The Biot-Savart Law:

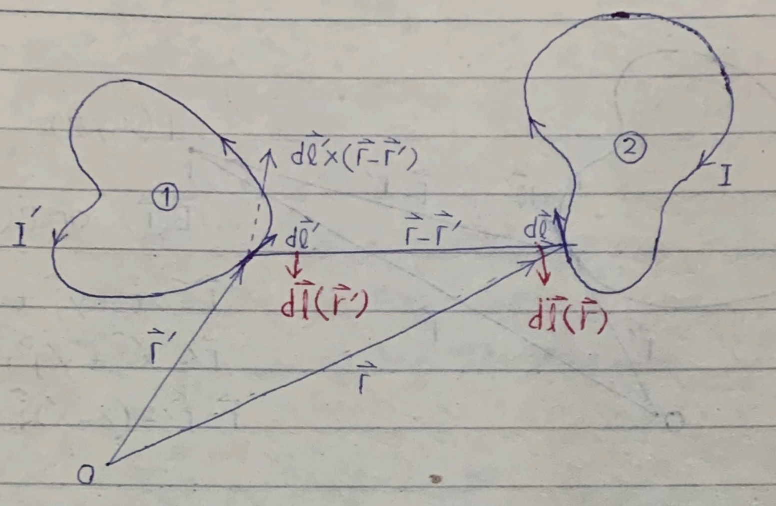
Stationary charges produce electric fields that are constant in time; hence the term electrostatics. Steady currents produce magnetic fields that are constant in time. That's why the theory of steady currents is called magnetostatics.

Stationary charges $\Rightarrow$ constant electric fields : electrostatics
Steady currents $\Rightarrow$ constant magnetic fields : magnetostatics

In 1820, Oersted first discovered that currents produce magnetic fields. Just after a few weeks of this discovery, Ampere presented the results of a series of experiments that may be generalized and expressed in modern mathematical language as

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I I' \oint \oint \frac{d\vec{l} \times \left[d\vec{l}' \times (\vec{r} - \vec{r}') \right]}{|\vec{r} - \vec{r}'|^3}$$

This expression can be understood with reference to the Figure.



The force $\vec{F}$ is the force exerted on circuit ② due to the influence of circuit ①. By definition,

$$\frac{\mu_0}{4\pi} = 10^{-7}\, N/A^2 \text{ (in mks units)}$$

It plays the same role as that of $\frac{1}{4\pi\epsilon_0}$ in electrostatics, and is a constant. 'μ_0' is called permeability of free space.

Now from our knowledge about current carrying conductors, we know that the force experienced by a current carrying element $Id\vec{l}$ of a closed circuit when placed in a magnetic field of magnetic induction $\vec{B}$ is given by

$$d\vec{F}(\vec{r}) = I d\vec{l} \times \vec{B}(\vec{r})$$

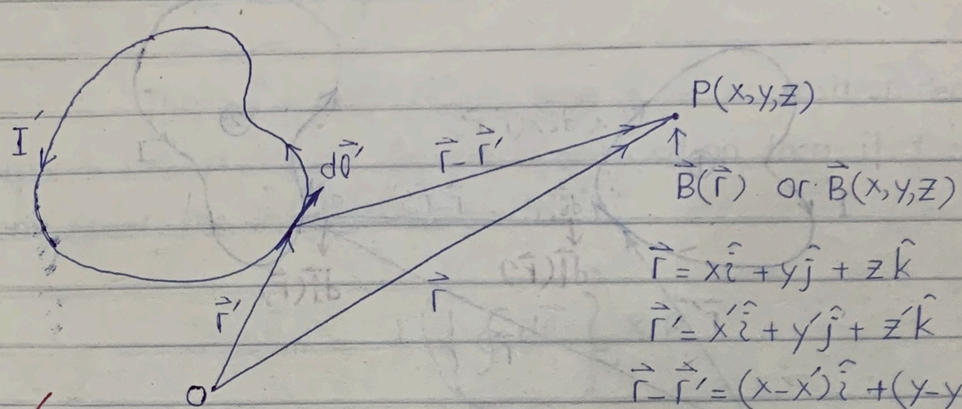
$$\Rightarrow \quad \underbrace{\vec{F} = \oint I d\vec{l} \times \vec{B}(\vec{r})}_{\text{Force experienced by whole closed loop}}$$

