Physical Significance of $\vec{A}$:

$$\int_S \vec{B}(\vec{r})\cdot d\vec{a} = \int_S \underbrace{(\vec{\nabla}\times\vec{A}(\vec{r}))\cdot d\vec{a}}_{\text{Applying Stoke's/Curl theorem}}$$

[redacted]

$$\int_S \underbrace{\vec{B}(\vec{r})\cdot d\vec{a}}_{\text{Weber}} = \oint_C \underbrace{\vec{A}(\vec{r})}_{\text{Wb/m (or T-m)}}\cdot d\vec{l}$$

Line integral of $\vec{A}(\vec{r})$ around any closed path equals the total magnetic flux passing through the area enclosed by the path.

Magnetic Vector Potential:

We know from Biot-Savart Law that

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0 \equiv \text{Maxwell's second equation} \quad \text{—— (1)}$$

i.e., Magnetic monopole doesn't exist.

We all know very well that the divergence of a curl of vector vanishes, i.e.,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}(\vec{r})) = 0 \quad \text{—— (2).}$$

Comparing Eqs. (1) and (2), we get

$$\underbrace{\vec{B}(\vec{r})}_{\text{Teslas}} = \underbrace{\vec{\nabla}}_{1/m}\times\underbrace{\vec{A}(\vec{r})}_{\text{Tesla-meters (or Wb/m)}}$$

weber is equivalent to $T.m^2$ (tesla-square-meter)

where,

$\vec{A} \equiv$ Magnetic Vector Potential (SI Unit: Tesla-meters)

Weber is SI unit of magnetic flux $\Rightarrow \Phi_B = \int_S \vec{B}(\vec{r})\cdot d\vec{a} \Rightarrow$ Weber $= T\text{-}m^2$

The differential form of Ampere's Circuital law is

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{r})) = \mu_0\vec{J}(\vec{r})$$

Using vector identity,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\therefore \quad \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

Note:-

In electrostatics,

$$\underbrace{\vec{\nabla}\times\vec{E}}_{\text{curl of a gradient is zero}} = 0$$

$$-\vec{\nabla}\times\vec{\nabla}V = 0$$

$$\Rightarrow \quad \vec{E} = -\vec{\nabla}V$$

$\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ permitted us to introduce electric scalar potential $V(\vec{r})$ such that:

$$\vec{E}(\vec{r}) \equiv -\vec{\nabla}V(\vec{r})$$

We have defined the vector quantity $\vec{A}$ as the magnetic vector potential whose curl yields $\vec{B}$. But a vector field is uniquely defined if and only if both its curl and divergence are defined. Therefore, we must still define the divergence of $\vec{A}$. In magnetostatics, we define $\vec{\nabla}\cdot\vec{A} = 0$ and refer to this constraint as Coulomb's Gauge.

$$\nabla^2\vec{A}(\vec{r}) = -\mu_0\vec{J}(\vec{r})$$

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The Uniqueness Theorem:

The uniqueness theorem states that a vector field $\vec{A}$ is uniquely determined in a region if the following requirements are satisfied:

(a) Its divergence is specified throughout the region,

(b) Its curl is specified throughout the region, and

(c) Its normal component is specified on the closed surface bounding the region.

Classification of Fields:

The divergence and curl of a vector field are independent operations; therefore, neither one is sufficient to describe a field completely. In fact, in our study of electromagnetic fields, we'll find that fields fall into four basic classifications; In solving field problems it is necessary to know which class of field we are working with because this will dictate the procedure we must use to solve the problem.

Class I Fields:

We'll consider a vector field $\vec{F}$ to be a class I field everywhere in a region if

$$\vec{\nabla}\cdot\vec{F} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{F} = 0$$

However, if the curl of a vector is zero, the vector can be written in terms of a gradient of a scalar function f. That is

$$\vec{F} = -\vec{\nabla}f$$

$$\therefore \quad \vec{\nabla}\cdot(-\vec{\nabla}f) = 0$$

$$\Rightarrow \quad \nabla^2 f = 0 \quad \equiv \text{Laplace's Equation}$$

Therefore, to obtain fields of class I, we need to solve Laplace's equation subjected to the conditions at the boundary of the region. Once, we know f, we can compute $\vec{F}$ by $\vec{F} = -\vec{\nabla}f$.

Examples of class I fields are electrostatic fields in charge-free medium and magnetic fields in current-free medium.

Class II Fields:

We'll refer to a vector field $\vec{F}$ as a class II field in a given region if

$$\vec{\nabla}\cdot\vec{F} \neq 0 \quad \text{and} \quad \underbrace{\vec{\nabla}\times\vec{F} = 0}_{\vec{F} = -\vec{\nabla}f}$$

Because $\vec{\nabla}\cdot\vec{F} \neq 0$, we can write it as $\vec{\nabla}\cdot\vec{F} = \rho$, where ρ is either a constant or a known function within the region. Thus,

$$\nabla^2 f \neq 0. \qquad \nabla^2 f = -\rho \quad \equiv \text{Poisson's Equation}$$

Thus, class II fields can be found by solving Poisson's Equation within the constraints of the boundary conditions. We can then find the vector field $\vec{F}$ as $\vec{F} = -\vec{\nabla} f$.

An electrostatic field in a charged region is an example of a class II field.

Class III Fields:

$$\vec{\nabla} \cdot \vec{F} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{F} \neq 0$$

If the divergence of a vector is zero, then the vector can be expressed in terms of the curl of another vector as

$$\vec{F} = \vec{\nabla} \times \vec{A}$$

where, $\vec{A}$ is another vector field.

Because $\vec{\nabla} \times \vec{F} \neq 0$, we can write it as $\vec{\nabla} \times \vec{F} = \vec{J}$

where, $\vec{J}$ is a known vector field.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{J} \neq 0.$$

$$\vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{A})}_{= 0} - \nabla^2 \vec{A} = \vec{J}$$

we must set an arbitrary constraint that $\vec{\nabla} \cdot \vec{A} = 0$ ≡ Coulomb's guage

$$\Rightarrow \quad \nabla^2 \vec{A} = -\vec{J} \equiv \text{Poisson's vector equation}$$

Read Further from:

"Electromagnetic Field Theory Fundamentals"

by Bhag Guru and Hüseyin Hiziroğlu — Second Edition.

Class IV Fields: $\vec{\nabla} \cdot \vec{F}(\vec{r}) \neq 0$; $\vec{\nabla} \times \vec{F}(\vec{r}) \neq 0$.