1-7

RANDOM VARIABLES:-

Q:- Explain the idea of "Random Variable"?

RANDOM VARIABL

Experiments have two kinds.

(i) Non Random (ii) Random.

The non random experiment have one particular result and therefore no series is possible.

In random experiment we have variety of results which may be described or may be assigned numerical number as they happen. When we assign real number to these non numerical events of random experiment then series of number is called random variable, e.g If 4 coins are thrown then we get heads & tails together and the series of events may be & given number as 0,1,2,3,4 heads. Likewise when two dices are thrown we may assign the sum number like 2,3, ——— 12. These series are Random variables. From these example we have seen that we assign number to all possible cases of the Sample Space. Therefore Random Variable is real numerical function of the sample space.

The random variable have two kinds.

(i) Discrete random Variable

(ii) Continous random Variable.

DISCRETE RANDOM VARIABLE.

Discrete random variable are those when there are jump points between the assigned numbers. As no. of accident per week. etc

DISCRETE PROBABILITY DISTRIBUTION.

The set of ordered pair $(x, f(x))$ is called the Probability function or Probability distribution of the discrete random variable X. A discrete Probability function may take the form of a table, a graph or a mathematical equation.

$$X: \quad x_1 \quad x_2 \quad x_3 \quad x_4 \text{ --- } x_n$$
$$f(x): \quad f(x_1) \quad f(x_2) \quad f(x_3) \quad f(x_4) \text{ --- } f(x_n)$$

Properties.. i) $0 < f(x_i) < 1$

ii) $\sum f(x_i) = 1$

CONTINOUS RANDOM VARIABLE.

In Continous r.v's there are all possible defined cases between the two limits a and b. when $a < b$ as Heights of the students, the monthly record of production eg

CONTINOUS PROBABILITY DISTRIBUTION.

Let X is a R.V whose range is the set of real number in the interval (a, b) $a < b$. We define a function $f(x)$ called the Prob density function or density function with the following properties

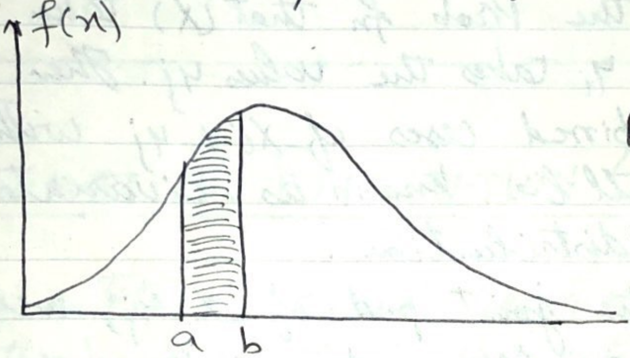
i) $f(x) \geq 0 \quad a \leq x \leq b$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

The Probability that X takes on a value in the interval (c, d) is

$$P(c \leq X \leq d) = \int_c^d f(x)\,dx$$

If $f(x)$ is the density function for a r.v X then we can represent $y = f(x)$ graphically



$P(a \leq X \leq b) = \int_a^b f(x)\,dx$

DISTRIBUTION FUNCTION FOR DISCRETE R.V.

The cumulative distribution function or briefly the distribution function for a r.v X is defined by

$$P(X \leq x) = F(x)$$

$F(x)$ is the Prob that X will assume a value less than or equal to x. As with $f(x)$, x is a real number $-\infty < x < \infty$ and the range of $F(x)$ is the closed interval $(0, 1)$

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

DISTRIBUTION FUNCTION FOR CONTINIOUS R.V.

The cummulative distribution fn in the Continous case

$$F(x) = P(X \le x) = P(-\infty \le X \le x) = \int_{-\infty}^{x} f(t)\,dt.$$

BI-VARIATE PROBABILITY FUNCTION OR DISTRIBUTION.

