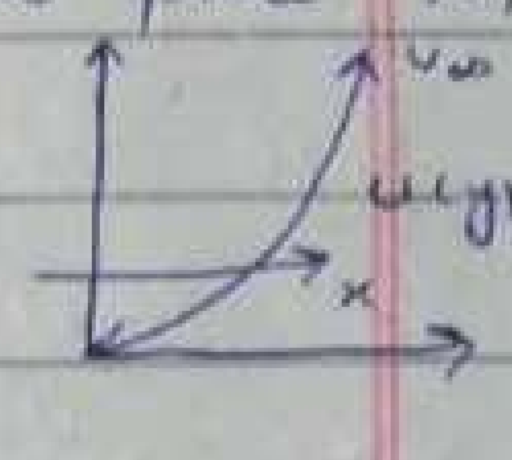
Lecture 3 **Fluid Dynamics - II**

Boundary layer thickness (δ)

Boundary layer thickness is defined as "the distance from the wall where the velocity differs by 1% from the external velocity (free-stream velocity U_∞)". The dimensionless boundary layer thickness, referred to the length of the plate l, becomes

$$\frac{\delta}{l} = 5\sqrt{\frac{\nu}{U l}} = \frac{5}{\sqrt{Re}}$$



δ is boundary layer thickness

l is characteristic length, U_∞ is free-stream velocity

$u(y)$ is the velocity of the fluid under influence of solid surface due to no-slip conditions.

Displacement boundary layer thickness

The displacement thickness is that distance by which the external potential field of flow is displaced outwards as a consequence of the decrease in velocity in the boundary layer. The decrease in volume flow due to the influence of friction is

$$\int_{y=0}^{\infty} (U_\infty - u)\, dy \quad (2)$$

so that for displacement thickness δ_1 we have the definition

$$U_\infty \delta_1 = \int_{y=0}^{\infty} (U_\infty - U)\, dy$$

$$\delta_1 U_\infty = U_\infty \int_{y=0}^{\infty} \left(1 - \frac{u}{U_\infty}\right) dy \quad - (3)$$

as U_∞ is free stream velocity and is constant.

So $$\delta_1 = \int_{y=0}^{\infty} \left(1 - \frac{u}{U_\infty}\right) dy \quad - (4)$$

as from flow along parallel plates.

We have $u = U_\infty f'(\eta)$ — (5) (previous lecture)

$$\frac{u}{U_\infty} = f'(\eta) \quad - (6)$$

using (6) in (1) $$\delta_1 = \int_{y=0}^{\infty} (1 - f'(\eta))\, dy$$

Now $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$, $$\frac{d\eta}{dy} = \sqrt{\frac{U_\infty}{\nu x}}$$

$$dy = \sqrt{\frac{\nu x}{U_\infty}}\, d\eta$$

$$\delta_1 = \int_{\eta=0}^{\infty} (1 - f'(\eta)) \sqrt{\frac{\nu x}{U_\infty}} \cdot d\eta$$

on integrations-

$$\Rightarrow \delta_1 = \sqrt{\frac{\nu x}{U_\infty}} \left[\eta - f(\eta) \Big|_0^\infty \right] \qquad \infty = \text{max. distance}$$

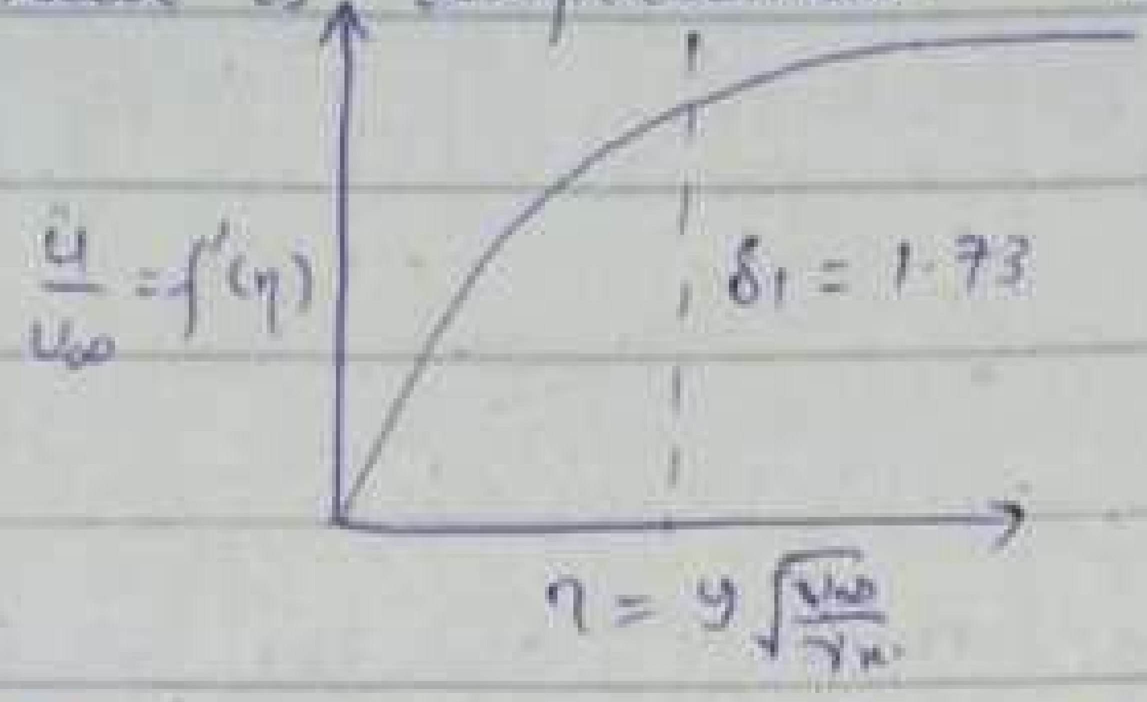
$$= \sqrt{\frac{\nu x}{U_\infty}} \left[\eta_1 - f(\eta_1) \right] \qquad \because f(0) = 0$$

