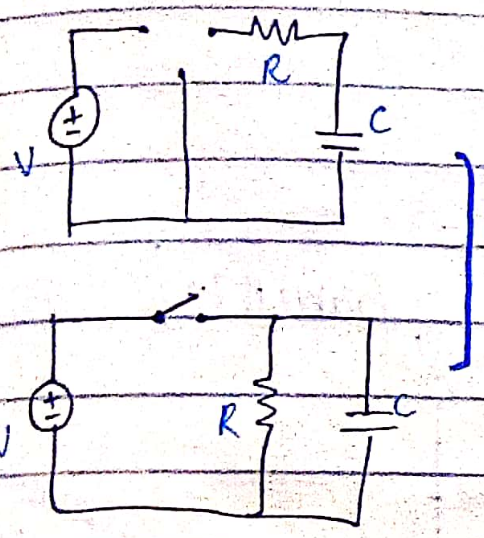
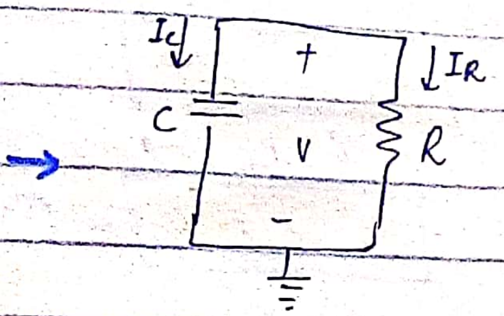
Ch#7 : First order Circuits :- → single element

7.2 The RC-source free ckt :- → transient response.





applying KCL :-

$$i_c + i_R = 0.$$

÷ by C on both sides :- ⟵ $C\frac{dv}{dt} + i_R = 0 \Rightarrow C\frac{dv}{dt} + \frac{V}{R} = 0.$

$$\boxed{\frac{dv}{dt} + \frac{v}{RC} = 0.}$$ → 1st ODE (1st order differential eqn.)

$$\frac{dv}{dt} = -\frac{v}{RC}$$

voltage :- $\frac{dv}{v} = -\frac{dt}{RC}$

integrate on both sides :-

$$\int \frac{1}{v}\,dv = -\frac{1}{RC}\int (1)\,dt$$

$$\ln v = -\frac{t}{RC} + \ln A / k / c \quad \text{constt.}$$

⇒ when $t=0$; $v(0) = V_0$

So, $\ln V_0 = 0 + \ln A \Rightarrow \ln V_0 = \ln A$ → put in above.

$$\ln v - \ln V_0 = \frac{-t}{RC}$$

$$\ln\left(\frac{v}{V_0}\right) = \frac{-t}{RC} \qquad \because e^{\ln x} = x$$

Taking exponential on both sides :-

$$e^{\ln\left(\frac{V}{V_0}\right)} = e^{-t/RC}$$

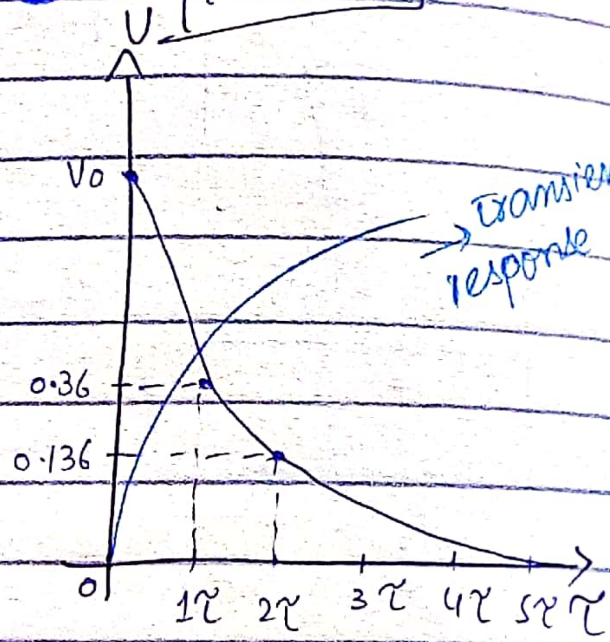
$$\frac{V}{V_0} = e^{-t/RC}$$

or $\boxed{V = V_0 e^{-t/RC}}$

$\because \tau = RC \rightarrow$ time req. for ~~...~~ response to decay to factor of $1/e$ or 36.8%.

