Example 1.33. *The half-width of the gain profile of a He-Ne laser material is about 2×10^{-3} nm. What should be the maximum length of the cavity in order to have a single longitudinal mode oscillation ?*

Solution : The separation between successive lines is given by

$$\Delta\lambda = \frac{\lambda^2}{2\mu L}$$

In order to obtain a single mode of oscillation,

$$\Delta\lambda = \text{Half-width of the gain profile}$$

$$\therefore \quad L = \frac{\lambda^2}{2\mu\Delta\lambda} = \frac{\left(6328\times10^{-10}\right)^2 m^2}{2\times1\times2\times10^{-12} m} \quad (\mu = 1)$$

$$= \frac{4\times10^{-13}}{4\times10^{-12}} m = \mathbf{10cm}$$

Example 1.34. *The half-width of the gain profile of a He-Ni laser material is about 2×10^{-3} nm. If the length of the cavity is 30 cm, how many longitudinal modes can be excited ? The emission wavelength of He-Ne laser is 6328 Å.*

Solution : The distance of separation between successive spectral lines is given by

$$\Delta\lambda = \frac{\lambda^2}{2\mu L} = \frac{(6328\times10^{-10} m)^2}{2\times1\times30\times10^{-2}\ m} = \frac{4\times10^{-13}}{6\times10^{-1} m}$$

$$= 0.66 \times 10^{-3} \text{ nm.}$$

μ is taken as 1.

$$\text{Number of modes} = \frac{\text{Half-width of the gain profile}}{\Delta\lambda}$$

$$= \frac{2\times10^{-3}\ nm}{0.66\times10^{-3}\ nm} = \mathbf{3}$$

1.48 LASER RATE EQUATIONS

We learnt in the previous sections that the populations of energy levels of the lasing medium change under the action of radiation. These changes can be described conveniently by means of rate equations. Rate equations help us in determining the steady state population difference and the threshold pumping rate required to maintain a steady state population inversion. Such a study shows that population inversion cannot take place in a two level system while a minimum pump power is required to obtain inversion in a three level system. On the other hand the attainment of population inversion in a four level system is not dependent on the pump power. We also draw information regarding the optimum power that could be extracted from the laser.

1.48.1 Two Level System

Let us consider a two level system consisting of energy levels E_1 and E_2 which are populated by N_1 and N_2 atoms per unit volume respectively. The total number of atoms which participate in the lasing action is given by

$$N_0 = N_1 + N_2 \qquad ...(1.112)$$

Lasing begins when the population of the level E_2 exceeds that of E_1. The threshold population inversion density is given by

$$N_{th} = N_2 - N_1 \quad \text{...(1.113)}$$

Combining eqs. (1.112) and (1.113), we get

$$N_2 = \frac{N_0}{2} + \frac{N_{th}}{2}$$

That is,

$$N_2 > N_0/2$$

The above condition implies that lasing can begin in a 2-level laser only when more than half of the total population is pumped up to the upper energy level.

We will now examine whether the population inversion state can be reached in a two-level system. Let $\rho(\nu)$ be the energy density of the light of frequency ν incident on the system. The number of atoms per unit volume per unit time which are excited to the upper level are

$$N_{ab} = B_{12}\,\rho(\nu)g(\nu)\,N_1 = W_{12}\,N_1 \quad \text{...(1.114)}$$

where

$$W_{12} = B_{12}\,\rho(\nu)g(\nu)$$

The number of atoms per unit volume per unit time undergoing stimulated emissions from E_2 to E_1 are

$$N_{st} = W_{21}\,N_2$$

Since the stimulated emission probability is equal to the absorption probability,

$$W_{12} = W_{21} \quad \text{...(1.115)}$$

While some of the excited atoms at level E_2 undergo stimulated emissions, some of the others undergo spontaneous emission transitions, which consist of both radiative and non-radiative types. The number of atoms undergoing spontaneous transitions form E_2 to E_1 will be

