CHAPTER

SEVENTEEN

HIGHER-ORDER DIFFERENCE EQUATIONS

The economic models in the preceding chapter involve difference equations that relate P_t and P_{t-1} to each other. As the P value in one period can uniquely determine the P value in the next, the time path of P becomes fully determinate once an initial value P_0 is specified. It may happen, however, that the value of an economic variable in period t (say, y_t) depends not only on y_{t-1} but also on y_{t-2}. Such a situation will give rise to a difference equation of the second order.

Strictly speaking, a *second-order difference equation* is one that involves an expression $\Delta^2 y_t$, called the *second difference* of y_t, but contains no differences of order higher than 2. The symbol Δ^2, the discrete-time counterpart of the symbol d^2/dt^2, is an instruction to "take the second difference" as follows:

$$\begin{aligned}\Delta^2 y_t &= \Delta(\Delta y_t) = \Delta(y_{t+1} - y_t) && \text{[by (16.1)]} \\ &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) && \text{[again by (16.1)]*} \\ &= y_{t+2} - 2y_{t+1} + y_t\end{aligned}$$

Thus a second difference of y_t is transformable into a sum of terms involving a

* That is, we first move the subscripts in the $(y_{t+1} - y_t)$ expression forward by one period, to get a new expression $(y_{t+2} - y_{t+1})$, and then we subtract from the latter the original expression. Note that, since the resulting difference may be written as $\Delta y_{t+1} - \Delta y_t$, we may infer the following rule of operation:

$$\Delta(y_{t+1} - y_t) = \Delta y_{t+1} - \Delta y_t$$

This is reminiscent of the rule applicable to the derivative of a sum or difference.

two-period time lag. Since expressions like $\Delta^2 y_t$ and Δy_t are quite cumbersome to work with, we shall simply redefine a second-order difference equation as one involving a two-period time lag in the variable. Similarly, a third-order difference equation is one that involves a three-period time lag, etc.

Let us first concentrate on the method of solving a second-order difference equation, leaving the generalization to higher-order equations for a later section. To keep the scope of discussion manageable, we shall only deal with linear difference equations with constant coefficients in the present chapter. However, both the constant-term and variable-term varieties will be examined below.

17.1 SECOND-ORDER LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS AND CONSTANT TERM

A simple variety of second-order difference equations takes the form

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c \tag{17.1}$$

You will recognize this equation to be linear, nonhomogeneous, and with constant coefficients (a_1, a_2) and constant term c.

Particular Integral

As before, the solution of (17.1) may be expected to have two components: a particular integral y_p representing the intertemporal equilibrium level of y, and a complementary function y_c specifying, for every time period, the deviation from the equilibrium. The particular integral, defined as any solution of the complete equation, can sometimes be found simply by trying a solution of the form $y_t = k$. Substituting this constant value of y into (17.1), we obtain

$$k + a_1 k + a_2 k = c \qquad \text{and} \qquad k = \frac{c}{1 + a_1 + a_2}$$

Thus, so long as $(1 + a_1 + a_2) \neq 0$, the particular integral is

$$y_p(= k) = \frac{c}{1 + a_1 + a_2} \qquad (\text{case of } a_1 + a_2 \neq -1) \tag{17.2}$$

Example 1 Find the particular integral of $y_{t+2} - 3y_{t+1} + 4y_t = 6$. Here we have $a_1 = -3$, $a_2 = 4$, and $c = 6$. Since $a_1 + a_2 \neq -1$, the particular integral can be obtained from (17.2) as follows:

$$y_p = \frac{6}{1 - 3 + 4} = 3$$

In case $a_1 + a_2 = -1$, then the trial solution $y_t = k$ breaks down, and we must try $y_t = kt$ instead. Substituting the latter into (17.1) and bearing in mind that we now have $y_{t+1} = k(t+1)$ and $y_{t+2} = k(t+2)$, we find that

$$k(t+2) + a_1 k(t+1) + a_2 kt = c$$

and $$k = \frac{c}{(1 + a_1 + a_2)t + a_1 + 2} = \frac{c}{a_1 + 2} \qquad [\text{since } a_1 + a_2 = -1]$$

Thus we can write the particular integral as

(17.2′) $$y_p(= kt) = \frac{c}{a_1 + 2} t \qquad (\text{case of } a_1 + a_2 = -1;\ a_1 \neq -2)$$

Example 2 Find the particular integral of $y_{t+2} + y_{t+1} - 2y_t = 12$. Here, $a_1 = 1$, $a_2 = -2$, and $c = 12$. Obviously, formula (17.2) is not applicable, but (17.2′) is. Thus,

$$y_p = \frac{12}{1+2} t = 4t$$

This particular integral represents a moving equilibrium.

If $a_1 + a_2 = -1$, but at the same time $a_1 = -2$ (that is, if $a_1 = -2$ and $a_2 = 1$), then we can adopt a trial solution of the form $y_t = kt^2$, which implies $y_{t+1} = k(t+1)^2$, etc. As you may verify, in this case the particular integral turns out to be

(17.2″) $$y_p = kt^2 = \frac{c}{2} t^2 \qquad (\text{case of } a_1 = -2;\ a_2 = 1)$$

However, since this formula applies only to the unique case of the difference equation $y_{t+2} - 2y_{t+1} + y_t = c$, its usefulness is rather limited.

Complementary Function

To find the complementary function, we must concentrate on the reduced equation

(17.3) $$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

Our experience with first-order difference equations has taught us that the expression Ab^t plays a prominent role in the general solution of such an equation. Let us therefore try a solution of the form $y_t = Ab^t$, which naturally implies that $y_{t+1} = Ab^{t+1}$, and so on. It is our task now to determine the values of A and b.

Upon substitution of the trial solution into (17.3), the equation becomes

$$Ab^{t+2} + a_1 Ab^{t+1} + a_2 Ab^t = 0$$

or, after canceling the (nonzero) common factor Ab^t,

(17.3′) $$b^2 + a_1 b + a_2 = 0$$

This quadratic equation—the *characteristic equation* of (17.3) or of (17.1)—which is comparable to (15.4″), possesses the two *characteristic roots*

(17.4) $$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

each of which is acceptable in the solution Ab^t. In fact, *both* b_1 *and* b_2 should appear in the general solution of the homogeneous difference equation (17.3) because, just as in the case of differential equations, this general solution must consist of two *linearly independent* parts, each with its own multiplicative arbitrary constant.

Three possible situations may be encountered in regard to the characteristic roots, depending on the square-root expression in (17.4). You will find these parallel very closely the analysis of second-order differential equations in Sec. 15.1.

Case 1 (distinct real roots) When $a_1^2 > 4a_2$, the square root in (17.4) is a real number, and b_1 and b_2 are real and distinct. In that event, b_1^t and b_2^t are linearly independent, and the complementary function can simply be written as a linear combination of these expressions; that is,

$$y_c = A_1 b_1^t + A_2 b_2^t \tag{17.5}$$

You should compare this with (15.7).

Example 3 Find the solution of $y_{t+2} + y_{t+1} - 2y_t = 12$. This equation has the coefficients $a_1 = 1$ and $a_2 = -2$; from (17.4), the characteristic roots can be found to be $b_1, b_2 = 1, -2$. Thus, the complementary function is

$$y_c = A_1(1)^t + A_2(-2)^t = A_1 + A_2(-2)^t$$

Since, in Example 2, the particular integral of the given difference equation has already been found to be $y_p = 4t$, we can write the general solution as

$$y_t = y_c + y_p = A_1 + A_2(-2)^t + 4t$$

There are still two arbitrary constants A_1 and A_2 to be definitized; to accomplish this, *two* initial conditions are necessary. Suppose that we are given $y_0 = 4$ and $y_1 = 5$. Then, since by letting $t = 0$ and $t = 1$ successively in the general solution we find

$$y_0 = A_1 + A_2 \qquad (= 4 \text{ by the first initial condition})$$

$$y_1 = A_1 - 2A_2 + 4 \qquad (= 5 \text{ by the second initial condition})$$

the arbitrary constants can be definitized to $A_1 = 3$ and $A_2 = 1$. The definite solution then can finally be written as

$$y_t = 3 + (-2)^t + 4t$$

Case 2 (repeated real roots) When $a_1^2 = 4a_2$, the square root in (17.4) vanishes, and the characteristic roots are repeated:

$$b(= b_1 = b_1) = -\frac{a_1}{2}$$

Now, if we express the complementary function in the form of (17.5), the two components will collapse into a single term:

$$A_1 b_1^t + A_2 b_2^t = (A_1 + A_2) b^t \equiv A_3 b^t$$

This will not do, because we are now short of one constant.