So, $\boxed{V = V_0 e^{-t/\tau}}$ — ①

Table # 7.1



when $t=0$:-

$V = V_0 e^{-0}$

or $V = V_0(1)$

$\boxed{V = V_0.}$

when $t=\tau=RC=1$:-

$V = V_0 e^{-t/\tau}$

$V = V_0 e^{-1}$

$V = V_0(0.368)$

~~$\frac{V}{V_0} = 0.36.70$~~

$\frac{V}{V_0} = 0.368$

discharge

or $\boxed{\frac{V}{V_0} = 36.8\%.}$

if : $V = V_0 e^{-\infty}$

$= V_0 \cdot \frac{1}{e^{\infty}} = V_0 \frac{1}{\infty}$

$= V_0 \cdot \frac{1}{1/0}$

$\boxed{V = 0}$

at $t=0$

$\hookrightarrow$ V starts from V_0 & decreases to zero. $t = \infty$

when $t=2\tau$:-

$V = V_0 e^{-2}$

$V = V_0(0.136)$

$V/V_0 = 13.6\%.$

when $t=3\tau$:-

$V = V_0 e^{-3}$

$V/V_0 = 0.049$

Current :-

$$i_R(t) = \frac{V}{R}$$

$$\boxed{i_R(t) = \frac{V_0 e^{-t/\tau}}{R}}$$

Power :-

$$P = I^2 R = V^2/4$$

if: $P = I^2 \cdot R$

$$= \left[\frac{V_0 e^{-t/\tau}}{R}\right]^2 \cdot R$$

$$P = \frac{V_0^2 e^{-2t/\tau}}{R^{\cancel{2}}} \cdot \cancel{R}$$

$$\boxed{P = \frac{V_0^2 e^{-2t/\tau}}{R}}$$

Energy :-

$$P = dw/dt$$

$$W = \int_0^t P \cdot dt \qquad \because \tau = RC$$

$$W = \int_0^t \frac{V_0^2}{R} \cdot e^{-2t/RC} \, dt$$

$$W = \frac{V_0^2}{\cancel{R}} \cdot \frac{\left(e^{-2t/RC}\right)}{\left(-2/\cancel{R}C\right)} \Bigg|_0^t$$

$$\cancel{W = -\frac{1}{2} \cdot CV_0^2 e^{-2t/RC}}$$

$$W = \frac{V_0^2}{\cancel{R}} \left[\frac{-\cancel{R}C}{2}\right] \cdot \left[e^{-2t/RC} - e^0\right]$$

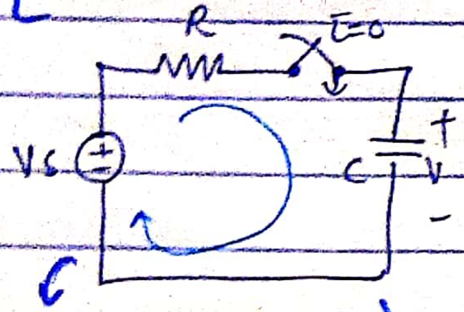
$$W = -\frac{1}{2} CV_0^2 \left[e^{-2t/RC} - 1\right]$$

$$\boxed{W = \frac{1}{2} CV_0^2 \left[1 - e^{-2t/RC}\right]}$$

steady-state response.

1.5 Step-response of an RC cct:- / RC-Transient response / temporary response / RC- steady-state:-

[complete response = transient response + steady-state response]



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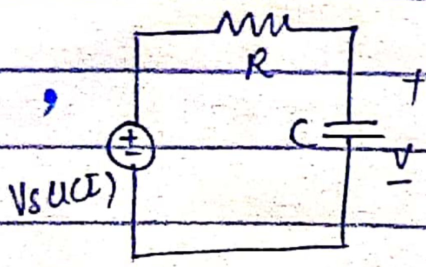

→ **KVL:- (voltage)**

[method#1]

$$V_s = V_R + V_c$$

$$V_s - V_c = iR$$

$$V_s - V_c = C\frac{dV_c}{dt} \cdot R$$

rearrange:-

$$\int \frac{dt}{RC} = \int \frac{dV_c}{V_s - V_c}$$

$$\int \frac{1}{RC} dt = \int \frac{1}{V_s - V_c} dV_c \Rightarrow \frac{1}{RC}\int 1 \cdot dt = -\int \frac{-1}{V_s - V_c} dV_c \Rightarrow -\ln|V_s - V_c| = +\frac{1}{RC}(t)$$

$$-\frac{t}{RC} + k = \ln|V_s - V_c|$$

[at $t = 0$:- $V_c = 0$.