$$N_{sp} = (A_{21} + S_{21})\,N_2 = T_{21}N_2 \quad \text{...(1.116)}$$

The rate of change of population of E_2 level is given by

$$\frac{dN_2}{dt} = W_{12}N_1 - W_{21}N_2 - T_{21}N_2$$

As

$$W_{12} = W_{21},$$

$$\frac{dN_2}{dt} = W_{12}(N_1 - N_2) - T_{21}N_2 \quad \text{...(1.117)}$$

Similarly, the rate of change of population at E_1 level is given by

$$\frac{dN_1}{dt} = -\,W_{12}\,(N_1 - N_2) + T_{21}N_2 \quad \text{...(1.118)}$$

In the steady state condition,

$$\frac{dN_2}{dt} = 0 \text{ and } \frac{dN_1}{dt} = 0$$

$$\therefore \quad W_{12}(N_1 - N_2) - T_{21}N_2 = 0$$

and

$$-\,W_{12}(N_1 - N_2) + T_{21}N_2 = 0$$

$$\therefore \quad W_{12}(N_1 - N_2) = T_{21}N_2$$

or

$$N_1 - N_2 = \frac{T_{21}}{W_{12}}N_2$$

$$\therefore \quad N_1 = N_2\left(1+\frac{T_{21}}{W_{12}}\right) = N_2\left(\frac{W_{12}+T_{21}}{W_{12}}\right)$$

or

$$\boxed{\frac{N_2}{N_1} = \frac{W_{12}}{W_{12}+T_{21}}} \quad ...(1.119)$$

$$N_2 < N_1 \text{ as } W_{12} + T_{21} > W_{12}$$

It implies that we can never attain a steady state population inversion by *optical pumping* in a two level system.

However, a pn-junction semiconductor laser is a two-level laser where pumping is done by direct electrical pumping.

1.48.2 Three Level Laser

Let us now consider a three level system consisting of energy levels E_1, E_2 and E_3 which are populated by N_1, N_2 and N_3 atoms per unit volume respectively. Let N_0 be the total number of active atoms per unit volume.

$$N_0 = N_1 + N_2 + N_3 \quad ...(1.120)$$

The rate of change of atomic density N_3 has the following components :

(*i*) The pump transition to E_3 which raises atoms from the ground level E_1 given by $W_p(N_1 - N_3)$,

(*ii*) The non-radiative spontaneous transition to the level E_2 given by $S_{32}N_3$, and

(*iii*) The spontaneous transitions to the level E_1 given by $A_{31}N_3$.

Therefore, the rate equation for N_3 can be written as

$$\frac{dN_3}{dt} = W_p(N_1 - N_3) - A_{31}N_3 - S_{32}N_3 \quad ...(1.121)$$

The rate of change of atomic density N_2 has the following components :

(*i*) The stimulated emissions to E_1 which produce laser light given by $W_{21}(N_2 - N_1)$,

(*ii*) The spontaneous emission to the level E_1 given by $A_{21}N_2$, and

(*iii*) The spontaneous transition from the level E_3 given by $S_{32}N_3$.

Therefore, the rate equation for N_2 can be written as

$$\frac{dN_2}{dt} = -W_{21}(N_2 - N_1) + S_{32}N_3 - A_{21}N_2 \quad ...(1.122)$$

The rate of change of atomic density N_1 has the following components:

(*i*) The pump transition transfers atoms to the level E_3 given by $W_p(N_1 - N_3)$.

(*ii*) The stimulated emission to the level E_1 given by $W_{21}(N_2 - N_1)$, and

(*iii*) The spontaneous transitions to the level E_1 given by $A_{21}N_2$.

The rate equation for N_1 can be written as

$$\frac{dN_1}{dt} = -W_p(N_1 - N_3) + W_{21}(N_2 - N_1) + A_{12}N_2 \quad ...(1.123)$$

Under steady state condition, we must have

$$\frac{dN_3}{dt} = 0, \frac{dN_2}{dt} = 0 \text{ and } \frac{dN_1}{dt} = 0$$

We write from Eq. (1.121), that

$$W_pN_1 = (W_p + A_{31} + S_{32})N_3$$

$$N_3 = \frac{W_p}{W_p + A_{31} + S_{32}} N_1 \quad ...(1.124)$$

As the probability of spontaneous transition from level 3 to level 2 is much higher than the probability of spontaneous transition from level 3 to level 1.

$$A_{31} << S_{32}.$$

$$\therefore \quad N_3 = \frac{W_p}{W_p + S_{32}} N_1 \quad ...(1.125)$$

From Eq. (1.122), we obtain

$$W_{21}N_1 + S_{32}N_3 = (W_{21} + A_{21})N_2$$

Using the value of N_3 from Eq. (1.125) into the above equation, we get

$$\left[W_{21} + \frac{W_p S_{32}}{W_p + S_{32}}\right] N_1 = (W_{21} + A_{21})\, N_2$$

$$\therefore \quad \frac{N_2}{N_1} = \frac{W_{21}(W_p + S_{32}) + W_p S_{32}}{(W_p + S_{32})(W_{21} + A_{21})}$$

$$\therefore \quad \frac{N_2 - N_1}{N_1} = \frac{W_p(S_{32} - A_{21}) - S_{32}A_{21}}{(W_p + S_{32})(W_{21} + A_{21})} \quad ...(1.126)$$