where η_1 denotes a point outside the boundary layer using table 7.1 (Pg 139)

$$\delta_1 = \sqrt{\frac{\nu x}{U_\infty}} \left(8.80000 - 7.07923\right)$$

$$\delta_1 = 1.7208\sqrt{\frac{\nu x}{U_\infty}}$$

which is displacement thickness.



Momentum boundary layer thickness

~~boundary layer thickness (δ_2) is the~~ The momentum boundary layer thickness (δ_2) is the loss of momentum in the boundary layer as compared with potential flow given by $\rho\int_0^\infty u(U_\infty - u)\,dy$. so that a new thickness can be defined by

$$\rho U_\infty^2 \delta_2 = \rho\int_{y=0}^\infty u(U_\infty - u)\,dy$$

$$\delta_2 = \int_{y=0}^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$$

the practise as given in displacement boundary layer thickness, numerical evaluation for the plate at zero incidence gives:

$$\delta_2 = \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty f'(1-f')\,d\eta$$

$$= \sqrt{\frac{\nu x}{U_\infty}}\left[\int_0^\infty f'(\eta)\,d\eta - \int_0^\infty (f'(\eta))^2\,d\eta\right] \quad \text{Let } f'(\eta) = g(\eta)$$

$$= \sqrt{\frac{\nu x}{U_\infty}}\left[f(\eta)\Big|_0^\infty - \frac{g(\eta)^3}{3}\Big|_0^\infty\right]$$

$\delta_2 = 0.664\sqrt{\frac{\nu x}{u_\infty}}$

(11)

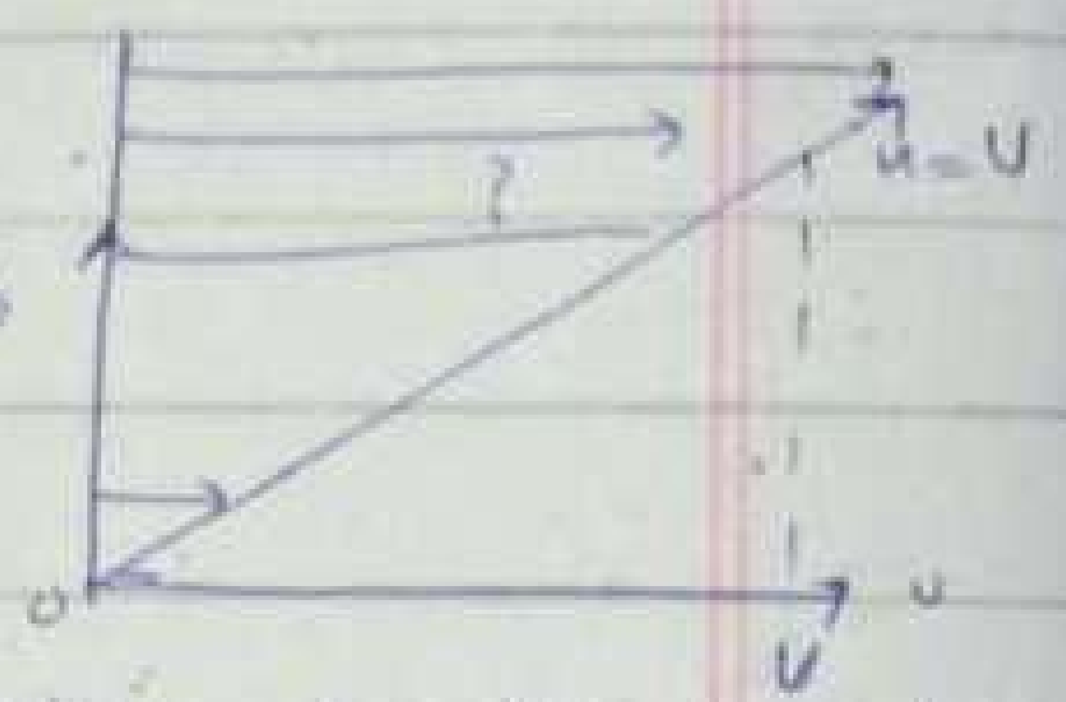
$$= \sqrt{\frac{\nu x}{u_\infty}}\left[f(\eta_1) - f(0) - \frac{f'(\eta_1)^3}{3} + \frac{f'(0)^3}{3} \right]$$