$$\ln(V_s - 0) = 0 + k$$

$$\ln V_s = k$$]

So,

$$\ln(V_s - V_c) = -\frac{t}{RC} + \ln V_s$$

$$\ln\left(\frac{V_s - V_c}{V_s}\right) = \frac{-t}{RC}$$

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So, $\frac{V_s - V_c}{V_s} = e^{-t/RC}$

$$V_s - V_c = V_s e^{-t/RC}$$

$$-V_C = -V_s + V_s e^{-t/RC}$$

or $V_c = V_s - V_s e^{-t/RC}$ $\quad \because e^{-t/RC} = \text{decay}$

$$\boxed{V_c = V_s\left(1 - e^{-t/RC}\right)}$$

when $t=0$:- $\quad \because e^0 = 1$

$$V_c = V_s\left(1 - e^0\right)$$

$$V_c = V_s(1-1)$$

$$\boxed{V_c = 0}$$

$\because \tau = RC$

when $t = 1\tau = RC$:-

$$V_c = V_s\left(1 - e^{-1}\right)$$

$$\boxed{\frac{V_c}{V_s} = 0.6321 = 63\% \text{ charge}}$$

when $t = 2\tau$:-

$$V_c = V_s\left(1 - e^{-2}\right)$$

$$\boxed{\frac{V_c}{V_s} = 0.8646 \text{ Charge}}$$

→ **current:-**

$$i = C\frac{dV}{dt}$$

$$i = C \cdot \frac{d}{dt}\left(V_s\left(1 - e^{-t/RC}\right)\right)$$

$$i = \cancel{C} \cdot V_s\left(0 + e^{-t/RC} \cdot + \frac{1}{R\cancel{C}}\right)$$

$$\boxed{i = \frac{V_s}{R} e^{-t/RC}}$$

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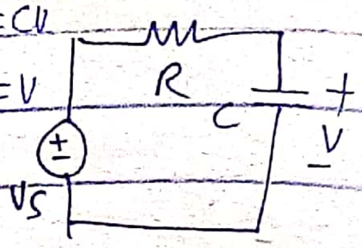
Method#2:- (Step-response of RC cct)

KVL:-

$\because q = CV$

$q/C = V$



$$V_s = V_R + V_C$$

$$V_s = iR + \frac{q}{C}$$

$$\boxed{V_s = \frac{dq}{dt} \cdot R + \frac{q}{C}} \xRightarrow{\times \text{ by } C} CV_s = RC\frac{dq}{dt} + q.$$

re-arrange:-

$$CV_s - q = RC\frac{dq}{dt}$$

multiply by -1:-

$$-RC\frac{dq}{dt} = q - CV_s$$

$$\frac{dq}{q - CV_s} = \frac{-1}{RC}\,dt$$

integrate:-

$$\int_0^q \frac{dq}{q - CV_s} = \frac{-1}{RC}\int_0^t dt$$

$$\ln(q - CV_s)\Big|_0^q = -\frac{1}{RC}\,|t|_0^t$$

$$\left[\ln(q - CV_s) - \ln(0 - CV_s)\right] = \frac{-t}{RC}$$

$$\ln\left(\frac{q - CV_s}{-CV_s}\right) = \frac{-t}{RC}$$

or $$\frac{q - CV_s}{-CV_s} = e^{-t/RC}$$

$$\frac{q - CV_s}{CV_s} = -e^{-t/RC}$$

$\because$ at $t = 0$; $q = 0$

$$\frac{q}{CV_s} - \frac{\cancel{CV_s}}{\cancel{CV_s}} = -e^{-t/RC}.$$

$$\frac{q}{CV_s} - 1 = -e^{-t/RC}$$

$$\frac{q}{CV_s} = 1 - e^{-t/RC}.$$

$\because q = CV$

$\frac{q}{C} = V$

$$\boxed{q = CV_s(1 - e^{-t/RC})}$$

or $\because q/C = V_C$

$$\boxed{V_C = V_s(1 - e^{-t/RC})}$$

current in capacitor:-

$$i_c = C\frac{dv}{dt}$$

$$i_c = C\frac{d}{dt}\left(V_s\left(1-e^{-t/RC}\right)\right)$$

$$\boxed{i_c = \frac{V_s}{R}\cdot e^{-t/RC}}$$ → current decays.

