CHAPTER-8

DISCRETE PROBABILITY DISTRIBUTIONS

Q:- What is the binomial experiments and what are its properties.

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Ans Several experiments have repeated independent trials contain only Two possible outcomes. They may be Success and failure, good and bad, alive and dead even and odd etc. If the probability of each outcomes remains same from trial to trial Then these trials are called Bernolli's trial.
The experiments having 'n' Bernolli's trial is called binomial experiment.

Properties of a binomial Experiments:-

(i) There are only two type of outcomes as success (S) or Failure (F).
(ii) The Prob of Success 'p' in each trial remain same
(iii) All trials are independent.
(iv) The experiment is repeated a fix no. of time say n.

DERIVATION OF BINOMIAL WITH PROBABILITY DISTRIBUTION.

Let a binomial experiment with 'n' independent trials. The two possible outcomes are S & F.

$P(S) = p, \quad P(F) = q$

Let x denote the no. of successes in n trials.

$p+q=1$
$q=1-p$
$p=1-q$

Case 1 First we find the Prob of zero success $P(X=0)$

$P(X=0) = P(FFF \text{ —— } F)$
$= P(F) \cdot P(F) \text{ —— } P(F)$
$= q \cdot q \cdot q \text{ —— } q$
$= q^n$
$P(X=0) = q^n$

<u>Case II</u> We find the Prob of 1 success $P(X=1)$ There are $\binom{n}{1}$ such combination

$P(X=1) = P(SFF \text{ — } F) + P(FSF \text{ — } F) +$
$+ \text{ —— } P(FF \text{ — } FS)$
$= (pqq \text{ — } q) + (qpq \text{ — } q) + \text{ —— } (qq \text{ — } p)$
$= pq^{n-1} + pq^{n-1} + \text{ —— } pq^{n-1}$
$P(X=1) = \binom{n}{1} pq^{n-1}$

<u>Case III</u> We find the Prob for 2 Successes $P(X=2)$. There are $\binom{n}{2}$ such combination

$P(X=2) = P(SSF \text{ — } F) + P(SFSF \text{ —— } F) + \text{ — —}$
$\text{— —} + P(FF \text{ —— } SS).$
$= (ppq \text{ — } q) + (pqpq \text{ — } q) + \text{ ——}$
$+ \text{ —— } (qq \text{ — } pp)$
$P(X=2) = p^2q^{n-2} + p^2q^{n-2} + \text{ —— } p^2q^{n-2}$
$P(X=2) = \binom{n}{2} p^2 q^{n-2}$

Similarly we find Probs for $X = 3, 4, \text{ — } n$

<u>Case IV</u>

We find Prob of x successes and $(n-x)$ failure $P(X=x)$
There are $\binom{n}{x}$ such combination
$P(X=x) = \binom{n}{x} p^x q^{n-x}$ which is known as binomial Prob distribution and denoted by
$b(x; n, p) = \binom{n}{x} p^x q^{n-x}$

First Four Moments of the binomial distribution $b(x; n, p)$

Let X bee a random variable with the binomial distribution $b(x; n, p)$ then the moments about the origin are defined as

$$\mu_r' = E(X^r) = \sum_{x=0}^{n} x^r \binom{n}{x} p^x q^{n-x}$$

$r = 1$

$$\mu_1' = E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, — n$$

$$= 0\binom{n}{0}p^0q^{n-0} + 1\cdot\binom{n}{1}p^1q^{n-1} + 2\cdot\binom{n}{2}p^2q^{n-2} + \cancel{3\binom{n}{3}p^3q^{n-3}}$$
$$+ — n\binom{n}{n}p^nq^{n-n}$$

$$= \left[npq^{n-1} + \frac{\cancel{2}n(n-1)}{\cancel{2}}p^2q^{n-2} + \frac{\cancel{3}\cdot n(n-1)(n-2)}{\cancel{3}\cdot 2}p^3q^{n-3} + \right.$$
$$\left. + — + np^n\right]$$

$$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2}p^2q^{n-3} + — \right.$$
$$\left. + — + p^{n-1}\right].$$

$$q + p = 1$$

mean $\mu_1' = np(q+p)^{n-1} = np$

$r = 2$

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} x^2 f(x) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

writing $x^2 = x(x-1) + x$ in factorial notation:

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} [x(x-1)+x]\binom{n}{x}p^xq^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\binom{n}{x}p^xq^{n-x} + \sum_{x=0}^{n} x\binom{n}{x}p^xq^{n-x}$$

$$= \left[2\cdot1\binom{n}{2}p^2q^{n-2} + 3\cdot2\binom{n}{3}p^3q^{n-3} + 4\cdot3\binom{n}{4}p^4q^{n-4}\right.$$
$$\left. + — n(n-1)\binom{n}{n}p^nq^{n-n}\right] + np$$

x^2

$x^2 = [x] + [x]$

$x^2 = x(x-1) + x$

$x^2 = x^2 - \cancel{x} + \cancel{x}$

$$= \left[\frac{\cancel{2}\cdot n(n-1)}{\cancel{2}} p^2 q^{n-2} + \frac{\cancel{6} n(n-1)(n-2)}{\cancel{6}} q^{n-3} p^3 + \frac{12 n(n-1)(n-2)(n-3)}{\cancel{24}\ 2} p^4 q^{n-4} \right.$$

$$\left. + \text{———} + n(n-1) p^n \right] + np$$

$$= n(n-1) p^2 \left[q^{n-2} + (n-2) p q^{n-3} + \frac{(n-2)(n-3)}{2} p^2 q^{n-4} + \text{———} \right.$$

$$\left. + \text{———} + \cancel{n(n-1)p^n} + p^{n-2} \right] + np$$

$$= n(n-1) p^2 (q+p)^{n-2} + np$$

$$= n(n-1) p^2 + np = n^2 p^2 - np^2 + np$$

$\underline{\underline{r=3}}$ $\quad \mu_3' = E(x^3) = \sum_{x=0}^{n} x^3 \binom{n}{x} p^x q^{n-x}$

$x^3 = x(x-1)(x-2) + 3x(x-1) + x$ in factorial notation.