The minimum pumping rate required to achieve population inversion may be given by

$$W_{pt} = \frac{S_{32}A_{21}}{S_{32} - A_{21}} \quad ...(1.127)$$

As

$$S_{32} << \mathrm{A}_{21}$$

$$W_{pt} \approx A_{21} \quad ...(1.128)$$

Similarly

$$\frac{N_2 + N_1}{N_1} = \frac{2W_{21}(W_p + S_{32}) + W_p S_{32} + W_p A_{21} + S_{32}A_{21}}{(W_p + S_{32})(W_{21} + A_{21})} \quad ...(1.129)$$

Dividing Eq. (1.126) with Eq. (1.128), we get

$$\frac{N_2 - N_1}{N_2 + N_1} = \frac{W_p(S_{32} - A_{21}) - S_{32}A_{21}}{2W_{21}(W_p + S_{32}) + (W_p + A_{21})S_{32} + W_p A_{21}}$$

Below threshold for laser oscillation, W_{21} is very small and the above equation may be approximated to

$$\frac{N_2 - N_1}{N_2 + N_1} \approx \frac{W_p\, S_{32} - S_{32}A_{21}}{W_p\, S_{32} + S_{32}\, A_{21} + W_p\, A_{21}}$$

$$= \frac{(W_p - A_{21})\, S_{32}}{(W_p + A_{21} + W_p A_{21} / S_{32})\, S_{32}}$$

As $S_{32} >> A_{21}$, the term $W_p A_{21}/S_{32}$ may be neglected.

$$\therefore \quad \frac{N_2 - N_1}{N_2 + N_1} = \frac{W_p - A_{21}}{W_p + A_{21}} \quad ...(1.130)$$

or

$$W_p N_1 = A_{21} N_2 \quad ...(1.131)$$

The condition necessary for laser oscillation to occur is that $(N_2 - N_1)$ must be positive. It requires that $W_p > A_{21}$.

Let us now estimate the threshold pumping power required to start laser oscillations. The number of atoms pumped per unit volume per unit time from E_1 to E_3 is $W_p N_1$. If the pump frequency is denoted by ν_p, then the power required per unit volume will be

$$P = W_p N_1 h\nu_p \quad ...(1.132)$$

Threshold pump power can be written as

$$P_{th} = A_{21} N_1 h\nu_p \quad ...(1.133)$$

As there will be very few atoms in E_3, $N_3 \approx 0$ and

$$N_0 \approx N_1 + N_2$$

$$N_0 >> N_2 - N_1$$

We can therefore assume $N_1 \approx N_2 \approx N_0/2$ and write Eq. (1. 133) as

$$\boxed{P_{th} = \frac{N_0 h\nu_p}{2\tau_{sp}}} \quad ...(1.134)$$

1.48.3 Four Level Laser

Let us consider the ideal four level laser shown in Fig. 1.23. Level E_1 is the ground level, E_4 the pumping level and E_3 and E_2 are the upper and lower lasing levels. Atoms are pumped from the ground level E_1 to the pumping level E_4. They make quick non-radiative transition to the metastable level E_3. We assume that $E_2 >> kT$ so that the level is not populated at ordinary temperatures by thermal process. It is further necessary that level E_2 has very small lifetime such that atoms are not accumulated in E_2 and spoil the population inversion condition between levels E_3 and E_2. Thus, the lifetime t_2 of atoms in level E_2 is short compared to the lifetime τ_3 of atoms in energy level E_3. We are interested only in the rate of change of the atomic density N_3 in level E_3 and that of N_2 in E_2.

Let R_2 denote the rate at which atoms are pumped to level E_4 and from there the atoms make a quick nonradiative transition to the level E_3. In effect, R_2 represents the rate at which atoms are arriving in E_3. Let R_1 denote the rate at which atoms are pumped into level E_2. Process R_1 is detrimental to laser action as it tends to reduce the population inversion condition between the levels E_3 and E_2. In gas lasers, pumping into the lower laser level E_2 is unavoidable and consequently, we take R_1 into consideration. The decrease in the number of atoms in level E_3 has the following components:

(i) The stimulated transition to E_2 which produces the lasing light given by $W_{32}(N_3 - N_2)$,

(ii) The spontaneous emission to the level E_2 given by $N_3 A_{32}$.

(iii) Process R_2 populates the level.