Compare it with

$$\vec{F} = \frac{\mu_0}{4\pi} \oint \oint' \frac{(I d\vec{l}) \times I' \left[d\vec{l}' \times (\vec{r}-\vec{r}') \right]}{|\vec{r}-\vec{r}'|^3}, \text{ it is clear that}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \oint' \frac{I' d\vec{l}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \vec{B}(P)$$

This is the general form of Biot-Savart Law, which gives us the value of magnetic induction $\vec{B}$ at any point 'P', a distance '$\vec{r}$' away from the origin and '$\vec{r}-\vec{r}'$' away from a conducting loop through which current 'I'' is flowing and $I'd\vec{l}'$ is its current carrying element.



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$

$\vec{r} - \vec{r}' = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$

? Check it! ✓

The direction of $\vec{B}(\vec{r})$ can be found by right hand rule, i.e, rotate $I'd\vec{l}'$ to coincide with $(\vec{r}-\vec{r}')$ through smallest angle, curl right hand fingers along the direction of rotation and the right hand thumb will point in the direction of $\vec{B}(\vec{r})$.

For surface currents, the Biot-Savart law becomes

$$I'd\vec{l}' \to \vec{K}(\vec{r}')\, da'$$

$$\therefore \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}(\vec{r}') \times (\vec{r}-\vec{r}')\, da'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\oint_C \frac{I'd\vec{l}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

and For the volume currents

$$I'd\vec{l}' \to \vec{J}(\vec{r}')\, dv' \text{ or } \vec{J}(\vec{r}')\, d\tau'$$

For open circuit.

$I' = I(\vec{r}')$

$d\vec{l}' = d\vec{l}(\vec{r}')$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')\, d\tau'}{|\vec{r}-\vec{r}'|^3}$$

Differential form of Equations are

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I'd\vec{l}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{K}(\vec{r}') \times (\vec{r}-\vec{r}')\, da'}{|\vec{r}-\vec{r}'|^3}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')\, d\tau'}{|\vec{r}-\vec{r}'|^3}$$

Proof : $\vec{\nabla}\cdot\vec{B} = 0$

As, $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau'$$

As, $\vec{r}-\vec{r}' = \vec{\imath}$

$$\therefore \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\vec{\imath}}{\imath^3}\,d\tau'$$

$$\Rightarrow \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{\hat{\imath}}{\imath^2}\,d\tau'$$

As, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$

$\vec{\imath} = \vec{r}-\vec{r}' = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$

So,

$\vec{B}$ is a function of (x, y, z),

$\vec{J}$ is a function of (x', y', z'),

$d\tau' = dx'dy'dz'$

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\cdot\int \vec{J}(\vec{r}')\times\frac{\hat{\imath}}{\imath^2}\,d\tau'$$

As, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\therefore \quad \vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left[\vec{J}(\vec{r}')\times\frac{\hat{\imath}}{\imath^2}\right]d\tau'$$

Using vector identity,

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\therefore \quad \vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\hat{\imath}}{\imath^2}\cdot\underbrace{\left(\vec{\nabla}\times\vec{J}(\vec{r}')\right)}_{=0}d\tau' - \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\cdot\underbrace{\left(\vec{\nabla}\times\frac{\hat{\imath}}{\imath^2}\right)}_{=0}d\tau'$$

Because $\vec{J}(\vec{r}')$ doesn't depend on the unprimed variables.

As, $$\frac{\hat{\imath}}{\imath^2} = -\vec{\nabla}\left(\frac{1}{\imath}\right)$$

$$\therefore \quad \vec{\nabla}\times\frac{\hat{\imath}}{\imath^2} = -\vec{\nabla}\times\vec{\nabla}\left(\frac{1}{\imath}\right) = 0$$ (Since, curl of a gradient is zero)

$$\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\therefore \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \equiv \text{Maxwell's second Equation}$$

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{B})\, d\tau = 0$$

Applying Divergence theorem,

$$\oint_S \vec{B} \cdot d\vec{a} = 0$$

i.e, the Magnetic flux through any closed surface is always equal to zero.

$\Rightarrow$ Magnetic monopoles donot exist.