$$= \sum_{x=0}^{n} \left[x(x-1)(x-2) + 3x(x-1) + x \right] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3 \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3n(n-1)p^2 + np$$

$$= \left(3\cdot 2\cdot 1 \binom{n}{3} p^3 q^{n-3} + 4\cdot 3\cdot 2 \binom{n}{4} p^4 q^{n-4} + 5\cdot 4\cdot 3 \binom{n}{5} p^5 q^{n-5} + \right.$$

$$\left. + \text{———} n(n-1)(n-2) \binom{n}{n} p^n q^{n-n} \right] + 3n(n-1)p^2 + np$$

$$= \left(\frac{\cancel{6} n(n-1)(n-2)}{\cancel{6}} p^3 q^{n-3} + \frac{\cancel{24} n(n-1)(n-2)(n-3)}{\cancel{24}} p^4 q^{n-4} + \right.$$

$$\frac{\cancel{60} n(n-1)(n-2)(n-3)(n-4)}{\cancel{120}\ 2} p^5 q^{n-5} + \text{———}$$

$$\left. n(n-1)(n-2) p^n \right] + 3n(n-1)p^2 + np.$$

$$= n(n-1)(n-2) p^3 \left[q^{n-3} + (n-3) p q^{n-4} + \frac{(n-3)(n-4)}{2} p^2 q^{n-5} + \right.$$

$$\left. + \text{———} + p^{n-3} \right] + 3n(n-1)p^2 + np.$$

$$\mu_3' = n(n-1)(n-2)p^3[q+p]^{n-3} + 3n(n-1)p^2 + np$$

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$\underline{r=4}$ $$\mu_4' = E(X^4) = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$$

$\underline{x^4}$

	1	0	0	0	0
1	↓	1	1	1	
	1	1	1	1	
2	↓	2	6		
	1	3	7		
3	↓	3			
		6			
4	↓				
	1				

$$x^4 = [x]^4 + 6[x]^3 + 7[x]^2 + x$$
$$= x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

$$\mu_4' = \sum_{x=0}^{n} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x]\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3)\binom{n}{x} p^x q^{n-x} + 6\sum x(x-1)(x-2)\binom{n}{x} p^x q^{n-x} + 7\sum x(x-1)\binom{n}{x} p^x q^{n-x} + \sum x\binom{n}{x} p^x q^{n-x}$$

$$= \sum\left[x(x-1)(x-2)(x-3)\binom{n}{x} p^x q^{n-x}\right] \cancel{+6n(n-1)p^2+}$$
$$+ 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$= 4\cdot3\cdot2\cdot1\binom{n}{4}p^4q^{n-4} + 5\cdot4\cdot3\cdot2\binom{n}{5}p^5q^{n-5} + 6\cdot5\cdot4\cdot3\binom{n}{6}p^6q^{n-6}$$
$$+ \text{———} + n(n-1)(n-2)(n-3)p^4] +$$
$$6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$$

$$= \frac{\cancel{24}\,n(n-1)(n-2)(n-3)}{\cancel{24}}p^4q^{n-4} + \frac{\cancel{120}\,n(n-1)(n-2)(n-3)(n-4)}{\cancel{120}}p^5q^{n-5}$$
$$+ \frac{\cancel{360}\,n(n-1)(n-2)(n-3)(n-4)(n-5)}{\cancel{720}}p^6q^{n-6} + \text{———}$$

$$= \left[\cancel{2} \cdot \frac{n(n-1)}{\cancel{2}} p^2 q^{n-2} + \frac{\cancel{6} n(n-1)(n-2)}{\cancel{6}} q^{n-3} p^3 + \frac{\cancel{12} n(n-1)(n-2)(n-3)}{\cancel{24}_2} p^4 q^{n-4} \right.$$

$$\left. + \text{———} + n(n-1) p^n \right] + np$$

$$= n(n-1) p^2 \left[q^{n-2} + (n-2) p q^{n-3} + \frac{(n-2)(n-3)}{2} p^2 q^{n-4} + \text{—} \right.$$

$$\left. + \text{———} + \cancel{n(n-1)p^n} + p^{n-2} \right] + np$$

$$= n(n-1) p^2 (q+p)^{n-2} + np$$

$$= n(n-1) p^2 + np = n^2 p^2 - np^2 + np$$

$r=3$ $\quad \mu_3' = E(x^3) = \sum_{x=0}^{n} x^3 \binom{n}{x} p^x q^{n-x}$

$x^3 = x(x-1)(x-2) + 3x(x-1) + x$ in factorial notation.

$$= \sum_{x=0}^{n} \left[x(x-1)(x-2) + 3x(x-1) + x \right] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3 \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3n(n-1)p^2 + np$$

$$= \left[3\cdot2\cdot1 \binom{n}{3} p^3 q^{n-3} + 4\cdot3\cdot2 \binom{n}{4} p^4 q^{n-4} + 5\cdot4\cdot3 \binom{n}{5} p^5 q^{n-5} + \right.$$

$$\left. + \text{———} \; n(n-1)(n-2) \binom{n}{n} p^n q^{n-n} \right] + 3n(n-1)p^2 + np$$

$$= \left[\frac{\cancel{6} n(n-1)(n-2)}{\cancel{6}} p^3 q^{n-3} + \frac{\cancel{24} n(n-1)(n-2)(n-3)}{\cancel{24}} p^4 q^{n-4} + \right.$$

$$\frac{\cancel{60} n(n-1)(n-2)(n-3)(n-4)}{\cancel{120}_2} p^5 q^{n-5} + \text{———}$$

$$\left. n(n-1)(n-2) p^n \right] + 3n(n-1)p^2 + np.$$

$$= n(n-1)(n-2) p^3 \left[q^{n-3} + (n-3) p q^{n-4} + \frac{(n-3)(n-4)}{2} p^2 q^{n-5} + \right.$$

$$\left. + \text{———} + p^{n-3} \right] + 3n(n-1)p^2 + np.$$

$$\mu_3' = n(n-1)(n-2)p^3(q+p)^{n-3} + 3n(n-1)p^2 + np$$

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$\underline{\underline{r=4}}$ $$\mu_4' = E(x^4) = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$$