Let us suppose r.v (X_i) takes the values $X_1, X_2 —— X_m$ with Prob fn as $g(X_i)$. Let us also suppose that r.v (Y_j) takes the values $Y_1, Y_2 —— Y_n$ with Prob fn as $h(Y_j)$. Let $f(X_i, Y_j)$ be the Prob fn that (X) takes the values X_i & Y_i takes the values Y_j. Then all possible combined cases of X_i & Y_j with P.F $f(X_i, Y_j)$ will be known as bivariate probability distribution.

Then the joint p.d of (X_i, Y_j) with Prob $f(x_i, Y_j)$ may represented in the form of table as below.

X \ Y	Y_1	Y_2	——	Y_j	——	Y_n	$g(X)$
X_1	$f(X_1, Y_1)$	$f(X_1, Y_2)$	——	$f(X_1, Y_j)$	——	$f(X_1, Y_n)$	$g(X_1)$
X_2	$f(X_2, Y_1)$	$f(X_2, Y_2)$	——	$f(X_2, Y_j)$	——	$f(X_2, Y_n)$	$g(X_2)$
|	|						|
X_i	$f(X_i, Y_1)$	$f(X_i, Y_2)$	——	$f(X_i, Y_j)$	——	$f(X_i, Y_n)$	$g(X_i)$
|	|	|		|		|	|
X_m	$f(X_m, Y_1)$	$f(X_m, Y_2)$	——	$f(X_m, Y_j)$	——	$f(X_m, Y_n)$	$g(X_m)$
$h(Y)$	$h(Y_1)$	$h(Y_2)$	——	$h(Y_j)$	——	$h(Y_n)$	1

Marginal p.d of X.

x_i: x_1 x_2 —— x_m

$g(x_i)$: $g(x_1)$ $g(x_2)$ —— $g(x_m)$

Marginal p.d of Y.

y_j: y_1 y_2 —— y_n

$h(y_j)$: $h(y_1)$ $h(y_2)$ —— $h(y_n)$

The joint p.d will have two properties.

i) $f(x_i, y_j) \geq 0$

ii) $\sum\sum f(x_i, y_j) = 1$

MATHEMATICAL EXPECTATION OF R.V.

For quantitative variable like height, weight etc we use the word (Mean) for average of such variate. This mean is the A.M. When the defined events on some sample space are given numerical values (known as R.V) then average value of this R.V is known as mathematical expectation of R.V(X) and is denoted by E(X)

Now ~~by~~ the mathematical definition of R.V(x_i) Expectation is as:

Let (x_i) takes the values x_1, x_2 —— x_m with corresponding probabilities as $f(x_1), f(x_2)$ —— $f(x_m)$

Then expectation of (X) is defined as "Sum of the product of R.V (x_i) with the probabilities $f(x_i)$

$$E(X) = x_1 f(x_1) + x_2 f(x_2) + \cdots x_n f(x_n)$$

$$E(X) = \sum_{i=1}^{n} x_i f(x_i)$$

In case of continous r.v

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

PROPERTIES OF EXPECTATION.

① Expectation of constant is constant itself.

Proof:- Let us ~~take~~ say that (a) is the constant and $f(x_i)$ is probability function when $i = 1, 2 ——— n$ therefore

$$E(a) = \sum_{i=1}^{n} a f(x_i)$$

$$= a f(x_1) + a f(x_2) + ——— a f(x_n)$$

$$= a[f(x_1) + f(x_2) + ——— f(x_n)]$$

$$= a \sum f(x_i) \qquad \sum f(x) = 1$$

$$E(a) = a \times 1 = a$$

②

$$E(ax+b) = aE(X) + b$$

Proof:- If $(ax+b)$ is a r.v then $f(x_i)$ is p.f where $i = 1, 2, 3 ——— n$

$$E(ax+b) = \sum_{i=1}^{n} (ax_i + b) f(x_i)$$

$$= \sum_{i=1}^{n} [a x_i f(x_i) + b f(x_i)]$$

$$= \sum_{i=1}^{n} a x_i f(x_i) + \sum_{i=1}^{n} b f(x_i)$$

$$= a \sum_{i=1}^{n} x_i f(x_i) + b \sum_{i=1}^{n} f(x_i)$$

(4)

$\sum x_i f(x_i) = E(x)$ and $\sum f(x) = 1$

$E(ax+b) = aE(x) + b.$

(3) Expectation of the product of two independent r.v's is equal to the product of their expectation

$$E(xy) = E(x) \cdot E(y)$$

Proof:-

Let (x_i) takes the values x_1, x_2 —— x_m with corresponding probability as $g(x_1), g(x_2)$ — $g(x_m)$

Let (y_j) takes the values y_1, y_2 —— y_n with corresponding probabilities $h(y_1), h(y_2)$ —— $h(y_n)$.