voltage across 'R':-

$$V_R = iR$$

$$V_R = \frac{V_s}{\cancel{R}}\cdot e^{-t/RC}\cdot \cancel{R}$$

$$\boxed{V_R = V_s e^{-t/RC}}$$ → V across R decreases & across C increase

$P = VI$

Energy:-

$$W = \int_0^t P\,dt$$

$$W = \int_0^t i^2 R\,dt$$

$$W = \int_0^t \left(\frac{V_s}{R}\cdot e^{-t/RC}\right)^2\cdot R\,dt$$

$$W = \int_0^t \left(\frac{V_s^2}{R^{\cancel{2}}}\cdot e^{-2t/RC}\cdot \cancel{R}\right)dt$$

$$\frac{V_s^2}{R} \div \frac{-2}{RC} \Rightarrow \frac{V_s^2}{\cancel{R}}\times\frac{\cancel{R}C}{-2}$$

$$W = \frac{V_s^2}{R}\int_0^t e^{-2t/RC}\,dt$$

$$W = \frac{-V_s^2}{\cancel{R}}\cdot \frac{e^{-2t/RC}}{-2/\cancel{R}C}\Bigg|_0^t$$

derivative of $-2t$

$$W = \frac{-V_s^2 C}{2}\left(e^{-2t/RC}-1\right)$$

x by -1:-

$$\boxed{W = \frac{1}{2}CV_s^2\left(-e^{-2t/RC}+1\right)}$$

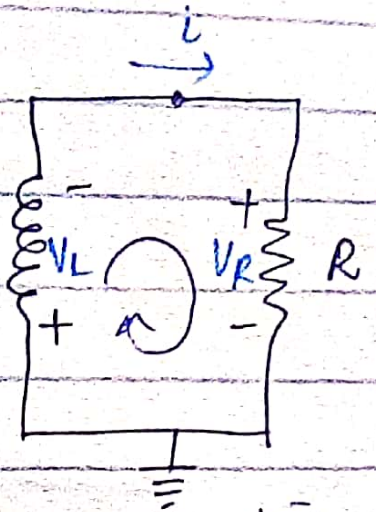
ch#7

7.3 Source - Free RL-circuit :-

at initial time t_0 inductor is charged. (by giving source and then take source away).

at $t=0$; $i(t) = I_0$

$\Rightarrow i(0) = I_0$



i) initial condition

ii) Time const. (T.C) $= \tau = L/R$

$$w(0) = \frac{1}{2} L I_0^2$$

$i(0^-) \rightarrow$ before switching

$i(0^+) \rightarrow$ after switching

$i(0) \rightarrow$ switching

KVL at cct:-

algeb. sum of all voltages in closed loop $= 0$

so, $V_L + V_R = 0$.

$$L\frac{di}{dt} + iR = 0$$

To remove L(co-efficient) :- ($\div$ by L).

$$\frac{di}{dt} + \frac{iR}{L} = 0$$

$$\frac{di}{dt} = -\frac{iR}{L}$$

re-arrange :-

$$\int_0^{i} \frac{1}{i} di = -\int_0^{t} \frac{R}{L} dt$$

$$\ln i \Big|_0^{i(t)} = -\frac{Rt}{L}\Big|_0^{t}$$

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$$\ln i(t) - \ln \underline{i(0)} = -\frac{R}{L}(t-0)$$