$\underline{\underline{x^4}}$

	1	0	0	0	0
1	↓	1	1	1	
	1	1	1	1	
2	↓	2	6		
	1	3	7		
3	↓	3			
4	↓	6			
	1				

$$x^4 = [x]^4 + 6[x]^3 + 7[x]^2 + x$$
$$= x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

$$\mu_4' = \sum_{x=0}^{n}\left[x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\right]\binom{n}{x}p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3)\binom{n}{x}p^x q^{n-x} + 6\sum x(x-1)(x-2)\binom{n}{x}p^x q^{n-x} + 7\sum x(x-1)\binom{n}{x}p^x q^{n-x} + \sum x\binom{n}{x}p^x q^{n-x}$$

$$= \sum\left[x(x-1)(x-2)(x-3)\binom{n}{x}p^x q^{n-x}\right] \cancel{+6n(n-1)p^2+}$$
$$+ 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$= 4\cdot3\cdot2\cdot1\binom{n}{4}p^4q^{n-4} + 5\cdot4\cdot3\cdot2\binom{n}{5}p^5q^{n-5} + 6\cdot5\cdot4\cdot3\binom{n}{6}p^6q^{n-6}$$
$$+ \text{———} + n(n-1)(n-2)(n-3)p^4\Big] +$$
$$6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$$

$$= \frac{\cancel{24}\,n(n-1)(n-2)(n-3)}{\cancel{24}}p^4q^{n-4} + \frac{\cancel{120}\,n(n-1)(n-2)(n-3)(n-4)}{\cancel{120}}p^5q^{n-5}$$
$$+ \frac{\cancel{360}\,n(n-1)(n-2)(n-3)(n-4)(n-5)}{\cancel{720}}p^6q^{n-6} + \text{———}$$

$\overline{\quad\quad} n(n-1)(n-2)(n-3)p^n\big] + 6n(n-1)(n-2)p^3 +$

$+ 7n(n-1)p^2 + np.$

~~$n(n-1)(n-2)(n-3)p^4(q+p)^{n-4} + 6n(n-1)(n-$~~

$= n(n-1)(n-2)(n-3)p^4\Big[q^{n-4} + (n-4)pq^{n-5} + \frac{(n-4)(n-5)}{2}p^2q^{n-6}$

$+ \overline{\quad\quad} + p^{n-4}\Big] + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

$= n(n-1)(n-2)(n-3)p^4(q+p)^{n-4} + 6n(n-1)(n-2)p^3 +$

$7n(n-1)p^2 + np.$

$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

Moments about origin are

$\mu_1' = np$

$\mu_2' = n(n-1)p^2 + np$

$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 +$
$7n(n-1)p^2 + np.$

Moments about A.M is calculated as

$\mu_1 = 0$

$\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = n(n-1)p^2 + np - (np)^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$

$\sigma^2 = np(1-p) = npq$

$\sigma = \sqrt{npq}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np - 3np[n(n-1)p^2 + np] +$
$2n^3p^3$

$$= np^3(n^2-3n+2)+3n^2p^2-3np^2+np-3np[n^2p^2-np^2+np]+2n^3p^3$$

$$= \cancel{n^3p^3}-\cancel{3n^2p^3}+2np^3+\cancel{3n^2p^2}-3np^2+np-\cancel{3n^3p^3}+\cancel{3n^2p^3}-\cancel{3n^2p^2}+\cancel{2n^3p^3}$$

$$= np-3np^2+2np^3 = np(1-3p+2p^2)$$

$$= np[(1-p)(1-2p)] = npq(1-2p)$$

$$\mu_3 = npq(q-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= n(n-1)(n-2)(n-3)p^4+6n(n-1)(n-2)p^3+7n(n-1)p^2+np-4np[n(n-1)(n-2)p^3+3n(n-1)p^2+np]+6n^2p^2[n(n-1)p^2+np]-3n^4p^4.$$

$$= np^4(n^2-3n+2)(n-3)+6np^3(n^2-3n+2)+7n^2p^2-7np^2+np-4np[np^3(n^2-3n+2)+3n^2p^2-3np^2+np]+6n^2p^2(n^2p^2-np^2+np)-3n^4p^4.$$

$$= \cancel{n^4p^4}-\cancel{6n^3p^4}+11n^2p^4-6np^4+\cancel{6n^3p^3}-18n^2p^3+12np^3+7n^2p^2-7np^2+np-\cancel{4n^4p^4}+\cancel{12n^3p^4}-8n^2p^4-\cancel{12n^3p^3}+12n^2p^3-4n^2p^2+\cancel{6n^4p^4}-\cancel{6n^3p^4}+\cancel{6n^3p^3}-\cancel{3n^4p^4}.$$

$$= 3n^2p^4-6np^4-6n^2p^3+12np^3+3n^2p^2-7np^2+np$$

$$= 3n^2p^2-6n^2p^3+3n^2p^4+np-7np^2+12np^3-6np^4$$

$$= [3n^2p^2-6n^2p^3+3n^2p^4]+[np-7np^2+12np^3-6np^4]$$

$$= 3n^2p^2(1-2p+p^2)+np(1-7p+12p^2-6p^3).$$

$$= 3n^2p^2(1-p)^2+np(1-7p+12p^2-6p^3)$$

$$\begin{array}{c|cccc} 1 & -6 & 12 & -7 & 1 \\ & \downarrow & -6 & 6 & -1 \\ \hline & -6 & 6 & -1 & \vert\, 0 \end{array}$$

$p=1$

$p-1=0 \qquad -6p^2+6p-1$

$(p-1)(-6p^2+6p-1)$

$(1-p)(6p^2-6p+1) = (1-p)(1-6p+6p^2)$

$= 3n^2p^2q^2 + np\,[(1-p)(1-6p+6p^2)]$

$= 3n^2p^2q^2 + npq\,(1-6p+6p^2]$

$= \cancel{3p}\ 3n^2p^2q^2 + npq\,[1-6p(1-p)]$

$\mu_4 = 3n^2p^2q^2 + npq\,(1-6pq)$

$\mu_1 = 0 \qquad\qquad \beta_1 = \frac{\mu_3^2}{u}$

$\mu_2 = npq$

$\mu_3 = npq(q-p)$

$\mu_4 = 3n^2p^2q^2 + npq\,(1-6pq)$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[npq\,(q-p)]^2}{n^3p^3q^3}$$

$$= \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2p^2q^2 + npq\,(1-6pq)}{n^2p^2q^2}$$

$$\beta_2 = 3 + \frac{1-6pq}{npq}$$

Moment Generating Function of the Binomial Distribution.