Let $f(x_i, y_j)$ be the probability that x takes the values (x_i) and (y) takes the values (y_j).

According to Expectation

$$E(XY) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i, y_j) f(x_i, y_j) \quad \text{—— (1)}$$

Since (x) & (y) are independent therefore for the joint happening their probabilities will multiply.

Thus $f(x_i, y_j) = g(x_i) . h(y_j)$ —— (2)

Put (2) in (1)

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i, y_j)\, g(x_i)\, h(y_j)$$

$$= \sum_{i=1}^{m} x_i g(x_i) \sum_{j=1}^{n} y_j h(y_j)$$

$$E(XY) = E(x) \cdot E(y).$$

(4) The expectation of the sum of two independent r.v is equal to the sum of their expectations

$$E(X+Y) = E(X) + E(Y)$$

<u>Proof</u>:- Let us consider that (X) takes the values (x_i) where $i = 1, 2, 3 \text{---} m$ with Probabilities $g(x_i)$. Let us also consider that (Y) takes the values y_j where $j = 1, 2, 3 \text{---} n$ with Probabilities $h(y_j)$. Let $f(x_i, y_j)$ be the joint Probability so that x takes the values x_i & y takes the values y_j. Therefore the table of joint prob is as

	y_1	y_2	---	y_j	---	y_n	$g(x_i)$
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	---	$f(x_1, y_j)$	---	$f(x_1, y_n)$	$g(x_1)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	---	$f(x_2, y_j)$	---	$f(x_2, y_n)$	$g(x_2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	$f(x_i, y_1)$	$f(x_i, y_2)$	---	$f(x_i, y_j)$	---	$f(x_i, y_n)$	$g(x_i)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	---	$f(x_m, y_j)$	---	$f(x_m, y_n)$	$g(x_m)$
$h(y_j)$	$h(y_1)$	$h(y_2)$	---	$h(y_j)$	---	$h(y_n)$	1

Now by definition of Expectation

$$E(X+Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) f(x_i, y_j) \text{ --- } \textcircled{A}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i f(x_i, y_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j f(x_i, y_j) \text{ --- (A)}$$

Now

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i f(x_i, y_j) = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} f(x_i, y_j)$$

$j = 1, 2, 3 \text{ --- } n$

$$= \sum_{i=1}^{m} x_i [f(x_i, y_1) + f(x_i, y_2) + \text{ --- } f(x_i, y_n)]$$

According to table

$$f(x_i, y_1) + f(x_i, y_2) + \text{ --- } f(x_i, y_n) = g(x_i)$$

$$= \sum_{i=1}^{m} x_i g(x_i) = E(X) \text{ --- } (i)$$

Likewise.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_j f(x_i, y_j) = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} f(x_i, y_j) \text{ --- }$$

$i = 1, 2, 3 \text{ --- } m$

$$= \sum_{j=1}^{n} y_j [f(x_1, y_j) + f(x_2, y_j) + \text{ --- } f(x_m, y_j)]$$

& According to table

$$f(x_1, y_j) + f(x_2, y_j) + \text{ --- } f(x_m, y_j) = h(y_j)$$

$$= \sum_{j=1}^{n} y_j h(y_j) = E(Y) \text{ --- } (ii)$$

Put (i) & (ii) in (A)

$$E(X+Y) = E(X) + E(Y)$$

(5) The covariance of two independent random variates is equal to zero.

Proof $Cov(X,Y) = E[X-E(X)][Y-E(Y)]$

$$= E[XY - XE(Y) - YE(X) + E(X).E(Y)]$$

$$= E(XY) - E(X)E(Y) - E(Y)\cancel{E}(X) + E(X).E(Y)$$

$$= 0$$

If the two values are co related then $Cov(X,Y)$ is not zero.