The rate equation for N_3 may be written as

$$\frac{dN_3}{dt} = R_2 - W_{32}(N_3 - N_2) - N_3 A_{32} \quad ...(1.135)$$

The change in the number of atoms in level E_2 has the following components :

(i) The stimulated emissions from level E_3 given by $W_{32}(N_3 - N_2)$;

(ii) The spontaneous emission from level E_3 given by $N_3 A_{32}$

(iii) The process R_1 which pumps atoms from level E_1 ;

(iv) The spontaneous emission to level E_1 given by $N_2 A_{21}$.

The rate equation for N_2 can be written as

$$\frac{dN_2}{dt} = R_1 + W_{32}(N_3 - N_2) + N_3A_{32} - N_3A_{21} \quad \text{...(1.136)}$$

In the steady state condition, $\frac{dN_3}{dt} = 0$ and $\frac{dN_2}{dt} = 0$

Thus from equations (1.135) and (1.136) we obtain

$$R_2 - W_{32}(N_3 - N_2) - N_3A_{32} = 0 \quad \text{...(1.137)}$$

$$R_1 + W_{32}(N_3 - N_2) + N_3A_{32} - N_2A_{21} = 0 \quad \text{...(1.138)}$$

Adding Eq. (1.137) to equation (1.138) we get

$$R_2 + R_1 = N_2 A_{21}$$

Since $R_2 >> R_1$, we write

$$R_2 = N_2 A_{21}$$

$$\therefore \quad N_2 = \frac{R_2}{A_{21}} \quad \text{...(1.139)}$$

Using Eq. (1.139) into Eq. (1.137) we obtain

$$R_2 = (W_{32} + A_{32})N_3 - W_{32}R_2/A_{21}$$

or

$$N_3 = R_2\left[1 + \frac{W_{32}}{A_{21}}\right]\left[\frac{1}{W_{32} + A_{32}}\right] \quad \text{...(1.140)}$$

$$\therefore \quad N_3 - N_2 = R_2\left[\frac{(A_{21} + W_{32})}{A_{21}(W_{32} + A_{32})} - \frac{1}{A_{21}}\right]$$

$$\therefore \quad N_3 - N_2 = R_2\left[\frac{1 - A_{32}/A_{21}}{(W_{32} + A_{32})}\right] \quad \text{...(1.141)}$$

From the above equation it is evident that

$$N_3 - N_2 > 0 \text{ if } A_{21} >> A_{32}$$

As

$$A_{21} = \frac{1}{\tau_{21}} \text{ and } A_{32} = \frac{1}{\tau_{32}},$$

the necessary condition for population inversion is thus

$$\tau_{21} < \tau_{32}$$

The above condition implies that atoms dropping to E_2 level by spontaneous emission must be removed at a faster rate than the arrival rate. If this condition is not satisfied atoms accumulate at E_2 and however hard we pump, population inversion cannot be attained between E_3 and E_2 levels. Hence lasing action does not occur.

Below the threshold, the stimulated transition rate

$$W_{32} = 0$$

$$\therefore \quad N_3 - N_2 = R_2\left[\frac{1 - A_{32}/A_{21}}{A_{32}}\right]$$

This condition continues upto the threshold level. Therefore, the threshold value is given by

$$N_{th} = (N_3 - N_2)_{th} = R_{th}\left[\frac{1 - A_{32}/A_{21}}{A_{32}}\right]. \quad \text{...(1.142)}$$

Since $A_{32} << A_{21}$, $1 - A_{32}/A_{21} \approx 1$,

$\therefore$

$$R_{th} = N_{th} A_{32}$$

or

$$R_{th} = \frac{N_{th}}{\tau_{32}} \quad ...(1.143)$$

Using Eq. (1.143) into Eq. (1.90), we get

$$N_{th} = \frac{8\pi \nu_0^2 \, \tau_{32} \gamma_{th} \, \Delta\nu}{\upsilon^2} \quad ...(1.144)$$

This is the stage at which the gain at ν_0 due to the population inversion is large enough to balance the cavity losses. Under steady state condition, $(N_3 - N_2)$ remains then on equal to N_{th} even if the rate of pumping is made greater than the pumping threshold. If an increase in $(N_3 - N_2)$ would occur, it would lead to an increase in stimulated emissions thereby an increase of stored energy with time in the cavity. This obviously violates the steady state assumption. Hence $(N_3 - N_2)$ remains equal to N_{th}.