The m.g.f of the binomial distribution $b(x; n, p)$ is derived as

$$M_0(t) = E(e^{tx_i}) = \sum e^{tx_i} f(x_i)$$

$$= \sum_{x=0}^{n} e^{tx_i} \binom{n}{x} q^{n-x} p^{x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} q^{n-x} (pe^t)^x$$

$$M_0(t) = (q + pe^t)^n$$

In order to get the first four moments of the function $M_0(t)$ we differentiate $M_0(t)$ once, twice, thrice and four times w.r.t 't' and putting $t=0$

$$M_0(t) = \left(\frac{d}{dt}(q+pe^t)^n\right)_{t=0}$$

$$= \left(n(q+pe^t)^{n-1} pe^t\right)_{t=0}$$

$$= \left(\underset{\text{I}}{npe^t}\, \underset{\text{II}}{(q+pe^t)^{n-1}}\right)_{t=0}$$

$$\mu_1' = np(q+p)^{n-1} = np$$

$$\mu_2' = E(x^2) = \left(\frac{d^2}{dt^2}(q+pe^t)^n\right)_{t=0}$$

$$= \left(npe^t.(n-1)(q+pe^t)^{n-2}.pe^t + (q+pe^t)^{n-1}.npe^t\right)_{t=0}$$

$$= \left(n(n-1)p^2e^{2t}(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}\right)_{t=0}$$

$$= n(n-1)p^2(q+p)^{n-2} + np(q+p)^{n-1}$$

$$\mu_2' = n(n-1)p^2 + np$$

$$\mu_2' = \underset{\text{I}}{n(n-1)p^2e^{2t}}\, \underset{\text{II}}{(q+pe^t)^{n-2}} + \underset{\text{I}}{npe^t}\, \underset{\text{II}}{(q+pe^t)^{n-1}}$$

$\mu_3' = E(X^3) = \left[\frac{d^3}{dt^3}(q+pe^t)^n\right]_{t=0}$

$= \Big[n(n-1)p^2e^{2t}(n-2)(q+pe^t)^{n-3}\cdot pe^t + (q+pe^t)^{n-2}\cdot$

$2n(n-1)p^2e^{2t} + npe^t(n-1)(q+pe^t)^{n-2}\cdot pe^t + (q+pe^t)^{n-1}\cdot pe^t\Big]_{t=0}$

$= \Big[\underset{\mathrm{I}}{n(n-1)(n-2)p^3e^{3t}}\underset{\mathrm{II}}{(q+pe^t)^{n-3}} + \underset{\mathrm{I}}{2n(n-1)p^2e^{2t}}\underset{\mathrm{II}}{(q+pe^t)^{n-2}} +$

$\underset{\mathrm{I}}{n(n-1)p^2e^{2t}}\underset{\mathrm{II}}{(q+pe^t)^{n-2}} + \underset{\mathrm{I}}{npe^t}\underset{\mathrm{II}}{(q+pe^t)^{n-1}}\Big]_{t=0}$

$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$\mu_4' = E(X^4) = \left[\frac{d^4}{dt^4}(q+pe^t)^n\right]_{t=0}$

$= \Big[n(n-1)(n-2)p^3e^{3t}(n-3)(q+pe^t)^{n-4}\cdot pe^t +$

$(q+pe^t)^{n-3}\cdot 3n(n-1)(n-2)p^3e^{3t} + 2n(n-1)p^2e^{2t}(n-2)$

$(q+pe^t)^{n-3}\cdot pe^t + (q+pe^t)^{n-2}\cdot 4n(n-1)p^2e^{2t} + n(n-1)p^2e^{2t}(n-2)$

$(q+pe^t)^{n-2}\cdot pe^t + (q+pe^t)^{n-2}\, 2n(n-1)p^2e^{2t} + npe^t(n-1)(q+pe^t)^{n-2}pe^t$

$+ (q+pe^t)^{n-1}\cdot npe^t\Big]_{t=0}$

$= n(n-1)(n-2)(n-3)p^4 + 3n(n-1)(n-2)p^3 + 2n(n-1)(n-2)p^3$

$+ 4n(n-1)p^2 + n(n-1)(n-2)p^3 + 2n(n-1)p^2 + n(n-1)p^2 + np.$

$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$

8.23(a) If the m.g.f of X is $M_0(t) = \left(\frac{1}{4} + \left(\frac{3}{4}\right)e^t\right)^{12}$

Find $E(X)$, $var(X)$ and $P(X \geq 10)$.

Sol $\quad M_0(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^{12}$

$$E(X) = \left[\frac{d}{dt}\left(\tfrac{1}{4}+\tfrac{3}{4}e^{t}\right)^{12}\right]_{t=0}$$

$$= \left[12\left(\tfrac{3}{4}e^{t}\right)\left(\tfrac{1}{4}+\tfrac{3}{4}e^{t}\right)^{11}\right]_{t=0}$$

$$= 12\left(\tfrac{3}{4}\right)(1) = 9$$

$$E(X^2) = \left[\frac{d^2}{dt^2}\left(\tfrac{1}{4}+\tfrac{3}{4}e^{t}\right)^{12}\right]_{t=0}$$

$$= \left[12\left(\tfrac{3}{4}e^{t}\right)\left(\tfrac{1}{4}+\tfrac{3}{4}e^{t}\right)^{11}\right]_{t=0} +$$

$$+ 12(11)\left(\tfrac{3}{4}\right)^2 e^{2t}\left(\tfrac{1}{4}+\tfrac{3}{4}e^{t}\right)^{10}\Big]_{t=0}$$

$$= 12\left(\tfrac{3}{4}\right)(1) + 12(11)\left(\tfrac{3}{4}\right)^2(1)$$

$$= 9 + \frac{297}{4} = 333/4$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{333}{4} - (9)^2 = 2.25$$

$$P(X \geq 10) = \sum_{x=10}^{12}\binom{12}{x}\left(\tfrac{3}{4}\right)^{x}\left(\tfrac{1}{4}\right)^{12-x}$$

Fitting a Binomial Distribution to the observed data.

The steps for fitting a binomial dist to the observed data are

(i) Find mean of the data by

$$\bar{X} = \frac{\sum fx}{\sum f}$$

(ii) Find the Parameter n & p by

$$\bar{X} = np \Longrightarrow p = \bar{X}/n$$

iii, Find all Prob for $X = 0, 1, 2 \ldots n$ and multiply each Prob by $N = \Sigma f$ we get the expected frequency.

The sum of the expected freq is equal to the sum of observed freq.