To supply the missing component—which, we recall, should be linearly independent of the term $A_3 b^t$—the old trick of multiplying b^t by the variable t will again work. The new component term is therefore to take the form $A_4 t b^t$. That this is linearly independent of $A_3 b^t$ should be obvious, for we can never obtain the expression $A_4 t b^t$ by attaching a constant coefficient to $A_3 b^t$. That $A_4 t b^t$ does indeed qualify as a solution of the homogeneous equation (17.3), just as $A_3 b^t$ does, can easily be verified by substituting $y_t = A_4 t b^t$ [and $y_{t+1} = A_4(t+1)b^{t+1}$, etc.] into (17.3)* and seeing that the latter will reduce to an identity $0 = 0$.

The complementary function for the repeated-root case is therefore

$$(17.6) \qquad y_c = A_3 b^t + A_4 t b^t$$

which you should compare with (15.9).

Example 4 Find the complementary function of $y_{t+2} + 6y_{t+1} + 9y_t = 4$. The coefficients being $a_1 = 6$ and $a_2 = 9$, the characteristic roots are found to be $b_1 = b_2 = -3$. We therefore have

$$y_c = A_3(-3)^t + A_4 t(-3)^t$$

If we proceed a step further, we can easily find $y_p = \frac{1}{4}$, so the general solution of the given difference equation is

$$y_t = A_3(-3)^t + A_4 t(-3)^t + \tfrac{1}{4}$$

Given two initial conditions, A_3 and A_4 can again be assigned definite values.

Case 3 (complex roots) Under the remaining possibility of $a_1^2 < 4a_2$, the characteristic roots are conjugate complex. Specifically, they will be in the form

$$b_1, b_2 = h \pm vi$$

where

$$(17.7) \qquad h = -\frac{a_1}{2} \qquad \text{and} \qquad v = \frac{\sqrt{4a_2 - a_1^2}}{2}$$

The complementary function itself thus becomes

$$y_c = A_1 b_1^t + A_2 b_2^t = A_1(h + vi)^t + A_2(h - vi)^t$$

As it stands, y_c is not easily interpreted. But fortunately, thanks to De Moivre's theorem, given in (15.23′), this complementary function can easily be transformed into trigonometric terms, which we have learned to interpret.

* In this substitution it should be kept in mind that we have in the present case $a_1^2 = 4a_2$ and $b = -a_1/2$.

According to the said theorem, we can write

$$(h \pm vi)^t = R^t(\cos\theta t \pm i\sin\theta t)$$

where the value of R (always taken to be positive) is, by (15.10),

$$R = \sqrt{h^2 + v^2} = \sqrt{\frac{a_1^2 + 4a_2 - a_1^2}{4}} = \sqrt{a_2} \tag{17.8}$$

and θ is the radian measure of the angle in the interval $[0, 2\pi)$, which satisfies the conditions

$$\cos\theta = \frac{h}{R} = \frac{-a_1}{2\sqrt{a_2}} \qquad \text{and} \qquad \sin\theta = \frac{v}{R} = \sqrt{1 - \frac{a_1^2}{4a_2}} \tag{17.9}$$

Therefore, the complementary function can be transformed as follows:

$$\begin{aligned} y_c &= A_1R^t(\cos\theta t + i\sin\theta t) + A_2R^t(\cos\theta t - i\sin\theta t) \\ &= R^t\left[(A_1 + A_2)\cos\theta t + (A_1 - A_2)i\sin\theta t\right] \\ &= R^t(A_5\cos\theta t + A_6\sin\theta t) \end{aligned} \tag{17.10}$$

where we have adopted the shorthand symbols

$$A_5 \equiv A_1 + A_2 \qquad \text{and} \qquad A_6 \equiv (A_1 - A_2)i$$

The complementary function (17.10) differs from its differential-equation counterpart (15.24′) in two important respects. First, the expressions $\cos\theta t$ and $\sin\theta t$ have replaced the previously used $\cos vt$ and $\sin vt$. Second, the multiplicative factor R^t (an exponential with base R) has replaced the natural exponential expression e^{ht}. In short, we have switched from the cartesian coordinates (h and v) of the complex roots to their polar coordinates (R and θ). The values of R and θ can be determined from (17.8) and (17.9) once h and v become known. It is also possible to calculate R and θ directly from the parameter values a_1 and a_2 via (17.8) and (17.9), provided we first make certain that $a_1^2 < 4a_2$ and that the roots are indeed complex.

Example 5 Find the general solution of $y_{t+2} + \frac{1}{4}y_t = 5$. With coefficients $a_1 = 0$ and $a_2 = \frac{1}{4}$, this constitutes an illustration of the complex-root case of $a_1^2 < 4a_2$. By (17.7), the real and imaginary parts of the roots are $h = 0$ and $v = \frac{1}{2}$. It follows from (17.8) that

$$R = \sqrt{0 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

Since the value of θ is that which can satisfy the two equations

$$\cos\theta = \frac{h}{R} = 0 \qquad \text{and} \qquad \sin\theta = \frac{v}{R} = 1$$

it may be concluded from Table 15.1 that

$$\theta = \frac{\pi}{2}$$

Consequently, the complementary function is

$$y_c = \left(\frac{1}{2}\right)^t \left(A_5 \cos\frac{\pi}{2}t + A_6 \sin\frac{\pi}{2}t\right)$$

To find y_p, let us try a constant solution $y_t = k$ in the complete equation. This yields $k = 4$; thus, $y_p = 4$, and the general solution can be written as

$$y_t = \left(\frac{1}{2}\right)^t \left(A_5 \cos\frac{\pi}{2}t + A_6 \sin\frac{\pi}{2}t\right) + 4 \tag{17.11}$$

Example 6 Find the general solution of $y_{t+2} - 4y_{t+1} + 16y_t = 0$. In the first place, the particular integral is easily found to be $y_p = 0$. This means that the general solution $y_t (= y_c + y_p)$ will be identical with y_c. To find the latter, we note that the coefficients $a_1 = -4$ and $a_2 = 16$ do produce complex roots. Thus we may substitute the a_1 and a_2 values directly into (17.8) and (17.9) to obtain

$$R = \sqrt{16} = 4$$

$$\cos\theta = \frac{4}{2 \cdot 4} = \frac{1}{2} \qquad \text{and} \qquad \sin\theta = \sqrt{1 - \frac{16}{4 \cdot 16}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

The last two equations enable us to find from Table 15.2 that

$$\theta = \frac{\pi}{3}$$

It follows that the complementary function—which also serves as the general solution here—is

$$y_c (= y_t) = 4^t \left(A_5 \cos\frac{\pi}{3}t + A_6 \sin\frac{\pi}{3}t\right) \tag{17.12}$$

The Convergence of the Time Path

As in the case of first-order difference equations, the convergence of the time path y_t hinges solely on whether y_c tends toward zero as $t \to \infty$. What we learned about the various configurations of the expression b^t, in Fig. 16.1, is therefore still applicable, although in the present context we shall have to consider *two* characteristic roots rather than one.

Consider first the case of distinct real roots: $b_1 \neq b_2$. If $|b_1| > 1$ and $|b_2| > 1$, then both component terms in the complementary function (17.5)—$A_1 b_1^t$ and $A_2 b_2^t$—will be explosive, and thus y_c must be divergent. In the opposite case of $|b_1| < 1$ and $|b_2| < 1$, both terms in y_c will converge toward zero as t is indefinitely increased, as will y_c also. What if $|b_1| > 1$ but $|b_2| < 1$? In this intermediate case, it is evident that the $A_2 b_2^t$ term tends to "die down," while the other term tends to deviate farther from zero. It follows that the $A_1 b_1^t$ term must eventually dominate the scene and render the path divergent.

Let us call the root with the higher *absolute* value the *dominant root*. Then it appears that it is the dominant root b_1 which really sets the tone of the time path, at least with regard to its ultimate convergence or divergence. Such is indeed the case. We may state, thus, that a *time path will be convergent—whatever the initial conditions may be—if and only if the dominant root is less than* 1 *in absolute value*. You can verify that this statement is valid for the cases where both roots are greater than or less than 1 in absolute value (discussed above), and where one root has an absolute value of 1 exactly (*not* discussed above). Note, however, that even though the eventual convergence depends on the dominant root alone, the *non*dominant root will exert a definite influence on the time path, too, at least in the beginning periods. Therefore, the exact configuration of y_t is still dependent on both roots.