Each atom raised into level E_3 absorbs an amount of energy E_4 so that the total pumping power per unit volume required at threshold is

$$P_{th} = \frac{N_{th}}{\tau_{32}} E_4 \quad ...(1.145)$$

Using Eq. (1.144) into the above, we get

$$\boxed{P_{th} = \frac{8\pi \nu_0^2 \, \gamma_{th} \, \Delta\nu}{\upsilon^2} E_4} \quad ...(1.146)$$

Comparison of three level and Four Level Lasers

In the case of a three-level laser, we can write

$$N_{th} = (N_2 - N_1)$$

and

$$N_0 = N_2 + N_1$$

For the laser to begin lasing $N_2 > \frac{N_0}{2} + \frac{N_{th}}{2}$

As $N_0 >> N_{th}$, $N_2 > N_0/2$ ....(1.147)

On the other hand in a four-level laser,

$$N_2 = N_1 \, e^{-(E_2 - E_1)/kT}$$

Assuming that $(E_2 - E_1)/kT >> 1$, $N_2 \approx 0$

For the laser to begin lasing $(N_3 - N_2) > N_{th}$ *i.e.* $N_3 > N_{th}$

$$\therefore \frac{(N_{th})_{3-level}}{(N_{th})_{4-level}} = \frac{N_0}{2N_{th}} = a \text{ very large quantity}$$

It is obvious that it is much easier to pump a four-level laser than a three-level laser. This is the reason why most of the lasers are of four-level.

1.49 OPTIMUM OUTPUT POWER

The population inversion $(N_3 - N_2)$ is given by Eq. (1.141) as

$$N_3 - N_2 = \frac{R_2 (1 - A_{32} / A_{21})}{W_{32} + A_{32}} \quad ...(1.148)$$

or
$$N_3 - N_2 = \frac{R_1}{W_{32} + A_{32}} \qquad ...(1.149)$$

where
$$R = R_2(1 - A_{32}/A_{21}) \qquad ...(1.150)$$

We rewrite Eq. (1.149) as

$$N_3 - N_2 = \frac{R / A_{32}}{1 + W_{32} / A_{32}} \qquad ...(1.151)$$

Since the gain coefficient γ (ν) is proportional to ($N_3 - N_2$), we use Eq. (1.151) to write

$$\gamma = \frac{\gamma_0}{1 + W_{32} / A_{32}} \qquad ...(1.152)$$

where $\gamma_0 = R/A_{32}$ is the gain coefficient in the absence of feedback.

The power emitted by the laser is given by

$$P_e = N_{th} V W_{32}\, h\nu \qquad ...(1.153)$$

where V is the active volume of the lasing material.

The amount of spontaneous light generated by the lasing material when it is just at the threshold, but not lasing, is given by

$$P_s = N_{th} V A_{32} h\nu = \frac{8\,\pi\mu_0^3\, h\Delta\nu V}{\lambda^3 t_c} \frac{\tau_{sp}}{\tau_{32}} \qquad ...(1.154)$$

P_s is called *critical fluorescence.*

Using eqs.(1.153) and (1.154) into eq.(1.152), we get

$$\gamma = \frac{\gamma_0}{1 + P_e / P_s} \qquad ...(1.155)$$

The expression (1.154) gives us the total power generated within the cavity by the atoms due to stimulated emission. However, only a fraction of the total emitted power P_0 is coupled out of the cavity as useful output laser beam through the output mirror. As far as the oscillation condition is concerned, the output power is a loss to the cavity. We would like to extract more power from the cavity, and it could be done by increasing the transmission coefficient of the output mirror.

If the transmission coefficient of the output mirror is increased, the light output increases but it means an increase in the cavity losses. Further, increasing transmission reduces mirror reflectivity. If the mirro reflectivity is smaller, the cavity losses exceed the gain and the laser ceases oscillating. On other hand, if the output mirror reflectivity is increased to say 100%, the laser oscillates but the output power will become zero. It means that for a given pumping rate, there exists an optimum output coupling which yields the maximum output power.

We recall that the oscillation condition is given by

$$r_1 r_2\, e^{[\gamma_t - \alpha]L} = 1 \qquad ...(1.156)$$

We rewrite the above equation as

$$e^{\gamma_t L}\,(1 - l_c) = 1 \qquad ...(1.157)$$

where $l_c = 1 - r_1 r_2 e^{-\alpha L}$ is the fractional loss per pass.

The term l_c consists of two types of losses : one due to the useful power output and the other due to the inherent losses. Thus,

$$l_c = T_0 + l_i \qquad ...(1.158)$$

$$P_0 = P_e \frac{T_0}{T_0 + \ell_i} \qquad ...(1.159)$$