According to Biot-Savart law, the magnetic field at point 'P' due to steady current for which $\vec{\nabla}\cdot\vec{J}=0$, is given by

$$\cancel{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau'}$$

As, $\vec{r}-\vec{r}' = \vec{\mathscr{r}}$

$$\therefore \cancel{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\, d\tau'}$$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau'$$

As, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$

$\vec{\mathscr{r}} = \vec{r}-\vec{r}' = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$

So,

$\vec{B}$ is a function of $(x, y, z)$

$\vec{J}$ is a function of $(x', y', z')$

$d\tau' = dx'dy'dz'$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau' = \frac{\mu_0}{4\pi}\int\left(\left(\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\cdot\vec{\nabla}\right)\vec{J}(\vec{r}')\right.$$

$$\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = ?$$

$$-\frac{\mu_0}{4\pi}\int_V\left(\vec{J}(\vec{r}')\cdot\vec{\nabla}\right)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d\tau'_V + \frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}')\left(\vec{\nabla}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right)$$

$$-\frac{\mu_0}{4\pi}\int_V\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\left(\vec{\nabla}\cdot\vec{J}(\vec{r}')\right)d\tau'$$

Using vector identity,

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

$$\therefore \vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \underbrace{\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\vec{\nabla}\right)\vec{J}}_{=0} - (\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \vec{J}\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - \underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}(\vec{\nabla}\cdot\vec{J})}_{=0}$$

I've dropped the terms involving derivatives of $\vec{J}$, because $\vec{J}$ doesn't depend on $x, y, z$.

* Assignment:

What is the contribution of the term $-(\vec{J}\cdot\vec{\nabla})\frac{\hat{r}}{r^2}$ ?

$$-\int_V (\vec{J}(\vec{r}')\cdot\vec{\nabla})\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau' = ?$$

$$-\int_V \left(\vec{J}(\vec{r}')\cdot\vec{\nabla}\right)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau' = +\int_V \left(\vec{J}(\vec{r}')\cdot\vec{\nabla}'\right)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau'$$

$$= \int_V \left(\vec{J}(\vec{r}')\cdot\vec{\nabla}'\right)\left[\frac{x-x'}{|\vec{r}-\vec{r}'|^3}\hat{x} + \frac{y-y'}{|\vec{r}-\vec{r}'|^3}\hat{y} + \frac{z-z'}{|\vec{r}-\vec{r}'|^3}\hat{z}\right]d\tau'$$

As, $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

$$\Rightarrow (\vec{A}\cdot\vec{\nabla})f = \vec{\nabla}\cdot(f\vec{A}) - f(\vec{\nabla}\cdot\vec{A})$$

$$\left(\vec{J}(\vec{r}')\cdot\vec{\nabla}'\right)\frac{x-x'}{|\vec{r}-\vec{r}'|^3} = \vec{\nabla}'\cdot\left(\frac{(x-x')}{|\vec{r}-\vec{r}'|^3}\vec{J}(\vec{r}')\right) - \frac{(x-x')}{|\vec{r}-\vec{r}'|^3}\underbrace{\left(\vec{\nabla}'\cdot\vec{J}(\vec{r}')\right)}_{=0}$$

(For steady currents)

$$= \int_V \vec{\nabla}'\cdot\left(\frac{(x-x')}{|\vec{r}-\vec{r}'|^3}\vec{J}(\vec{r}')\right)d\tau'\,\hat{x} + \cdots$$

$$= \oint_S \frac{(x-x')}{|\vec{r}-\vec{r}'|^3}\vec{J}(\vec{r}')\cdot d\vec{a}'\,\hat{x} + \cdots = \oint_S \frac{1}{|\vec{r}-\vec{r}'|^3}\left[(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}\right]\vec{J}(\vec{r}')\cdot d\vec{a}'$$

If volume V and corresponding enclosing surface $S\to\infty$, then $\exists$ no currents on surface, $\Rightarrow$ $-\int\left(\vec{J}(\vec{r}')\cdot\vec{\nabla}\right)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau' = 0 = \oint_S \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\vec{J}(\vec{r}')\cdot d\vec{a}'$

$\vec{J}(\vec{r}')=0$, all space out there.

* Assignment:

Prove that, $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$

$$\therefore \quad \vec{\nabla} \times \vec{B} = \frac{\mu_0}{\cancel{4\pi}} \int_{\text{all space}} \vec{J}(\vec{r}') \, \cancel{4\pi} \, \delta^3(\vec{r} - \vec{r}') \, d\tau'$$

$$= \mu_0 \int_{\text{all space}} \vec{J}(\vec{r}') \, \delta^3(\vec{r} - \vec{r}') \, d\tau'$$

Using the property of Delta function.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r}) \quad ; \quad \text{where, } \vec{B} = \vec{B}(\vec{r})$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

Applying Stoke's theorem,

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I}$$

<- End of Lecture # 43 ...