Turning to the repeated-root case, we find the complementary function to consist of the terms $A_3 b^t$ and $A_4 t b^t$, as shown in (17.6). The former is already familiar to us, but a word of explanation is still needed for the latter, which involves a multiplicative t. If $|b| > 1$, the b^t term will be explosive, and the multiplicative t will simply serve to intensify the explosiveness as t increases. If $|b| < 1$, on the other hand, the b^t part (which tends to zero as t increases) and the t part will run counter to each other; i.e., the value of t will offset rather than reinforce b^t. Which force will prove the stronger? The answer is that the damping force of b^t will always win over the exploding force of t. For this reason, the basic requirement for convergence in the repeated-root case is still that the root be less than 1 in absolute value.

Example 7 Analyze the convergence of the solutions in Examples 3 and 4 above. For Example 3, the solution is

$$y_t = 3 + (-2)^t + 4t$$

where the roots are 1 and -2, respectively $[3(1)^t = 3]$, and where there is a moving equilibrium $4t$. The dominant root being -2, the time path is divergent.

For Example 4, where the solution is

$$y_t = A_3(-3)^t + A_4 t(-3)^t + \frac{1}{4}$$

and where $|b| = 3$, we also have divergence.

Let us now consider the complex-root case. From the general form of the complementary function in (17.10),

$$y_c = R^t(A_5 \cos\theta t + A_6 \sin\theta t)$$

it is clear that the parenthetical expression, like the one in (15.24′), will produce a fluctuating pattern of a periodic nature. However, since the variable t can only take integer values $0, 1, 2, \ldots$ in the present context, we shall catch and utilize only a subset of the points on the graph of a circular function. The y value at each such point will always prevail for a whole period, till the next relevant point is

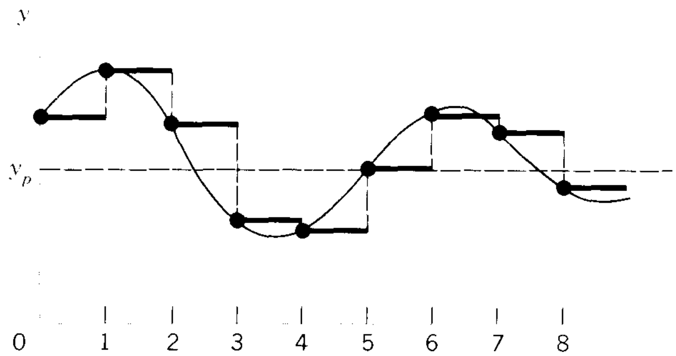

Figure 17.1

reached. As illustrated in Fig. 17.1, the resulting path is neither the usual oscillatory type (not alternating between values above and below y_p in consecutive periods), nor the usual fluctuating type (not smooth); rather, it displays a sort of *stepped* fluctuation. As far as convergence is concerned, though, the decisive factor is really the R^t term, which, like the e^{ht} term in (15.24′), will dictate whether the stepped fluctuation is to be intensified or mitigated as t increases. In the present case, the fluctuation can be gradually narrowed down if and only if $R < 1$. Since R is by definition the absolute value of the conjugate complex roots $(h \pm vi)$, the condition for convergence is again that the characteristic roots be less than unity in absolute value.

To summarize: For all three cases of characteristic roots, the time path will converge to a (stationary or moving) intertemporal equilibrium—regardless of what the initial conditions may happen to be—if and only if the absolute value of every root is less than 1.

Example 8 Are the time paths (17.11) and (17.12) convergent? In (17.11) we have $R = \frac{1}{2}$; therefore the time path will converge to the stationary equilibrium $(= 4)$. In (17.12), on the other hand, we have $R = 4$, so the time path will not converge to the equilibrium $(= 0)$.

EXERCISE 17.1

1 Write out the characteristic equation for each of the following, and find the characteristic roots:

(*a*) $y_{t+2} - y_{t+1} + \frac{1}{2}y_t = 2$ (*c*) $y_{t+2} + \frac{1}{2}y_{t+1} - \frac{1}{2}y_t = 5$

(*b*) $y_{t+2} - 4y_{t+1} + 4y_t = 7$ (*d*) $y_{t+2} - 2y_{t+1} + 3y_t = 4$

2 For each of the difference equations in the preceding problem, state on the basis of its characteristic roots whether the time path involves oscillation or stepped fluctuation, and whether it is explosive.

3 Find the particular integrals of the equations in Exercise 17.1-1 above. Do these represent stationary or moving equilibria?

4 Solve the following difference equations:

(*a*) $y_{t+2} + 3y_{t+1} - \frac{7}{4}y_t = 9 \qquad (y_0 = 6;\ y_1 = 3)$

(*b*) $y_{t+2} - 2y_{t+1} + 2y_t = 1 \qquad (y_0 = 3;\ y_1 = 4)$

(*c*) $y_{t+2} - y_{t+1} + \frac{1}{4}y_t = 2 \qquad (y_0 = 4;\ y_1 = 7)$

5 Analyze the time paths obtained in the preceding problem.

17.2 SAMUELSON MULTIPLIER-ACCELERATION INTERACTION MODEL

As an illustration of the use of second-order difference equations in economics, let us cite Professor Samuelson's classic *interaction* model, which seeks to explore the dynamic process of income determination when the acceleration principle is in operation along with the Keynesian multiplier.* Among other things, that model serves to demonstrate that the mere interaction of the multiplier and the accelerator is capable of generating cyclical fluctuations endogenously.

The Framework

Suppose that national income Y_t is made up of three component expenditure streams: consumption, C_t; investment, I_t; and government expenditure, G_t. Consumption is envisaged as a function not of current income but of the income of the prior period, Y_{t-1}; for simplicity, it is assumed that C_t is strictly proportional to Y_{t-1}. Investment, which is of the "induced" variety, is a function of the prevailing trend of consumer spending. It is through this induced investment, of course, that the acceleration principle enters into the model. Specifically, we shall assume I_t to bear a fixed ratio to the consumption increment $\Delta C_{t-1} \equiv C_t - C_{t-1}$. The third component, G_t, on the other hand, is taken to be exogenous; in fact, we shall assume it to be a constant and simply denote it by G_0.

These assumptions can be translated into the following set of equations:

$$\begin{aligned} &Y_t = C_t + I_t + G_0 \\ (17.13) \qquad &C_t = \gamma Y_{t-1} && (0 < \gamma < 1) \\ &I_t = \alpha(C_t - C_{t-1}) && (\alpha > 0) \end{aligned}$$

where γ (the Greek letter gamma) represents the marginal propensity to consume, and α stands for the accelerator (short for *acceleration coefficient*). Note that, if

* Paul A. Samuelson, "Interactions between the Multiplier Analysis and the Principle of Acceleration," *Review of Economic Statistics*, May, 1939, pp. 75–78; reprinted in American Economic Association, *Readings in Business Cycle Theory*, Richard D. Irwin, Inc., Homewood, Ill., 1944, pp. 261–269.

induced investment is expunged from the model, we are left with a first-order difference equation which embodies the dynamic multiplier process (cf. Example 2 of Sec. 16.2). With induced investment included, however, we have a second-order difference equation that depicts the interaction of the multiplier and the accelerator.

By virtue of the second equation, we can express I_t in terms of income as follows:

$$I_t = \alpha(\gamma Y_{t-1} - \gamma Y_{t-2}) = \alpha\gamma(Y_{t-1} - Y_{t-2})$$

Upon substituting this and the C_t equation into the first equation in (17.13) and rearranging, the model can be condensed into the single equation

$$Y_t - \gamma(1 + \alpha)Y_{t-1} + \alpha\gamma Y_{t-2} = G_0$$

or, equivalently (after shifting the subscripts forward by two periods),

$$Y_{t+2} - \gamma(1 + \alpha)Y_{t+1} + \alpha\gamma Y_t = G_0 \tag{17.14}$$

Because this is a second-order linear difference equation with constant coefficients and constant term, it can be solved by the method just learned.

The Solution

As the particular integral, we have, by (17.2),

$$Y_p = \frac{G_0}{1 - \gamma(1 + \alpha) + \alpha\gamma} = \frac{G_0}{1 - \gamma}$$

It may be noted that the expression $1/(1 - \gamma)$ is merely the multiplier that would prevail in the absence of induced investment. Thus $G_0/(1 - \gamma)$—the exogeneous expenditure item times the multiplier—should give us the equilibrium income in the sense that this income level satisfies the equilibrium condition "national income = total expenditure" [cf. (3.24)]. Being the particular integral of the model, however, it also gives us the intertemporal equilibrium income.

With regard to the complementary function, there are three possible cases. Case 1 ($a_1^2 > 4a_2$), in the present context, is characterized by

$$\gamma^2(1 + \alpha)^2 > 4\alpha\gamma \qquad \text{or} \qquad \gamma(1 + \alpha)^2 > 4\alpha$$

or

$$\gamma > \frac{4\alpha}{(1 + \alpha)^2}$$

Similarly, to characterize Cases 2 and 3, we only need to change the $>$ sign in the last inequality to $=$ and $<$, respectively. In Fig. 17.2, we have drawn the graph of the equation $\gamma = 4\alpha/(1 + \alpha)^2$. According to the above discussion, the (α, γ) pairs that are located exactly *on* this curve pertain to Case 2. On the other hand, the (α, γ) pairs lying *above* this curve (involving higher γ values) have to do with Case 1, and those lying *below* the curve with Case 3.