initial condition $\because i(0) = I_0$

$$\ln i(t) - \ln I_0 = \frac{-Rt}{L}$$

or $\boxed{\text{Lnf}}$ inch#8

$$\ln \frac{i(t)}{I_0} = \frac{-Rt}{L}$$

Taking e both sides

$$\frac{i(t)}{I_0} = e^{-Rt/L}$$

$$i(t) = I_0 e^{-Rt/L}$$

or

$$i(t) = I_0 e^{-\frac{R}{L}t}$$ max. volt. across R

$\because \tau = \frac{L}{R}$

$\frac{1}{\tau} = R/L$

So, $$\boxed{i(t) = I_0 e^{-t/\tau}}$$

at $t=0$:-

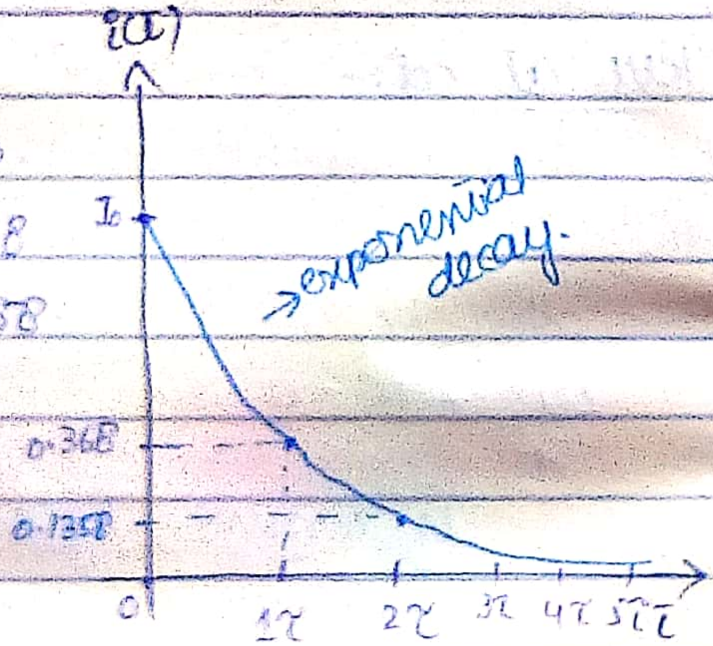
$$i(0) = I_0 e^0 = I_0$$

at $t=\tau$: $i(1) = I_0 e^{-1} = I_0\, 0.368$

at $t=2\tau$: $i(t) = I_0 e^{-2} = I_0\, 0.1358$

at $t=\infty$: $i(\infty) = I_0 e^{-\infty} = I_0 \frac{1}{e^{\infty}}$

$\neq 0$



Transient state

Voltage across R (V_R):-

$$V_R = iR$$

So, $$V_R = R(I_0 e^{-t/\tau})$$

$$\boxed{V_R = I_0 R e^{-t/\tau}}$$

Power:-

$$P_R = iV_R$$

$$P_R = (I_0 e^{-t/\tau})(I_0 R e^{-t/\tau})$$

$$\boxed{P_R = I_0^2 R \cdot e^{-2t/\tau}}$$

Energy:-

$$W = \int_0^t P \, dt$$

So,

$$W = \int_0^t I_0^2 R e^{-2t \cdot \frac{R}{L}}$$

$$W = I_0^2 R \int_0^t e^{-2t \cdot R/L} \, dt$$

$$= I_0^2 R \left| \frac{e^{-2t \cdot R/L}}{-2R/L} \right|_0^t$$

$$= -\frac{1}{2} L I_0^2 \left(e^{-2R/L \cdot t} - 1 \right). \qquad \because e^0 = 1.$$

x by -1.

$$= \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau} \right) \qquad \because \tau = L/R. \quad \frac{1}{\tau} = R/L$$

at $t=0$ $\quad W = \frac{1}{2} L I_0^2 (1 - e^0)$

$$W(0) = 0$$

at $t = \infty$ $\quad W(\infty) = \frac{1}{2} L I_0^2 (1 - e^{-\infty})$

So, $\boxed{W(\infty) = \frac{1}{2} L I_0^2}$ → (last para from book)

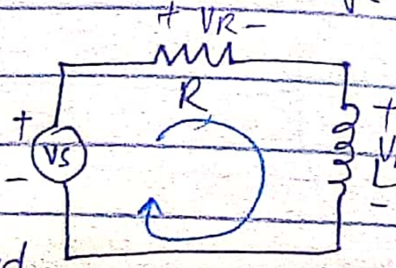
inductors & capacitor don't dessipate power, only resistor dessipate power.

steady state.

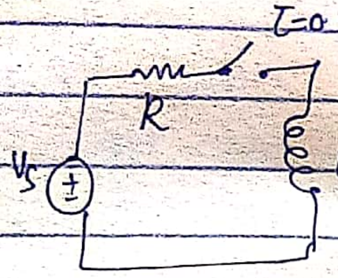
7.6 | Step-response of RL-cct.

→ **RL - with source :-** $\{ i_{total} = i_{tran.} + i_{steady} \}$ discharge / charge

at $t=0$; $i(0)=0$ $t<0$ / $i(0^-)$ before switching



at $i(0^+)$ → after switching (closed cct.)