THE POISSON DISTRIBUTION.

The Poisson distribution is a limiting case of the binomial distribution when n becomes very large and p becomes small such that $np = \mu$ remains constant then the limiting form of the binomial $b(x; n, p)$ is

$$P(X) = \frac{e^{-\mu} \mu^{x}}{x!} \quad \text{where } x = 0, 1, 2 \ldots$$

This is called the Poisson distribution with Parameter μ and is denoted by $P(x; \mu)$. The Parameter μ may be interpreted as the mean rate of occurance of events

$$\sum_{x=0}^{\infty} P(x; \mu) = \sum_{x=0}^{\infty} \frac{e^{-\mu} \mu^{x}}{x!}$$

$$= e^{-\mu} + e^{-\mu}\mu + \frac{e^{-\mu}\mu^2}{2!} + \frac{e^{-\mu}\mu^3}{3!} + \ldots$$

$$= e^{-\mu}\left[1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \ldots\right]$$

$$= \cancel{e^{-\mu}} \; e^{-\mu} \cdot e^{\mu} = e^{-\mu+\mu} = e^{0} = 1$$

This distribution was first discovered by S.D. Poisson. The Poisson Prob dist is also called the law of small numbers

Some examples obeying the Poisson Prob law are given below

① The number of person born blind per year in a large city.

② The number of typing error per page.

③ The number of telephone calls per minute at some switch board etc.

DERIVATION OF POISSON PROB DISTRIBUTION.

The Prob of x successes in n trials is given by

$$P(x) = \binom{n}{x} p^x q^{n-x} \quad \text{where}$$

$$\binom{n}{x} = \frac{n!}{x!\,n-x!} \qquad \mu = np. \qquad p = \frac{\mu}{n} \qquad q = 1 - \mu/n$$

Subsituting these values we get

$$P(x) = \frac{n!}{x!\,n-x!} (1-\mu/n)^{n-x} (\mu/n)^x$$

$$= \frac{\mu^x}{n^x x!} (1-\mu/n)^n (1-\mu/n)^{-x} \frac{n!}{n-x!}$$

By making n tends to infinity

$$\lim_{n\to\infty} P(x) = \frac{\mu^x}{n^x x!} \lim_{n\to\infty} \left[(1-\mu/n)^n (1-\mu/n)^{-x} \frac{n!}{n-x!} \right]$$

Since $\lim_{n\to\infty} (1-\mu/n)^n = e^{-\mu}$

$\lim_{n\to\infty} (1-\mu/n)^{-x} = 1$

$$\lim_{n\to\infty} p(x) = \frac{\mu^x e^{-\mu}}{n^x\, x!} \lim_{n\to\infty} \frac{n!}{n-x!}$$ ①

According to Stirling Approximation

$$n! \sim \sqrt{2\pi}\, e^{-n}\, n^{n+1/2}$$

$$n-x! \sim \sqrt{2\pi}\, e^{-(n-x)}\, (n-x)^{n-x+1/2}$$

Substituting these values we get

$$\lim_{n\to\infty} P(x) = \frac{\mu^x e^{-\mu}}{n^x\, x!} \lim_{n\to\infty} \left(\frac{\cancel{\sqrt{2\pi}} \cdot e^{-n} \cdot n^{n+1/2}}{\cancel{\sqrt{2\pi}} \cdot e^{-n+x} (n-x)^{n-x+1/2}} \right)$$

$$= \frac{\mu^x e^{-\mu}}{x!\, e^x} \lim_{n\to\infty} \left(\frac{\cancel{e^{-n}}\, n^{-x}\, n^{n+1/2}}{\cancel{e^{-n}} \cdot (n-x)^{n-x+1/2}} \right)$$

$$= \frac{\mu^x e^{-\mu}}{x!\, e^x} \lim_{n\to\infty} \left(\frac{(n)^{n-x+1/2}}{(n-x)^{n-x+1/2}} \right)$$

$$= \frac{\mu^x e^{-\mu}}{x!\, e^x} \lim_{n\to\infty} \left[\frac{1}{(1-x/n)^{n-x+1/2}} \right]$$

Since $\lim_{n\to\infty} (1-x/n)^n = e^{-x}$

$$\lim_{n\to\infty} (1-x/n)^{-x+1/2} = 1$$

$$= \frac{\mu^x e^{-\mu}}{x!\, e^x} \lim_{n\to\infty} \left(\frac{1}{(1-x/n)^n (1-x/n)^{-x+1/2}} \right)$$

$$= \frac{\mu^x e^{-\mu}}{x!\, e^x} \frac{1}{e^{-x}} = \frac{e^{-\mu} \mu^x}{x!}$$

$\lim_{n\to\infty} p(x) = \frac{e^{-\mu}\mu^x}{x!}$ which is the required Poisson distribution.

PROPERTIES OF THE POISSON DISTRIBUTION/
(First Four Moments of Poisson distribution)

(1) The Poisson distribution has an interesting property that its mean & variance are equal.

Proof Let 'X' be a random variable with the Poisson distribution $P(x;\mu)$ then the moments about origin is defined as

$$\mu_r' = E(X^r) = \sum_{x=0}^{\infty} x^r \frac{e^{-\mu}\mu^x}{x!}$$

$r=1$

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\mu}\mu^x}{x!}$$

$$= 0.\frac{e^{-\mu}\mu^0}{0!} + 1.\frac{e^{-\mu}\mu^1}{1!} + \cancel{2}.\frac{e^{-\mu}\mu^2}{\cancel{2!}} + \cancel{3}\frac{e^{-\mu}\mu^3}{\cancel{3!}_2} + \cdots$$

$$= e^{-\mu}\mu + e^{-\mu}\mu^2 + \frac{e^{-\mu}\mu^3}{2!} + \text{---}$$

$$= e^{-\mu}\mu\left(1 + \mu + \frac{\mu^2}{2!} + \text{---}\right)$$

$$\mu_1' = \cancel{e^{-\mu}}\mu . \cancel{e^{\mu}} = \mu = \text{Mean.}$$

$r=2$

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\mu}\mu^x}{x!}$$

writing $x^2 = x(x-1) + x$

$$= \sum_{x=0}^{\infty} \left(x(x-1)+x\right) \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\mu}\mu^x}{x!} + \sum_{x=0}^{\infty} x.\frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\mu}\mu^x}{x!} + \mu$$

$$= \left[\cancel{2.1}\frac{e^{-\mu}\mu^2}{\cancel{2!}} + \cancel{3.2}\frac{e^{-\mu}\mu^3}{\cancel{3!}} + \cancel{4.3}\frac{e^{-\mu}\mu^4}{\cancel{4!}_2} + \text{---}\right] + \mu$$

$$= e^{-\mu}\mu^2\left[1 + \mu + \frac{\mu^2}{2!} + \text{---}\right] + \mu$$

$= \cancel{e^{-\mu}} \mu^2 \cdot \cancel{e^{\mu}} + \mu$

$E(X^2) = \mu^2 + \mu$

$\sigma^2 = E(X^2) - [E(X)]^2$

$= \mu^2 + \mu - (\mu)^2$

Variance $= \cancel{\mu^2} + \mu - \cancel{\mu^2} = \mu$

Higher degree moments are found as below.