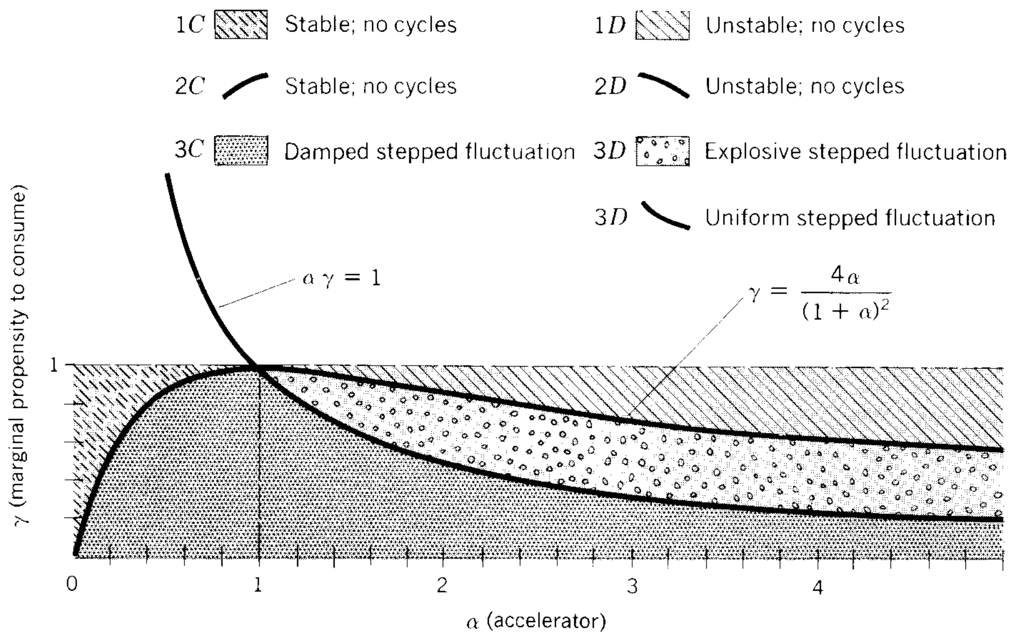

Figure 17.2

This tripartite classification, with its graphical representation in Fig. 17.2, is of interest because it reveals clearly the conditions under which cyclical fluctuations can emerge endogenously from the interaction of the multiplier and the accelerator. But this tells nothing about the convergence or divergence of the time path of Y. It remains, therefore, for us to distinguish, under each case, between the *damped* and the *explosive* subcases. We could, of course, take the easy way out by simply illustrating such subcases by citing specific numerical examples. But let us attempt the more rewarding, if also more arduous, task of delineating the general conditions under which convergence and divergence will prevail.

Convergence versus Divergence

The difference equation (17.14) has the characteristic equation

$$b^2 - \gamma(1 + \alpha)b + \alpha\gamma = 0$$

which yields the two roots

$$b_1, b_2 = \frac{\gamma(1 + \alpha) \pm \sqrt{\gamma^2(1 + \alpha)^2 - 4\alpha\gamma}}{2}$$

Since the question of convergence versus divergence depends on the values of b_1 and b_2, and since b_1 and b_2, in turn, depend on the values of the parameters α and γ, the conditions for convergence and divergence should be expressible in terms of the values of α and γ. To do this, we can make use of the fact that—by (15.6)

—the two characteristic roots are always related to each other by the following two equations:

$$(17.15)\qquad b_1 + b_2 = \gamma(1+\alpha)$$

$$(17.15')\qquad b_1 b_2 = \alpha\gamma$$

On the basis of these two equations, we may observe that

$$(17.16)\qquad (1-b_1)(1-b_2) = 1 - (b_1 + b_2) + b_1 b_2 = 1 - \gamma(1+\alpha) + \alpha\gamma = 1 - \gamma$$

In view of the model specification that $0 < \gamma < 1$, it becomes necessary to impose on the two roots the condition

$$(17.17)\qquad 0 < (1-b_1)(1-b_2) < 1$$

Let us now examine the question of convergence under Case 1, where the roots are real and distinct. Since, by assumption, α and γ are both positive, (17.15′) tells us that $b_1 b_2 > 0$, which implies that b_1 and b_2 possess the same algebraic sign. Furthermore, since $\gamma(1+\alpha) > 0$, (17.15) indicates that both b_1 and b_2 must be positive. Hence, the time path Y_t cannot have oscillations in Case 1.

Even though the signs of b_1 and b_2 are now known, there actually exist under Case 1 as many as five possible combinations of (b_1, b_2) values, each with its own implication regarding the corresponding values for α and γ:

(*i*)	$0 < b_2 < b_1 < 1$	$\Rightarrow$	$0 < \gamma < 1;\ \alpha\gamma < 1$
(*ii*)	$0 < b_2 < b_1 = 1$	$\Rightarrow$	$\gamma = 1$
(*iii*)	$0 < b_2 < 1 < b_1$	$\Rightarrow$	$\gamma > 1$
(*iv*)	$1 = b_2 < b_1$	$\Rightarrow$	$\gamma = 1$
(*v*)	$1 < b_2 < b_1$	$\Rightarrow$	$0 < \gamma < 1;\ \alpha\gamma > 1$

Possibility *i*, where both b_1 and b_2 are positive fractions, duly satisfies condition (17.17) and hence conforms to the model specification $0 < \gamma < 1$. The product of the two roots must also be a positive fraction under this possibility, and this, by (17.15′), implies that $\alpha\gamma < 1$. In contrast, the next three possibilities all violate condition (17.17) and result in inadmissible γ values (see Exercise 17.2-3). Hence they must be ruled out. But Possibility *v* is again acceptable. With both b_1 and b_2 greater than one, (17.17) is again satisfied, although this time we have $\alpha\gamma > 1$ (rather than < 1) from (17.15′). The upshot is that there are only two admissible subcases under Case 1. The first—Possibility *i*—involves fractional roots b_1 and b_2, and therefore yields a convergent time path of Y. The other subcase—Possibility *v*—features roots greater than one, and thus produces a divergent time path. As far as the values of α and γ are concerned, however, the question of convergence and divergence only hinges on whether $\alpha\gamma < 1$ or $\alpha\gamma > 1$. This information is summarized in the top part of Table 17.1, where the convergent subcase is labeled 1C, and the divergent subcase 1D.

The analysis of Case 2, with repeated roots, is similar in nature. The roots are now $b = \gamma(1+\alpha)/2$, with a positive sign because α and γ are positive. Thus, there is again no oscillation. This time we may classify the value of b into three possibilities only:

$$\begin{array}{llll} (vi) & 0 < b < 1 & \Rightarrow & \gamma < 1;\ \alpha\gamma < 1 \\ (vii) & b = 1 & \Rightarrow & \gamma = 1 \\ (viii) & b > 1 & \Rightarrow & \gamma < 1;\ \alpha\gamma > 1 \end{array}$$

Under Possibility *vi*, b $(= b_1 = b_2)$ is a positive fraction, thus the implications regarding α and γ are entirely identical with those of Possibility *i* under Case 1. In an analogous manner, Possibility *viii*, with b $(= b_1 = b_2)$ greater than one, yields the same results as Possibility *v*. On the other hand, Possibility *vii* violates (17.17) and must be ruled out. Thus there are again only two admissible subcases. The first—Possibility *vi*—yields a convergent time path, whereas the other—Possibility *viii*—gives a divergent one. In terms of α and γ, the convergent and divergent subcases are again associated, respectively, with $\alpha\gamma < 1$ and $\alpha\gamma > 1$. These results are listed in the middle part of Table 17.1, where the two subcases are labeled 2C (convergent) and 2D (divergent).