KVL :-

$$V_S = V_R + V_L$$

$$V_S - V_R - V_L = 0.$$

or $$V_S - iR = L\frac{di}{dt}$$

÷ by R on both sides :-

$$\frac{V_S}{R} - i = \frac{L}{R}\cdot\frac{di}{dt}$$

re-arrange :-

$$\left(\frac{V_S}{R} - i\right)dt = \frac{L}{R}\,di$$

$$\int_0^t \frac{R}{L}\,dt = -\int_0^{i(t)/t} \frac{-1}{\left(\frac{V_S}{R} - i\right)}\,di.$$

$\because \frac{V_S}{R} = \text{const} = 0$

$$\frac{R}{L}\cdot t\Big|_0^t = -\ln\left(\frac{V_S}{R} - i\right)\Big|_0^{i(t)/t}$$

(t-0)

⇒ or $$-\frac{R}{L}\cdot t = \left[+\ln\left(\frac{V_S}{R} - i(t)\right) - \ln\left(\frac{V_S}{R} - 0\right)\right]$$

or $$\ln\frac{\left(\frac{V_S}{R} - i(t)\right)}{V_S/R} = -\frac{R}{L}t.$$

or take e both sides.

$$\frac{\left(\frac{V_s}{R} - i(t)\right)}{V_s/R} = e^{-\frac{R}{L} \cdot t}$$

$\because \tau = L/R$

$\frac{1}{\tau} = R/L$

$$\frac{V_s}{R} - i(t) = \frac{V_s}{R} \cdot e^{-t/\tau}$$

$$-i(t) = -\frac{V_s}{R} + \frac{V_s}{R}\left(e^{-t/\tau}\right)$$

x by -1

$$+i(t) = \frac{V_s}{R} - \frac{V_s}{R}\left(e^{-t/\tau}\right)$$

$$\boxed{i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right)}$$

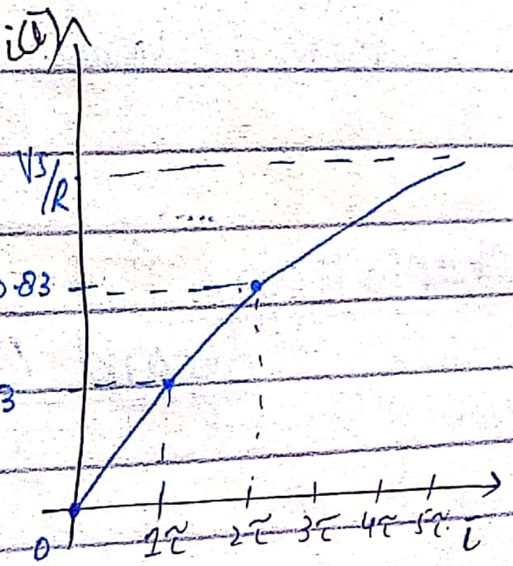
Analysis:-

at $t=0$: $i(0) = \frac{V_s}{R}(1 - e^{0}) = 0$.

at $t=1\tau$: $i(1) = \frac{V_s}{R}(1 - e^{-1}) = \frac{V_s}{R} \cdot 0.63$

at $t=2\tau$: $i(2) = \frac{V_s}{R}(1 - e^{-2}) = \frac{V_s}{R}(0.86)$

at $t=\infty$: $i(\infty) = \frac{V_s}{R}(1 - e^{-\infty}) = \frac{V_s}{R}$ (max) $\rightarrow$ max. voltage across L.



VR:-

$$V_R = iR$$

$$V_R = \frac{V_s}{\cancel{R}}(1 - e^{-t/\tau}) \cdot \cancel{R}$$

$$\boxed{V_R = V_s(1 - e^{-t/\tau})}$$

V_L :-

$$V_L = L\frac{di}{dt} \qquad 1/\tau = R/L$$

$$V_L = L\left[\frac{d}{dt}\left(\frac{V_s}{R}\left(1-e^{-t/\tau}\right)\right)\right]$$

$$V_L = \frac{V_s}{R}\cdot L\left[-e^{-t/\tau}\right]\cdot\frac{d}{dt}\left(-Rt/L\right)$$

$$V_L = \frac{V_s}{\cancel{R}}\cdot\cancel{L}\left[(-)e^{-\frac{Rt}{L}}\right]\cdot(-)\frac{\cancel{R}}{\cancel{L}}$$

$$(+)$$

$$V_L = V_s e^{-\frac{Rt}{L}}$$

or $\boxed{V_L = V_s e^{-t/\tau}}$

So,

$$i_{(total)} = i_{(trans.)} + i_{(steady)}$$

$$i_{(tot)} = \left(I_0 e^{-t/\tau}\right) + \left(\frac{V_s}{R}\left(1-e^{-t/\tau}\right)\right)$$

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