$r=3$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty} x^3 \frac{e^{-\mu}\mu^x}{x!}$$

writing $x^3 = x(x-1)(x-2) + 3x(x-1) + x$

$$= \sum_{x=0}^{\infty} [x(x-1)(x-2) + 3x(x-1) + x] \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\mu}\mu^x}{x!} + 3\sum_{x=0}^{\infty} x(x-1)\frac{e^{-\mu}\mu^x}{x!} + \sum_{x=0}^{\infty} x \cdot \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\mu}\mu^x}{x!} + 3\mu^2 + \mu$$

$$= \left[\cancel{3 \cdot 2 \cdot 1} \frac{e^{-\mu}\mu^3}{\cancel{3!}} + \cancel{4 \cdot 3 \cdot 2} \frac{e^{-\mu}\mu^4}{\cancel{4!}} + \cancel{5 \cdot 4 \cdot 3} \frac{e^{-\mu}\mu^5}{\cancel{5!}_2} + \text{---} \right] + 3\mu^2 + \mu$$

$$= e^{-\mu}\mu^3 \left(1 + \mu + \mu^2/2! + \text{---} \right) + 3\mu^2 + \mu$$

$$= \cancel{e^{-\mu}} \mu^3 \cancel{e^{\mu}} + 3\mu^2 + \mu$$

$$\mu_3' = \mu^3 + 3\mu^2 + \mu$$

$r=4$

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty} x^4 \frac{e^{-\mu}\mu^x}{x!}$$

writing $x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$

$$= \sum_{x=0}^{\infty} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x] \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3)\frac{e^{-\mu}\mu^3}{3!} + 6\sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\mu}\mu^x}{x!}$$

$$+ 7\sum_{x=0}^{\infty} x(x-1)\frac{e^{-\mu}\mu^x}{x!} + \sum x \cdot \frac{e^{-\mu}\mu^x}{x!}$$

$$= \left[\cancel{4\cdot3\cdot2\cdot1}\frac{e^{-\mu}\mu^4}{\cancel{24}} + \cancel{5\cdot4\cdot3\cdot2}\frac{e^{-\mu}\mu^5}{\cancel{120}} + \cancel{6\cdot5\cdot4\cdot3}\frac{e^{-\mu}\mu^6}{\cancel{720}_{2}} + \cdots\right]$$

$$+ 6\mu^3 + 7\mu^2 + \mu$$

$$= e^{-\mu}\mu^4\left[1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots\right] + 6\mu^3 + 7\mu^2 + \mu$$

$$= \cancel{e^{-\mu}}\mu^4 \cdot \cancel{e^{\mu}} + 6\mu^3 + 7\mu^2 + \mu$$

$$\mu_4' = \mu^4 + 6\mu^3 + 7\mu^2 + \mu$$

Moments about origin are

$$\mu_1' = \mu$$
$$\mu_2' = \mu^2 + \mu$$
$$\mu_3' = \mu^3 + 3\mu^2 + \mu$$
$$\mu_4' = \mu^4 + 6\mu^3 + 7\mu^2 + \mu.$$

Moments about mean are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \cancel{\mu^2} + \mu - \cancel{\mu^2} = \mu = \sigma^2$$

$$\sigma = \sqrt{\mu}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \mu^3 + 3\mu^2 + \mu - 3\mu(\mu^2 + \mu) + 2\mu^3$$

$$= \cancel{\mu^3} + \cancel{3\mu^2} + \mu - \cancel{3\mu^3} - \cancel{3\mu^2} + \cancel{2\mu^3}$$

$$= \mu$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \mu^4 + 6\mu^3 + 7\mu^2 + \mu - 4\mu(\mu^3 + 3\mu^2 + \mu) + 6\mu^2(\mu^2 + \mu) - 3\mu^4$$

$$= \cancel{\mu^4} + \cancel{6\mu^3} + 7\mu^2 + \mu - \cancel{4\mu^4} - \cancel{12\mu^3} - 4\mu^2 + \cancel{6\mu^4} + \cancel{6\mu^3} - \cancel{3\mu^4}$$

$$\mu_4 = 3\mu^2 + \mu = \mu(3\mu + 1)$$

$$\mu_1 = 0$$

$$\mu_2 = \mu$$

$$\mu_3 = \mu$$

$$\mu_4 = \mu(3\mu + 1)$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu^2}{\mu^3} = 1/\mu$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\mu^2 + \mu}{\mu^2}$$

$$= 3 + 1/\mu$$

Moment Generating Function of the Poisson Distribution.

The m.g.f of the Poisson distribution $P(x;\mu)$ about the origin is defined as

$$M_0(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx_i} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\mu} \mu^{x}}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^{t})^{x}}{x!}$$

$$= e^{-\mu} \left[1 + \mu e^{t} + \frac{(\mu e^{t})^{2}}{2!} + \frac{(\mu e^{t})^{3}}{3!} + \cdots \right]$$

$$= e^{-\mu} e^{\mu e^{t}} = e^{\mu e^{t} - \mu} = e^{\mu(e^{t}-1)}$$

$$M_0(t) = e^{\mu(e^{t}-1)}$$

In order to get the moments about the origin we differentiate $M_0(t)$ once, twice, thrice and four times w.r.t 't' & Putting $t=0$

$$\mu_1' = E(X) = \left[\frac{d}{dt} \left(e^{\mu(e^{t}-1)} \right) \right]_{t=0}$$

$$= \left(\underset{I}{\mu e^{t}} \cdot \underset{II}{e^{\mu(e^{t}-1)}} \right)_{t=0}$$