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(6) The variance of sum or differences of two independent random variate is equal to the sum of their expectation.

$V(X) = E[X-E(X)]^2$

$$V(X \pm Y) = V(X) + V(Y)$$

Proof :- Let us we consider $(X-Y)$

$$V(X-Y) = E[(X-Y) - E(X-Y)]^2$$

$$= E[X - Y - E(X) + E(Y)]^2$$

$$= E[(X-E(X)) - (Y-E(Y))]^2$$

$$= E([X-E(X)]^2 + [Y-E(Y)]^2 - 2[X-E(X)][Y-E(Y)])$$

$$= E[X-E(X)]^2 + E[Y-E(Y)]^2 - 2E[X-E(X)][Y-E(Y)]$$

But $E[X-E(X)][Y-E(Y)] = 0$

$$V(X-Y) = E(X-E(X))^2 + E(Y-E(Y))^2$$

$$\boxed{V(X-Y) = V(X) + V(Y)}$$

Moment Generating Function:

The m.g.f usually denoted by $M_0(t)$ of a random variable x about the origin is defined as the expected value of the function e^{tx}, where t is real number.

$$M_0(t) = E(e^{tx}) = \sum_{i=1}^{\infty} e^{tx_i} f(x_i)$$

$$= \sum \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} \cdots \frac{(tx)^r}{r!} \cdots\right) f(x)$$

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots$$

$$= \sum f(x) + t \sum x f(x) + \frac{t^2}{2!} \sum x^2 f(x) +$$

$$+ \cdots + \frac{t^r}{r!} \sum x^r f(x) + \cdots$$

$$= 1 + t E(x) + \frac{t^2}{2!} E(x^2) + \cdots + \frac{t^r}{r!} E(x^r) + \cdots$$

$$= 1 + t \mu_1' + \frac{t^2}{2!} \mu_2' + \cdots + \frac{t^r}{r!} \mu_r' + \cdots$$

$$\frac{M_0(t)}{\cancel{E(x)}} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

Thus we find that coefficient of $\frac{t^r}{r!}$ in the expansion of $M_0(t)$ is $E(X^r)$ or μ_r', the rth moment about zero. We call the function $M_0(t)$ the moment Generating function because it generates moments of the r.v. 'X'.

PROPERTIES OF MOMENT GENRATING FUNCTION.

If $M_x(t)$ is the m.g.f of the r.v 'x' and a & b $(b \neq 0)$

① $$M_{x \pm a}(t) = E\left[e^{(X \pm a)t}\right]$$

$$= E\left[e^{Xt} e^{\pm at}\right]$$

$$M_x(t) = E(e^{tx})$$

$$= e^{\pm at} E(e^{Xt})$$

$$= e^{\pm at} M_x(t)$$

② $$M_{bx}(t) = E\left[e^{(bX)t}\right]$$

$$= E\left[e^{X(bt)}\right] = M_x(bt)$$

③ $$M_{\left(\frac{x \pm a}{b}\right)}(t) = E\left[e^{\left(\frac{X \pm a}{b}\right)t}\right]$$

$$= E\left[e^{\frac{X}{b}t \pm \frac{at}{b}}\right]$$

$$= e^{\pm \frac{at}{b}} E\left[e^{\left(\frac{X}{b}\right)t}\right]$$

$$= e^{\pm \frac{at}{b}} E\left[e^{X(t/b)}\right]$$

$$= e^{\pm \frac{at}{b}} M_x\left(\frac{t}{b}\right)$$

④ If X & Y are independent r.v having m.g.f's $M_x(t)$ & $M_y(t)$ resp. then

$$M_{(x+y)}(t) = E\left[e^{(X+Y)t}\right]$$

$$= E(e^{Xt}) E(e^{Yt})$$

$$= M_x(t) \cdot M_y(t)$$ i.e the m.g.f of sum of two independent r.v is equal to the Product of their m.g.f's