Finally, in Case 3, with complex roots, we have stepped fluctuation, and hence endogenous business cycles. In this case, we should look to the absolute value $R = \sqrt{a_2}$ [see (17.8)] for the clue to convergence and divergence, where a_2 is the coefficient of the y_t term in the difference equation (17.1). In the present

Table 17.1 Cases and subcases of the Samuelson model

Case	Subcase	Values of α and γ	Time path Y_t
1 Distinct real roots			
$\gamma > \dfrac{4\alpha}{(1+\alpha)^2}$	1C: $0 < b_2 < b_1 < 1$ 1D: $1 < b_2 < b_1$	$\alpha\gamma < 1$ $\alpha\gamma > 1$	Nonoscillatory and nonfluctuating
2 Repeated real roots			
$\gamma = \dfrac{4\alpha}{(1+\alpha)^2}$	2C: $0 < b < 1$ 2D: $b > 1$	$\alpha\gamma < 1$ $\alpha\gamma > 1$	Nonoscillatory and nonfluctuating
3 Complex roots			
$\gamma < \dfrac{4\alpha}{(1+\alpha)^2}$	3C: $R < 1$ 3D: $R \geq 1$	$\alpha\gamma < 1$ $\alpha\gamma \geq 1$	With stepped fluctuation

model, we have $R = \sqrt{\alpha\gamma}$, which gives rise to the following three possibilities:

$$(ix) \qquad R < 1 \quad \Rightarrow \quad \alpha\gamma < 1$$

$$(x) \qquad R = 1 \quad \Rightarrow \quad \alpha\gamma = 1$$

$$(xi) \qquad R > 1 \quad \Rightarrow \quad \alpha\gamma > 1$$

Even though all of these happen to be admissible (see Exercise 17.2-4), only the $R < 1$ possibility entails a convergent time path and qualifies as Subcase 3C in Table 17.1. The other two are thus collectively labeled as Subcase 3D.

In sum, we may conclude from Table 17.1 that a convergent time path can obtain if and only if $\alpha\gamma < 1$.

A Graphical Summary

The above analysis has resulted in a somewhat complex classification of cases and subcases. It would help to have a visual representation of the classificatory scheme. This is supplied in Fig. 17.2.

The set of all admissible (α, γ) pairs in the model is shown in Fig. 17.2 by the variously shaded rectangular area. Since the values of $\gamma = 0$ and $\gamma = 1$ are excluded, as is the value $\alpha = 0$, the shaded area is a sort of rectangle without sides. We have already graphed the equation $\gamma = 4\alpha/(1 + \alpha)^2$ to mark off the three major cases of Table 17.1: The points on that curve pertain to Case 2; the points lying to the north of the curve (representing higher γ values) belong to Case 1; those lying to the south (with lower γ values) are of Case 3. To distinguish between the convergent and divergent subcases, we now add the graph of $\alpha\gamma = 1$ (a rectangular hyperbola) as another demarcation line. The points lying to the north of this rectangular hyperbola satisfy the inequality $\alpha\gamma > 1$, whereas those located below it correspond to $\alpha\gamma < 1$. It is then possible to mark off the subcases easily. Under Case 1, the broken-line shaded region, being below the hyperbola, corresponds to Subcase 1C, but the solid-line shaded region is associated with Subcase 1D. Under Case 2, which relates to the points lying on the curve $\gamma = 4\alpha/(1 + \alpha)^2$, Subcase 2C covers the upward-sloping portion of that curve, and Subcase 2D, the downward-sloping portion. Finally, for Case 3, the rectangular hyperbola serves to separate the dot-shaded region (Subcase 3C) from the pebble-shaded region (Subcase 3D). The latter, you should note, also includes the points located *on* the rectangular hyperbola itself, because of the *weak inequality* in the specification $\alpha\gamma \geq 1$.

Since Fig. 17.2 is the repository of all the qualitative conclusions of the model, given any ordered pair (α, γ), we can always find the correct subcase graphically by plotting the ordered pair in the diagram.

Example 1 If the accelerator is 0.8 and the marginal propensity to consume is 0.7, what kind of interaction time path will result? The ordered pair (0.8, 0.7) is located in the dot-shaded region, Subcase 3C; thus the time path is characterized by damped stepped fluctuation.

Example 2 What kind of interaction is implied by $\alpha = 2$ and $\gamma = 0.5$? The ordered pair (2, 0.5) lies exactly on the rectangular hyperbola, under Subcase 3D. The time path of Y will again display stepped fluctuation, but it will be neither explosive nor damped. By analogy to the cases of uniform oscillation and uniform fluctuation, we may term this situation as "uniform stepped fluctuation." However, the uniformity feature in this latter case cannot in general be expected to be a perfect one, because, similarly to what was done in Fig. 17.1, we can only accept those points on a sine or cosine curve that correspond to integer values of t, but these values of t may hit an entirely different set of points on the curve in each period of fluctuation.

EXERCISE 17.2

1 By consulting Fig. 17.2, find the subcases to which the following sets of values of α and γ pertain, and describe the interaction time path qualitatively.

(*a*) $\alpha = 3.5$; $\gamma = 0.8$ (*c*) $\alpha = 0.2$; $\gamma = 0.9$

(*b*) $\alpha = 2$; $\gamma = 0.7$ (*d*) $\alpha = 1.5$; $\gamma = 0.6$

2 From the values of α and γ given in parts (*a*) and (*c*) of the preceding problem, find the numerical values of the characteristic roots in each instance, and analyze the nature of the time path. Do your results check with those obtained earlier?

3 Verify that Possibilities *ii*, *iii*, and *iv* in Case 1 imply inadmissible values of γ.

4 Show that in Case 3 we can never encounter $\gamma \geq 1$.

17.3 INFLATION AND UNEMPLOYMENT IN DISCRETE TIME

The interaction of inflation and unemployment, discussed earlier in the continuous-time framework, can also be couched in discrete time. Using essentially the same economic assumptions, we shall illustrate in this section how that model can be reformulated as a difference-equation model.

The Model

The earlier continuous-time formulation (Sec. 15.5) consisted of three differential equations:

$$p = \alpha - T - \beta U + h\pi \qquad (15.33)$$ [expectations-augmented Phillips relation]

$$\frac{d\pi}{dt} = j(p - \pi) \qquad (15.34)$$ [adaptive expectations]

$$\frac{dU}{dt} = -k(m - p) \qquad (15.35)$$ [monetary policy]

Three endogenous variables are present: p (actual rate of inflation), π (expected rate of inflation), and U (rate of unemployment). As many as six parameters appear in the model; among these, the parameter m—the rate of growth of nominal money (or, the rate of monetary expansion)—differs from the others in that its magnitude is set as a policy decision.

When cast into the period-analysis mold, the Phillips relation (15.33) simply becomes

$$(17.18) \qquad p_t = \alpha - T - \beta U_t + h\pi_t \qquad (\alpha, \beta > 0; 0 < h \leq 1)$$

In the adaptive-expectations equation, the derivative must be replaced by a difference expression:

$$(17.19) \qquad \pi_{t+1} - \pi_t = j(p_t - \pi_t) \qquad (0 < j \leq 1)$$

By the same token, the monetary-policy equation should be changed to*

$$(17.20) \qquad U_{t+1} - U_t = -k(m - p_{t+1}) \qquad (k > 0)$$

These three equations constitute the new version of the inflation-unemployment model.

The Difference Equation in p

As the first step in the analysis of this new model, we again try to condense the model into a single equation in a single variable. Let that variable be p. Accordingly, we shall focus our attention on (17.18). However, since (17.18)—unlike the other two equations—does not by itself describe a pattern of change, it is up to us to create such a pattern. This is accomplished by *differencing* p_t, i.e., by taking the first difference of p_t, according to the definition

$$\Delta p_t \equiv p_{t+1} - p_t$$

Two steps are involved in this. First, we shift the time subscripts in (17.18) forward one period, to get

$$(17.18') \qquad p_{t+1} = \alpha - T - \beta U_{t+1} + h\pi_{t+1}$$

Then we subtract (17.18) from (17.18′), to obtain the first difference of p_t that gives the desired pattern of change:

$$(17.21) \qquad \begin{aligned} p_{t+1} - p_t &= -\beta(U_{t+1} - U_t) + h(\pi_{t+1} - \pi_t) \\ &= \beta k(m - p_{t+1}) + hj(p_t - \pi_t) \qquad \text{[by (17.20) and (17.19)]} \end{aligned}$$

Note that, on the second line of (17.21), the patterns of change of the other two variables as given in (17.19) and (17.20) have been incorporated into the pattern of change of the p variable. Thus (17.21) now embodies all the information in the present model.

* We have assumed that the change in U_t depends on $(m - p_{t+1})$, the rate of growth of real money in period $(t + 1)$. As an alternative, it is possible to make it depend on the rate of growth of real money in period t, $(m - p_t)$ (see Exercise 17.3-4).