$$\mu_1' = \mu$$

$$\mu_2' = E(X^2) = \left(\frac{d^2}{dt^2} e^{\mu(e^{t}-1)} \right)_{t=0}$$

$$= \left[\mu e^{t} \cdot \mu e^{t} \cdot e^{\mu(e^{t}-1)} + e^{\mu(e^{t}-1)} \cdot \mu e^{t} \right]_{t=0}$$

$$= \left(\underset{I}{\mu^{2} e^{2t}} \cdot \underset{II}{e^{\mu(e^{t}-1)}} + \underset{I}{\mu e^{t}} \cdot \underset{II}{e^{\mu(e^{t}-1)}} \right)_{t=0}$$

$$\mu_2' = \mu^2 + \mu$$

$$\mu_3' = E(X^3) = \frac{d^3}{dt^3} \left(e^{\mu(e^{t}-1)} \right)_{t=0}$$

$$= \left[\mu^{2} e^{2t} \cdot \mu e^{t} e^{\mu(e^{t}-1)} + e^{\mu(e^{t}-1)} \cdot 2\mu^{2} e^{2t} + \mu e^{t} \cdot e^{\mu(e^{t}-1)} \mu e^{t} + e^{\mu(e^{t}-1)} \cdot \mu e^{t} \right]_{t=0}$$

$$= \left[\mu^3 e^{3t} \cdot e^{\mu(e^t-1)} + 2\mu^2 e^{2t} \cdot e^{\mu(e^t-1)} + \mu^2 e^{2t} e^{\mu(e^t-1)} \right.$$

$$\left. + \mu e^t \cdot e^{\mu(e^t-1)} \right]_{t=0}$$

$$\mu_3' = \mu^3 + 2\mu^2 + \mu^2 + \mu = \mu^3 + 3\mu^2 + \mu$$

$$\mu_4' = \frac{d^4}{dt^4}\left[e^{\mu(e^t-1)} \right]_{t=0}$$

$$= \left[\mu^3 e^{3t} \cdot \mu e^t \cdot e^{\mu(e^t-1)} + e^{\mu(e^t-1)} \cdot 3\mu^3 e^{3t} + \right.$$

$$2\mu^2 e^{2t} \cdot \mu e^t \cdot e^{\mu(e^t-1)} + e^{\mu(e^t-1)} \cdot \mu^2 e^{2t} +$$

$$\mu^2 e^{2t} \cdot \mu e^t \cdot e^{\mu(e^t-1)} + e^{\mu(e^t-1)} \cdot 2\mu^2 e^{2t} + \mu e^t \cdot \mu e^t \cdot e^{\mu(e^t-1)}$$

$$\left. + e^{\mu(e^t-1)} \cdot \mu e^t \right]_{t=0}$$

$$= \mu^4 + 3\mu^3 + 2\mu^3 + 4\mu^2 + \mu^3 + 2\mu^2 + \mu^2 + \mu$$

$$\mu_4' = \mu^4 + 6\mu^3 + 7\mu^2 + \mu$$

HYPERGEOMETRIC DISTRIBUTION.

Q:- What is hypergeometric experiment and what are its properties.

Ans:- There are many experiments in which the condition of independence does not exist and the Prob of success 'P' changes from trial to trial. Such experiments are called ~~the~~ hypergeometric experiments. The Properties of hyp experiments are

Properties

① There are only two possible outcomes

Success (S) and failure (F).

② The Prob of Success 'P' changes from trial to trial

③ The Successive trials are dependent.

④ The experiment is repeated a fix no. of time (n is fixed)

$$P(X=x) = h(x; N, n, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

N = no. of units in the set or Population

n = no. of units of sample Taken from Population.

K = total no. of Successes in popn

Parameters are N, n, K.

Q:- Find Mean and Variance of hypergeometric distn.

OR.

Prove that the mean and variance of hypergeometric dist are $\mu = nP$ & $\sigma^2 = npq \frac{N-n}{N-1}$ where $P = \frac{K}{N}$, $q = \frac{N-K}{N}$

Proof $\mu = E(X) = \sum_{x=0}^{n} x P(x)$

$$= \sum_{x=0}^{n} x \cdot \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} x \cdot \frac{K!}{x!\,(K-x)!} \frac{\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} x \cdot \frac{K(K-1)!}{x(x-1)!\,(K-x)!} \frac{\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{K}{\binom{N}{n}} \sum_{x=1}^{n} \binom{K-1}{x-1}\binom{N-K}{n-x}$$

Let $Y = X - 1 \Rightarrow X = Y + 1$

If $X = 1$ then $Y = 0$

$X = n$ then $Y = n-1$

$$\binom{K-1}{x-1} \quad \frac{(K-1)!}{(x-1)!\,(K-x)!}$$

$$E(X) = \frac{K}{\binom{N}{n}} \sum_{Y=0}^{n-1} \binom{K-1}{Y}\binom{N-K}{n-Y-1}$$

$$= \frac{K}{\binom{N}{n}} \binom{\cancel{K}-1+N-\cancel{K}}{\cancel{Y}+n-\cancel{Y}-1}$$

$$= \frac{K}{\binom{N}{n}} \binom{N-1}{n-1}$$

$$= \frac{K}{N!}\, n!\,(N-n)!\, \frac{(N-1)!}{(n-1)!\,(N-n)!}$$

$$= \frac{K \cdot \cancel{n(n-1)!}\,\cancel{(N-n)!}}{N\cancel{(N-1)!}} \frac{\cancel{(N-1)!}}{\cancel{(n-1)!}\,\cancel{(N-n)!}}$$

$$= \frac{nK}{N} \qquad P = K/N$$

$$\mu = E(X) = nP$$

$$\sigma^2 = E(X^2) - [E(X)]^2 \quad \text{------} \quad (1)$$

$$E(X^2) = E[X(X-1) + X]$$

$$= E(X) + E X(X-1)$$

$$= \frac{nK}{N} + \sum_{x=0}^{n} x(x-1) \cdot P(x)$$

$$= \frac{nK}{N} + \sum_{x=0}^{n} x(x-1) \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{nK}{N} + \sum_{x=0}^{n} x(x-1) \cdot \frac{K!}{x!\,(K-x)!} \frac{\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{nK}{N} + \sum_{x=0}^{n} \cancel{x(x-1)} \cdot \frac{K(K-1)(K-2)!}{\cancel{x(x-1)}(x-2)!\,(K-x)!} \frac{\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{nK}{N} + \frac{K(K-1)}{\binom{N}{n}} \sum_{x=2}^{n} \binom{K-2}{x-2}\binom{N-K}{n-x}$$