$$= \sqrt{\frac{\nu x}{u_\infty}}\left(7.07923 - \frac{1}{3}\right) = 6.7458\sqrt{\frac{\nu x}{u_\infty}}$$

which is known as momentum thickness.

Exercise: Consider the laminar flow of an incompressible fluid past a flat plate at $y=0$. The boundary layer velocity profile is approximated as $u = \frac{Uy}{\delta}$ for $0 \le y \le \delta$, $u = U$ for $y > \delta$

Calculate momentum boundary layer thickness from the concept of shear stress.



Sol:- The momentum integral solution for the boundary layer flow on a flat plate

$$\tau_w = \rho U^2 \frac{d\delta_1}{dx} \quad — \quad (1)$$

In our case for laminar flow

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$u = \frac{Uy}{\delta} \Rightarrow \frac{du}{dy} = \frac{U}{\delta} \quad - (3)$$

$$\tau_w = \mu\frac{U}{\delta}$$

The boundary layer thickness that is

$$\delta_1 = \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dy \qquad u = \frac{Uy}{\delta}, \quad \frac{u}{U} = \frac{y}{\delta}$$

$$\delta_1 = \int_0^{\delta} \left(\frac{y}{\delta}\right)\left(1-\frac{y}{\delta}\right) dy$$

$$= \int_0^{\delta} \frac{y}{\delta}\, dy - \int_0^{\delta} \frac{y^2}{\delta^2}\, dy$$

$$= \frac{1}{\delta} \cdot \frac{y^2}{2}\Big|_0^{\delta} - \frac{1}{\delta^2} \cdot \frac{y^3}{3}\Big|_0^{\delta}$$

$$\delta_1 = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

$$\delta_1 = \frac{\delta}{6}$$

Now we have to find δ which is function of x. Now $\tau_w = \mu \frac{U}{\delta} = \rho U^2 \frac{d\delta_1}{dx}$

$$\mu \frac{U}{\delta} = \rho \frac{U^2}{6} \frac{d\delta}{dx}$$

$$\delta\, d\delta = \frac{6\mu}{\rho U} dx$$

is can be integrated from the leading edge of the plate $=0$, when $\delta = 0$ to an arbitrary location x, where the undary layer thickness is δ

$$\Rightarrow \frac{\delta^2}{2} = \frac{6\mu x}{\rho U}$$

$$\delta^2 = \frac{12\mu x}{\rho U} = 3.464\sqrt{\frac{\mu x}{\rho U}} \quad , \nu = \frac{\mu}{\rho}$$

$$\delta = 3.464\sqrt{\frac{\nu x}{U}}$$

$$\delta_1 = \frac{3.464}{6}\sqrt{\frac{\nu x}{U}}$$

Q1:- A viscous fluid flow past a flat plate such that the boundary layer thickness at a distance 1.3 m from the leading edge is 12 mm. Determine the boundary layer thickness at a distance of 0.20, 2.0, 20 m from the leading edge. Assume laminar flow.

Sol :- Given thickness

$\delta = 12\ mm = 12 \times 10^{-3}\ m = 0.012\ m$

distance $= l = 1.3\ m$

find $\delta = ?$ at $l = 0.20,\ 2.0$ and $20.0\ m$

we know

$$\frac{\delta}{l} = 5\sqrt{\frac{\nu}{ul}}$$

$$\frac{\delta}{l} = 5\sqrt{\frac{\nu}{U}}\sqrt{\frac{1}{l}}$$

$$5\sqrt{\frac{\nu}{U}} = \frac{\delta}{l}\sqrt{l}$$

$$5\sqrt{\frac{\nu}{U}} = \frac{\delta l}{\sqrt{l}} = \frac{0.012\,m}{\sqrt{1.3}} = \frac{0.012}{1.14017}$$

$$5\sqrt{\frac{\nu}{U}} = 0.01052$$

① Now at $l = 0.20$

$$\delta = 5\sqrt{\frac{V}{U}} \cdot \sqrt{l} = 0.01052 \times \sqrt{0.20}$$

$$\delta = 4.704 \times 10^{-3} m = 4.70 mm.$$

② at $l = 2m$

$$\delta = 0.0105 \times \sqrt{2} = 0.01487 = 14.8 mm.$$

③ at $l = 20m$

$$\delta = 0.0105 \times \sqrt{20} = 0.04704 = 47.04 mm$$