However, the π_t term is extraneous to the study of p and needs to be eliminated from the above equation. To that end, we make use of the fact that

$$h\pi_t = p_t - (\alpha - T) + \beta U_t \qquad \text{[by (17.18)]} \tag{17.22}$$

Substituting this into (17.21) and collecting terms, we obtain

$$(1 + \beta k)p_{t+1} - [1 - j(1 - h)]p_t + j\beta U_t = \beta km + j(\alpha - T) \tag{17.23}$$

But there now appears a U_t term to be eliminated. To do that, we difference (17.23) to get a $(U_{t+1} - U_t)$ term and then use (17.20) to eliminate the latter. Only after this rather lengthy process of substitutions do we get the desired difference equation in the p variable alone, which, when duly normalized, takes the form

$$p_{t+2} \underbrace{- \frac{1 + hj + (1 - j)(1 + \beta k)}{1 + \beta k}}_{a_1} p_{t+1} + \underbrace{\frac{1 - j(1 - h)}{1 + \beta k}}_{a_2} p_t = \underbrace{\frac{j\beta km}{1 + \beta k}}_{c} \tag{17.24}$$

The Time Path of p

The intertemporal equilibrium value of p, given by the particular integral of (17.24), is

$$\bar{p} = \frac{c}{1 + a_1 + a_2} = \frac{j\beta km}{\beta kj} = m \qquad \text{[by (17.2)]}$$

As in the continuous-time model, therefore, the equilibrium rate of inflation is exactly equal to the rate of monetary expansion.

As to the complementary function, there may arise either distinct real roots (Case 1), or repeated real roots (Case 2), or complex roots (Case 3), depending on the relative magnitudes of a_1^2 and $4a_2$. In the present model,

$$a_1^2 \gtreqless 4a_2 \quad \text{iff} \quad [1 + hj + (1 - j)(1 + \beta k)]^2 \gtreqless 4[1 - j(1 - h)](1 + \beta k) \tag{17.25}$$

If $h = \frac{1}{2}, j = \frac{1}{3}$ and $\beta k = 5$, for instance, then $a_1^2 = (5\frac{1}{6})^2$ whereas $4a_2 = 20$; thus Case 1 results. But if $h = j = 1$, then $a_1^2 = 4$ while $4a_2 = 4(1 + \beta k) > 4$, and we have Case 3 instead. In view of the larger number of parameters in the present model, however, it is not feasible to construct a classificatory graph like Fig. 17.2 in the Samuelson model.

Nevertheless, the analysis of convergence can still proceed along the same line as in the last section. Specifically, we recall from (15.6) that the two characteristic roots b_1 and b_2 must satisfy the following two relations

$$b_1 + b_2 = -a_1 = \frac{1 + hj}{1 + \beta k} + 1 - j > 0 \tag{17.26}$$

$$b_1 b_2 = a_2 = \frac{1 - j(1 - h)}{1 + \beta k} \in (0, 1) \tag{17.26'}$$

[see (17.24)]

Furthermore, we have in the present model

$$(17.27) \qquad (1 - b_1)(1 - b_2) = 1 - (b_1 + b_2) + b_1 b_2 = \frac{\beta jk}{1 + \beta k} > 0$$

Now consider Case 1, where the two roots b_1 and b_2 are real and distinct. Since their product $b_1 b_2$ is positive, b_1 and b_2 must take the same sign. Because their sum is positive, moreover, b_1 and b_2 must both be positive, implying that no oscillation can occur. From (17.27), we can infer that neither b_1 nor b_2 can be equal to one; for otherwise $(1 - b_1)(1 - b_2)$ would be zero, in violation of the indicated inequality. This means that, in terms of the various possibilities of (b_1, b_2) combinations enumerated in the Samuelson model, Possibilities *ii* and *iv* cannot arise here. It is also unacceptable to have one root greater, and the other root less, than one; for otherwise $(1 - b_1)(1 - b_2)$ would be negative. Thus Possibility *iii* is ruled out as well. It follows that b_1 and b_2 must be *either* both greater than one, *or* both less than one. If $b_1 > 1$ and $b_2 > 1$ (Possibility *v*), however, (17.26′) would be violated. Hence the only viable eventuality is Possibility *i*, with b_1 and b_2 both being positive fractions, so that the time path of p is convergent.

The analysis of Case 2 is basically not much different. By practically identical reasoning, we can conclude that the repeated root b can only turn out to be a positive fraction in this model; that is, Possibility *vi* is feasible, but not Possibilities *vii* and *viii*. The time path of p in Case 2 is again nonoscillatory and convergent.

For Case 3, convergence requires that R (the absolute value of the complex roots) be less than one. By (17.8), $R = \sqrt{a_2}$. Inasmuch as a_2 is a positive fraction [see (17.26′)], we do have $R < 1$. Thus the time path of p in Case 3 is also convergent, although this time there will be stepped fluctuation.

The Analysis of U

If we wish to analyze instead the time path of the rate of unemployment, we may take (17.20) as the point of departure. To get rid of the p term in that equation, we first substitute (17.18′) to get

$$(17.28) \qquad (1 + \beta k)U_{t+1} - U_t = k(\alpha - T - m) + kh\pi_{t+1}$$

Next, to prepare for the substitution of the other equation, (17.19), we difference (17.28) to find that

$$(17.29) \qquad (1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = kh(\pi_{t+2} - \pi_{t+1})$$

In view of the presence of a difference expression in π on the right, we can substitute for it a forward-shifted version of the adaptive-expectations equation. The result of this,

$$(17.30) \qquad (1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = khj(p_{t+1} - \pi_{t+1})$$

is the embodiment of all the information in the model.

However, we must eliminate the p and π variables before a proper difference equation in U will emerge. For this purpose, we note from (17.20) that

$$kp_{t+1} = U_{t+1} - U_t + km \tag{17.31}$$

Moreover, by multiplying (17.22) through by $(-kj)$ and shifting the time subscripts, we can write

$$\begin{aligned} -kjh\pi_{t+1} &= -kjp_{t+1} + kj(\alpha - T) - \beta kjU_{t+1} \\ &= -j(U_{t+1} - U_t + km) + kj(\alpha - T) - \beta kjU_{t+1} \qquad [\text{by } (17.31)] \\ &= -j(1 + \beta k)U_{t+1} + jU_t + kj(\alpha - T - m) \end{aligned} \tag{17.32}$$

These two results express p_{t+1} and π_{t+1} in terms of the U variable and can thus enable us, on substitution into (17.30), to obtain—at long last!—the desired difference equation in the U variable alone:

$$U_{t+2} - \frac{1 + hj + (1 - j)(1 + \beta k)}{1 + \beta k} U_{t+1} + \frac{1 - j(1 - h)}{1 + \beta k} U_t = \frac{kj[\alpha - T - (1 - h)m]}{1 + \beta k} \tag{17.33}$$

It is noteworthy that the two constant coefficients on the left (a_1 and a_2) are identical with those in the difference equation for p [i.e., (17.24)]. As a result, the earlier analysis of the complementary function of the p path should be equally applicable to the present context. But the constant term on the right of (17.33) does differ from that of (17.24). Consequently, the particular integrals in the two situations will be different. This is as it should be, for, coincidence aside, there is no inherent reason to expect the intertemporal equilibrium rate of unemployment to be the same as the equilibrium rate of inflation.

The Long-Run Phillips Relation

It is readily verified that the intertemporal equilibrium rate of unemployment is

$$\overline{U} = \frac{1}{\beta}[\alpha - T - (1 - h)m]$$

But since the equilibrium rate of inflation has been found to be $\bar{p} = m$, we can link $\overline{U}$ to $\bar{p}$ by the equation

$$\overline{U} = \frac{1}{\beta}[\alpha - T - (1 - h)\bar{p}] \tag{17.34}$$

Because this equation is concerned only with the *equilibrium* rates of unemployment and inflation, it is said to depict the *long-run* Phillips relation.

A special case of (17.34) has received a great deal of attention among economists: the case of $h = 1$. If $h = 1$, the $\bar{p}$ term will have a zero coefficient and thus drop out of the picture. In other words, $\overline{U}$ will become a constant function of

$\bar{p}$. In the standard Phillips diagram, where the rate of unemployment is plotted on the horizontal axis, this outcome gives rise to a vertical long-run Phillips curve. The $\bar{U}$ value in this case, referred to as the *natural rate of unemployment*, is then consistent with any equilibrium rate of inflation, with the notable policy implication that, in the long run, there is no trade-off between the twin evils of inflation and unemployment as exists in the short run.

But what if $h < 1$? In that event, the coefficient $\bar{p}$ in (17.34) will be negative. Then the long-run Phillips curve will turn out to be downward-sloping, thereby still providing a trade-off relation between inflation and unemployment. Whether the long-run Phillips curve is vertical or negatively sloped is, therefore, critically dependent on the value of the h parameter, which, according to the expectations-augmented Phillips relation, measures the extent to which the expected rate of inflation can work its way into the wage structure and the actual rate of inflation. All of this may sound familiar to you. This is because we discussed the topic in Example 1 in Sec. 15.5, and you have also worked on it in Exercise 15.5-4.