Let $Y = X-2 \Rightarrow X = Y+2$

If $X=2 \quad Y=0$

$X=n \quad Y=n-2$

$$E(X^2) = \frac{nK}{N} + \frac{K(K-1)}{\binom{N}{n}} \sum_{Y=0}^{n-2} \binom{K-2}{Y}\binom{N-K}{n-Y-2}$$

$$= \frac{nK}{N} + \frac{K(K-1)}{\binom{N}{n}} \binom{\cancel{K}-2+N-\cancel{K}}{\cancel{Y}+n-\cancel{Y}-2}$$

$$= \frac{nK}{N} + \frac{K(K-1)}{\binom{N}{n}} \binom{N-2}{n-2}$$

$$= \frac{nK}{N} + \frac{K(K-1)\,n!\,\cancel{(N-n)!}}{N!} \frac{(N-2)!}{(n-2)!\,\cancel{(N-n)!}}$$

$$\therefore E(X^2) = \frac{nK}{N} + \frac{K(K-1)\,n(n-1)\cancel{(n-2)!}\,\cancel{(N-2)!}}{N(N-1)\cancel{(N-2)!}\;\cancel{(n-2)!}}$$

$$E(X^2) = \frac{nK}{N} + \frac{nK(K-1)(n-1)}{N(N-1)}$$

Put in ①

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$\sigma^2 = \frac{nK}{N} + \frac{nK(K-1)(n-1)}{N(N-1)} - \left(\frac{nK}{N}\right)^2$$

$$\sigma^2 = \frac{nK}{N}\left[1 + \frac{nK-K-n+1}{(N-1)} - \frac{nK}{N}\right]$$

$$= \frac{nK}{N}\left[\frac{N^2-\cancel{N}+\cancel{nNK}-NK-nN+\cancel{N}-\cancel{nNK}+nK}{N(N-1)}\right]$$

$$= \frac{nK}{N}\left[\frac{N^2-NK-nN+nK}{N(N-1)}\right]$$

$$= \frac{nK}{N}\left[\frac{N(N-K)-n(N-K)}{N(N-1)}\right]$$

$$= \frac{nK}{N}\left(\frac{(N-K)(N-n)}{N(N-1)}\right)$$

$$= n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1}$$

$$p = \frac{K}{N}$$

$$q = \frac{N-K}{N}$$

$$\sigma^2 = n \cdot p \cdot q \cdot \frac{N-n}{N-1}$$

Q:- If in a hypergeometric dist 'N' is very large then hyp. geometric dist approaches the binomial distribution.

<u>Proof</u> In hypergeometric dist

$$p = \frac{K}{N} \Rightarrow K = Np$$

$$q = \frac{N-K}{N} \Rightarrow N-K = Nq$$

$$h(x; N, n, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$$

$$= \frac{Np! \quad Nq! \quad n! \quad N-n!}{x!\,(Np-x)!\,(n-x)!\,(Nq-n+x)! \quad N!}$$

$$= \binom{n}{x} \frac{(Np)!\,(Nq)!\,(N-n)!}{(Np-x)!\,(Nq-n+x)!\,N!}$$

Now Applying Striling formula

$$n! \simeq e^{-n}\, n^{n+1/2} \cdot \sqrt{2\pi}$$

$$= \binom{n}{x} \frac{\left[e^{-NP} (NP)^{NP+1/2} \cancel{\sqrt{2\pi}}\right]\left[e^{-Nq} (Nq)^{Nq+1/2} \cancel{\sqrt{2\pi}}\right]\left[e^{-(N-n)} (N-n)^{N-n+1/2} \cancel{\sqrt{2\pi}}\right]}{\left[e^{-(NP-x)} (NP-x)^{NP-x+1/2} \cancel{\sqrt{2\pi}}\right]\left[e^{-(Nq-n+x)} (Nq-n+x)^{Nq-n+x+1/2} \cancel{\sqrt{2\pi}}\right]\left[e^{-N} \cdot N^{N+1/2} \cancel{\sqrt{2\pi}}\right]}$$

$$= \binom{n}{x} \frac{e^{\cancel{-NP}\cancel{-Nq}\cancel{-n}\cancel{+x}\cancel{+NP}\cancel{-x}\cancel{+Nq}\cancel{-n}\cancel{+x}\cancel{+N}} (NP)^{NP+1/2} (Nq)^{Nq+1/2} N^{N-n+1/2} (1-n/N)^{N-n+1/2}}{(NP)^{NP-x+1/2} (1-x/NP)^{NP-x+1/2} (Nq)^{Nq-n+x+1/2} \left(1-\frac{n-x}{Nq}\right)^{Nq-n+x+1/2} N^{N+1/2}}$$

$$= \binom{n}{x} e^{0} \frac{N^{\cancel{NP+1/2}\cancel{+N}\cancel{-n}\cancel{+1/2}\cancel{+Nq}\cancel{+1/2}\cancel{-NP}\cancel{+n}\cancel{-1/2}\cancel{-Nq}\cancel{+n}\cancel{-x}\cancel{-1/2}\cancel{-N}\cancel{-1/2}} P^{\cancel{NP+1/2}\cancel{-NP}+x\cancel{-1/2}} q^{\cancel{Nq+1/2}\cancel{-Nq}+n-x\cancel{-1/2}} (1-n/N)^{N-n+1/2}}{(1-x/NP)^{NP-x+1/2} \left(1-\frac{n-x}{Nq}\right)^{Nq-n+x+1/2}}$$

$$= \binom{n}{x} \frac{1 \; N^{0} \, P^{x} q^{n-x} (1-n/N)^{N-n+1/2}}{(1-x/NP)^{NP-x+1/2} \left(1-\frac{n-x}{Nq}\right)^{Nq-n+x+1/2}}$$

So $\lim_{N\to\infty}$ then $1-n/N,\ 1-x/NP,\ 1-\frac{n-x}{Nq} \longrightarrow 1$

$$\lim_{N\to\infty} h(x; N, n, K) = \binom{n}{x} P^{x} q^{n-x}$$

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