$\nabla^2 A_x(\vec{r}) = -\mu_0 J_x(\vec{r})$

$\nabla^2 V(\vec{r}) = -\dfrac{\rho(\vec{r})}{\epsilon_0}$ Poisson's Eq. in electrostatics

As, $\vec{J}(\vec{r}) = J_x(\vec{r})\hat{x} + J_y(\vec{r})\hat{y} + J_z(\vec{r})\hat{z}$

$\left.\begin{array}{l}\nabla^2 A_y(\vec{r}) = -\mu_0 J_y(\vec{r}) \\ \nabla^2 A_z(\vec{r}) = -\mu_0 J_z(\vec{r})\end{array}\right\}$

Three separate/independent scalar forms of Poisson's Equation

$\therefore \nabla^2 \vec{A} = -\mu_0 \vec{J}$ ≡ Poisson's vector equation/Vector form of Poisson's Equation for magnetostatics

According to Biot-Savart Law, the magnetic field, $\vec{B}$ at an arbitrary point 'P' due to volume currents is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d\tau' \quad \text{— (3)}$$

The famous vector identity is

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

where, $\vec{A}$ ≡ Any vector f ≡ any scalar function

$\nabla^2\vec{A}(\vec{r}) = (\nabla^2 A_x(\vec{r}))\hat{x} + (\nabla^2 A_y(\vec{r}))\hat{y} + (\nabla^2 A_z(\vec{r}))\hat{z}$; in cartesian coordinates of course

Let, $f = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$ and $\vec{A} = \vec{J}(\vec{r}\,')$

In current-free regions, $\nabla^2\vec{A}(\vec{r}) = 0$

$\Rightarrow \nabla^2 A_x(\vec{r}) = 0$; $\nabla^2 A_y(\vec{r}) = 0$; $\nabla^2 A_z(\vec{r}) = 0$

Same as solution of Laplace eq. in electrostatics except that there you've only one solution $V(\vec{r})$, here three: $A_x(\vec{r})$; $A_y(\vec{r})$; $A_z(\vec{r})$.

$$\Rightarrow \vec{\nabla}f = \vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|} = -\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\therefore \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = \frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\nabla}\times\vec{J}(\vec{r}\,')\right) + \vec{J}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$\vec{\nabla}\times\vec{J}(\vec{r}\,') = 0$ since, $\vec{J}$ is independent of (x, y, z)

$$\Rightarrow \vec{J}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) \quad \text{— (4)}$$

Using Eq. (4) in Eq. (3), we get

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) d\tau'$$

Solution of $\nabla^2 V(\vec{r}) = -\dfrac{\rho(\vec{r})}{\epsilon_0}$ is $V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\tau'$

since, the integration is over primed variables, i.e., $d\tau' = dx'\,dy'\,dz'$ and $\vec{\nabla} = \dfrac{\partial}{\partial x}\hat{i} + \dfrac{\partial}{\partial y}\hat{j} + \dfrac{\partial}{\partial z}\hat{k}$

→ Solution of $\nabla^2 A_x(\vec{r}) = -\mu_0 J_x(\vec{r})$ is $A_x(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int_V \frac{J_x(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\tau'$

$$\therefore \vec{B}(\vec{r}) = \vec{\nabla}\times\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\tau'$$

$\Rightarrow \vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int_V \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\tau'$

Comparing it with $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$

$I\,d\vec{l} \sim \vec{K}\,da \sim \vec{J}\,d\tau$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\tau'$$

$V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{|\vec{r}-\vec{r}\,'|}$

$\nabla^2 V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{all space}} \rho(\vec{r}\,')\,\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|}\,d\tau'$

$= \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{all space}} \rho(\vec{r}\,')\,\vec{\nabla}\cdot\left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)d\tau'$

$= -\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{all space}} \rho(\vec{r}\,')\left(\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right)d\tau'$

$= -\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{all space}} \rho(\vec{r}\,')\,4\pi\,\delta^3(\vec{r}-\vec{r}\,')\,d\tau'$

$= -\dfrac{\rho(\vec{r})}{\epsilon_0}$.

For surface currents, $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\vec{K}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}da'$ For line currents, $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\vec{I}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dl' = \dfrac{\mu_0 I}{4\pi}\int\frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|}$ (for steady current)

∴ [redacted] 6 :→ Biot-Savart law for MVP