EXERCISE 17.3

1 Supply the intermediate steps leading from (17.23) to (17.24).

2 Show that if the model discussed in this section is condensed into a difference equation in the variable π, the result will be the same as (17.24) except for the substitution of π for p.

3 The time paths of p and U in the model discussed in this section have been found to be consistently convergent. Can divergent time paths arise if we drop the assumption that $h \leq 1$? If yes, which divergent "possibilities" in Cases 1, 2, and 3 will now become feasible?

4 Retain equations (17.18) and (17.19), but change (17.20) to

$$U_{t+1} - U_t = -k(m - p_t)$$

(*a*) Derive a new difference equation in the variable p.

(*b*) Does the new difference equation yield a different $\bar{p}$?

(*c*) Assume that $j = h = 1$. Find the conditions under which the characteristic roots will fall under Cases 1, 2, and 3, respectively.

(*d*) Let $j = h = 1$. Describe the time path of p (including convergence or divergence) when $\beta k = 3$, 4, and 5, respectively.

17.4 GENERALIZATIONS TO VARIABLE-TERM AND HIGHER-ORDER EQUATIONS

We are now ready to extend our methods in two directions, to the variable-term case, and to difference equations of higher orders.

Variable Term in the Form of cm^t

When the constant term c in (17.1) is replaced by a variable term—some function of t—the only effect will be on the particular integral. (Why?) To find the new particular integral, we can again apply the method of undetermined coefficients. In the differential-equation context (Sec. 15.6), that method requires that the variable term and its successive derivatives together take only a finite number of distinct types of expression, apart from multiplicative constants. Applied to difference equations, the requirement should be amended to read: "the variable term and its successive *differences* must together take only a finite number of distinct expression types, apart from multiplicative constants." Let us illustrate this method by concrete examples, first taking a variable term in the form cm^t, where c and m are constants.

Example 1 Find the particular integral of

$$y_{t+2} + y_{t+1} - 3y_t = 7^t$$

Here, we have $c = 1$ and $m = 7$. First, let us ascertain whether the variable term 7^t yields a finite number of expression types on successive differencing. According to the rule of differencing ($\Delta y_t = y_{t+1} - y_t$), the *first* difference of the term is

$$\Delta 7^t = 7^{t+1} - 7^t = (7-1)7^t = 6(7)^t$$

Similarly, the *second* difference, $\Delta^2(7^t)$, can be expressed as

$$\Delta(\Delta 7^t) = \Delta 6(7^t) = 6(7)^{t+1} - 6(7)^t = 6(7-1)7^t = 36(7)^t$$

Moreover, as can be verified, all successive differences will, like the first and second, be some multiple of 7^t. Since there is only a single expression type, we can try a solution $y_t = B(7)^t$ for the particular integral, where B is an undetermined coefficient.

Substituting the trial solution and its corresponding versions for periods $(t+1)$ and $(t+2)$ into the given difference equation, we obtain

$$B(7)^{t+2} + B(7)^{t+1} - 3B(7)^t = 7^t \qquad \text{or} \qquad B(7^2 + 7 - 3)(7)^t = 7^t$$

Thus,

$$B = \frac{1}{49 + 7 - 3} = \frac{1}{53}$$

and we can write the particular integral as

$$y_p = B(7)^t = \frac{1}{53}(7)^t$$

This, of course, represents a moving equilibrium. You can verify the correctness of the solution by substituting it into the difference equation and seeing to it that there will result an identity, $7^t = 7^t$.

The result reached in Example 1 can be easily generalized from the variable term 7^t to that of cm^t. From our experience, we expect all the successive differences of cm^t to take the same form of expression: namely, Bm^t, where B is some multiplicative constant. Hence we can try a solution $y_t = Bm^t$ for the particular integral, when given the difference equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = cm^t \tag{17.35}$$

Using the trial solution $y_t = Bm^t$, which implies $y_{t+1} = Bm^{t+1}$, etc., we can rewrite equation (17.35) as

$$Bm^{t+2} + a_1 Bm^{t+1} + a_2 Bm^t = cm^t$$

or

$$B(m^2 + a_1 m + a_2)m^t = cm^t$$

Hence the coefficient B in the trial solution should be

$$B = \frac{c}{m^2 + a_1 m + a_2}$$

and the desired particular integral of (17.35) can be written as

$$y_p = Bm^t = \frac{c}{m^2 + a_1 m + a_2} m^t \qquad (m^2 + a_1 m + a_2 \neq 0) \tag{17.36}$$

Note that the denominator of B is not allowed to be zero. If it happens to be,* we must then use the trial solution $y_t = Btm^t$ instead; or, if that too fails, $y_t = Bt^2 m^t$.

Variable Term in the Form of ct^n

Let us now consider variable terms in the form ct^n, where c is any constant, and n is a positive integer.

Example 2 Find the particular integral of

$$y_{t+2} + 5y_{t+1} + 2y_t = t^2$$

The first three differences of t^2 (a special case of ct^n with $c = 1$ and $n = 2$) are found as follows:†

$$\Delta t^2 = (t+1)^2 - t^2 = 2t + 1$$

$$\Delta^2 t^2 = \Delta(\Delta t^2) = \Delta(2t+1) = \Delta 2t + \Delta 1$$

$$= 2(t+1) - 2t + 0 = 2 \qquad [\Delta \text{ constant} = 0]$$

$$\Delta^3 t^2 = \Delta(\Delta^2 t^2) = \Delta 2 = 0$$

* Analogous to the situation in Example 3 of Sec. 15.6, this eventuality will materialize when the constant m happens to be equal to a characteristic root of the difference equation. The characteristic roots of the difference equation above are the values of b that satisfy the equation $b^2 + a_1 b + a_2 = 0$. If one root happens to have the value m, then it must follow that $m^2 + a_1 m + a_2 = 0$.

† These results should be compared with the first three derivatives of t^2:

$$\frac{d}{dt}t^2 = 2t \qquad \frac{d^2}{dt^2}t^2 = 2 \qquad \text{and} \qquad \frac{d^3}{dt^3}t^2 = 0$$

Since further differencing will only yield zero, there are altogether three distinct types of expression: t^2 (from the variable term itself), t, and a constant (from the successive differences).

Let us therefore try the solution

$$y_t = B_0 + B_1 t + B_2 t^2$$

for the particular integral, with undetermined coefficients B_0, B_1, and B_2. Note that this solution implies

$$\begin{aligned} y_{t+1} &= B_0 + B_1(t+1) + B_2(t+1)^2 \\ &= (B_0 + B_1 + B_2) + (B_1 + 2B_2)t + B_2 t^2 \\ y_{t+2} &= B_0 + B_1(t+2) + B_2(t+2)^2 \\ &= (B_0 + 2B_1 + 4B_2) + (B_1 + 4B_2)t + B_2 t^2 \end{aligned}$$

When these are substituted into the difference equation, we obtain

$$(8B_0 + 7B_1 + 9B_2) + (8B_1 + 14B_2)t + 8B_2 t^2 = t^2$$

Equating the two sides term by term, we see that the undetermined coefficients are required to satisfy the following simultaneous equations:

$$\begin{aligned} 8B_0 + 7B_1 + 9B_2 &= 0 \\ 8B_1 + 14B_2 &= 0 \\ 8B_2 &= 1 \end{aligned}$$

Thus, their values must be $B_0 = \frac{13}{256}$, $B_1 = -\frac{7}{32}$, and $B_2 = \frac{1}{8}$, giving us the particular integral

$$y_p = \frac{13}{256} - \frac{7}{32}t + \frac{1}{8}t^2$$

Our experience with the variable term t^2 should enable us to generalize the method to the case of ct^n. In the new trial solution, there should obviously be a term $B_n t^n$, to correspond to the given variable term. Furthermore, since successive differencing of the term yields the distinct expressions $t^{n-1}, t^{n-2}, \ldots, t$, and B_0 (constant), the new trial solution for the case of the variable term ct^n should be written as

$$y_t = B_0 + B_1 t + B_2 t^2 + \cdots + B_n t^n$$

But the rest of the procedure is entirely the same.

It must be added that such a trial solution may also fail to work. In that event, the trick—already employed on countless other occasions—is again to multiply the original trial solution by a sufficiently high power of t. That is, we can instead try $y_t = t(B_0 + B_1 t + B_2 t^2 + \cdots + B_n t^n)$, etc.

Higher-Order Linear Difference Equations

The *order* of a difference equation indicates the highest-order difference present in the equation; but it also indicates the maximum number of periods of time lag involved. An nth-order linear difference equation (with constant coefficients and constant term) may thus be written in general as

$$y_{t+n} + a_1 y_{t+n-1} + \cdots + a_{n-1} y_{t+1} + a_n y_t = c \tag{17.37}$$

The method of finding the particular integral of this does not differ in any substantive way. As a starter, we can still try $y_t = k$ (the case of stationary intertemporal equilibrium). Should this fail, we then try $y_t = kt$ or $y_t = kt^2$, etc., in that order.

In the search for the complementary function, however, we shall now be confronted with a characteristic equation which is an nth-degree polynomial equation:

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0 \tag{17.38}$$

There will now be n characteristic roots b_i $(i = 1, 2, \ldots, n)$, all of which should enter into the complementary function thus:

$$y_c = \sum_{i=1}^{n} A_i b_i^t \tag{17.39}$$

provided, of course, that the roots are all real and distinct. In case there are repeated real roots (say, $b_1 = b_2 = b_3$), then the first three terms in the sum in (17.39) must be modified to

$$A_1 b_1^t + A_2 t b_1^t + A_3 t^2 b_1^t \qquad \text{[cf. (17.6)]}$$

Moreover, if there is a pair of conjugate complex roots—say, b_{n-1}, b_n—then the last two terms in the sum in (17.39) are to be combined into the expression

$$R^t (A_{n-1} \cos \theta t + A_n \sin \theta t)$$

A similar expression can also be assigned to any other pair of complex roots. In case of two *repeated* pairs, however, one of the two must be given a multiplicative factor of tR^t instead of R^t.

After y_p and y_c are both found, the general solution of the complete difference equation (17.37) is again obtained by summing; that is,

$$y_t = y_p + y_c$$

But since there will be a total of n arbitrary constants in this solution, no less than n initial conditions will be required to definitize it.

Example 3 Find the general solution of the third-order difference equation

$$y_{t+3} - \frac{7}{8}y_{t+2} + \frac{1}{8}y_{t+1} + \frac{1}{32}y_t = 9$$

By trying the solution $y_t = k$, the particular integral is easily found to be $y_p = 32$. As for the complementary function, since the cubic characteristic equation

$$b^3 - \frac{7}{8}b^2 + \frac{1}{8}b + \frac{1}{32} = 0$$

can be factored into the form

$$\left(b - \frac{1}{2}\right)\left(b - \frac{1}{2}\right)\left(b + \frac{1}{8}\right) = 0$$

the roots are $b_1 = b_2 = \frac{1}{2}$ and $b_3 = -\frac{1}{8}$. This enables us to write

$$y_c = A_1\left(\frac{1}{2}\right)^t + A_2 t\left(\frac{1}{2}\right)^t + A_3\left(-\frac{1}{8}\right)^t$$

Note that the second term contains a multiplicative t; this is due to the presence of repeated roots. The general solution of the given difference equation is then simply the sum of y_c and y_p.

In this example, all three characteristic roots happen to be less than 1 in their absolute values. We can therefore conclude that the solution obtained represents a time path which converges to the stationary equilibrium level 32.

Convergence and the Schur Theorem

When we have a high-order difference equation that is not easily solved, we can nonetheless determine the convergence of the relevant time path qualitatively without having to struggle with its actual quantitative solution. You will recall that the time path can converge if and only if every root of the characteristic equation is less than 1 in absolute value. In view of this, the following theorem—known as the *Schur theorem**—becomes directly applicable:

The roots of the nth-degree polynomial equation

$$a_0 b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$$

will all be less than unity in absolute value if and only if the following n

* For a discussion of this theorem and its history, see John S. Chipman, *The Theory of Inter-Sectoral Money Flows and Income Formation*, The Johns Hopkins Press, Baltimore, 1951, pp. 119–120.

determinants

$$\Delta_1 = \left|\begin{array}{c|c} a_0 & a_n \\ \hline a_n & a_0 \end{array}\right| \qquad \Delta_2 = \left|\begin{array}{cc|cc} a_0 & 0 & a_n & a_{n-1} \\ a_1 & a_0 & 0 & a_n \\ \hline a_n & 0 & a_0 & a_1 \\ a_{n-1} & a_n & 0 & a_0 \end{array}\right| \qquad \cdots$$

$$\Delta_n = \left|\begin{array}{cccc|cccc} a_0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & 0 & 0 & a_n & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ \hline a_n & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots & 0 & 0 & a_0 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & a_0 \end{array}\right|$$

are all positive.

Note that, since the condition in the theorem is given on the "if and only if" basis, it is a necessary-and-sufficient condition. Thus the Schur theorem is a perfect difference-equation counterpart of the Routh theorem introduced earlier in the differential-equation framework.

The construction of these determinants is based on a simple procedure. This is best explained with the aid of the dashed lines which partition each determinant into four *areas*. Each area of the kth determinant, Δ_k, always consists of a $k \times k$ subdeterminant. The *upper-left* area has a_0 alone in the diagonal, zeros above the diagonal, and progressively larger subscripts for the successive coefficients in each column below the diagonal elements. When we transpose the elements of the upper-left area, we obtain the *lower-right* area. Turning to the *upper-right* area, we now place the a_n coefficient alone in the diagonal, with zeros below the diagonal, and progressively smaller subscripts for the successive coefficients as we go up each column from the diagonal. When the elements of this area are transposed, we get the *lower-left* area.

The application of this theorem is straightforward. Since the coefficients of the characteristic equation are the same as those appearing on the left side of the original difference equation, we can introduce them directly into the determinants cited. Note that, in our context, we always have $a_0 = 1$.

Example 4 Does the time path of the equation $y_{t+2} + 3y_{t+1} + 2y_t = 12$ converge? Here we have $n = 2$, and the coefficients are $a_0 = 1$, $a_1 = 3$, and $a_2 = 2$. Thus we get

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 < 0$$

Since this already violates the convergence condition, there is no need to proceed to Δ_2.

Actually, the characteristic roots of the given difference equation are easily found to be $b_1, b_2 = -1, -2$, which indeed imply a divergent time path.

Example 5 Test the convergence of the path of $y_{t+2} + \frac{1}{6}y_{t+1} - \frac{1}{6}y_t = 2$ by the Schur theorem. Here the coefficients are $a_0 = 1$, $a_1 = \frac{1}{6}$, $a_2 = -\frac{1}{6}$ (with $n = 2$). Thus we have

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & -\frac{1}{6} \\ -\frac{1}{6} & 1 \end{vmatrix} = \frac{35}{36} > 0$$

$$\Delta_2 = \begin{vmatrix} a_0 & 0 & a_2 & a_1 \\ a_1 & a_0 & 0 & a_2 \\ a_2 & 0 & a_0 & a_1 \\ a_1 & a_2 & 0 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 1 & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & 0 & 1 \end{vmatrix} = \frac{1176}{1296} > 0$$

These do satisfy the necessary-and-sufficient condition for convergence.

EXERCISE 17.4

1 Apply the definition of the "differencing" symbol Δ, to find:
(a) Δt (b) $\Delta^2 t$ (c) Δt^3
Compare the results of differencing with those of differentiation.

2 Find the particular integral of each of the following:
(a) $y_{t+2} + 2y_{t+1} + y_t = 3^t$
(b) $y_{t+2} - 5y_{t+1} - 6y_t = 2(6)^t$
(c) $3y_{t+2} + 9y_t = 3(4)^t$

3 Find the particular integrals of:
(a) $y_{t+2} - 2y_{t+1} + 5y_t = t$
(b) $y_{t+2} - 2y_{t+1} + 5y_t = 4 + 2t$
(c) $y_{t+2} + 5y_{t+1} + 2y_t = 18 + 6t + 8t^2$

4 Would you expect that, when the variable term takes the form $m^t + t^n$, the trial solution should be $B(m)^t + (B_0 + B_1 t + \cdots + B_n t^n)$? Why?

5 Find the characteristic roots and the complementary function of:
(a) $y_{t+3} - \frac{1}{2}y_{t+2} - y_{t+1} + \frac{1}{2}y_t = 0$
(b) $y_{t+3} - 2y_{t+2} + \frac{5}{4}y_{t+1} - \frac{1}{4}y_t = 1$
[*Hint:* Try factoring out $(b - \frac{1}{2})$ in both characteristic equations.]

6 Test the convergence of the solutions of the following difference equations by the Schur theorem:

(*a*) $y_{t+2} + \frac{1}{2}y_{t+1} - \frac{1}{2}y_t = 3$

(*b*) $y_{t+2} - \frac{1}{9}y_t = 1$

7 In the case of a third-order difference equation

$$y_{t+3} + a_1 y_{t+2} + a_2 y_{t+1} + a_3 y_t = c$$

what are the exact forms of the determinants required by the Schur theorem?