2. In spontaneous emission, there is no phase relation between the photons emitted while in stimulated emission, photons have same frequency and are *in phase* within the incident photons.
3. In spontaneous emission *incoherent* radiation is achieved while stimulated emission the radiations are unidirectional and coherent.
4. In spontaneous emission, the probable rate of transition from excited level E_2 to lower level E_1 is proportional to the number of excited electrons remaining in E_2 level, while in case of stimulated emission, the rate of emission is proportional to the number of atoms in the excited state and the energy density of the incident radiation.

1.22 EINSTEIN RELATIONS

Under thermal equilibrium, the mean population N_1 and N_2 in the lower and upper energy levels respectively must remain constant. This condition requires that the number of transitions from E_2 to E_1 must be equal to the number of transitions from E_1 to E_2. Thus,

$$\left.\begin{matrix}\text{The number of atoms absorbing}\\ \text{photons per second per unit volume}\end{matrix}\right\} = \left\{\begin{matrix}\text{The number of atoms emitting photons}\\ \text{per second per unit volume}\end{matrix}\right.$$

$$\left.\begin{matrix}\text{The number of atoms absorbing}\\ \text{photons per second per unit volume}\end{matrix}\right\} = B_{12}\,\rho(\nu)\,N_1$$

$$\left.\begin{matrix}\text{The number of atoms emitting}\\ \text{photons per second per unit volume}\end{matrix}\right\} = A_{21}\,N_2 + B_{21}\,\rho(\nu)\,N_2$$

In equilibrium condition, the number of transitions from E_2 to E_1 must be equal to the number of transitions from E_1 to E_2. Thus,

$$B_{12}\,\rho(\nu)\,N_1 = A_{21}N_2 + B_{21}\,\rho(\nu)\,N_2 \quad \text{...(1.58)}$$

$$\rho(\nu)\,(B_{12}\,N_1 - B_{21}N_2) = A_{21}N_2$$

$$\therefore \quad \rho(\nu) = A_{21}\,N_2/B_{12}N_1 - B_{21}\,N_2$$

On dividing both the numerator and denominator on the right hand side of the above equation with $B_{12}\,N_2$, we get

$$\boxed{\rho(\nu) = \frac{A_{21}/B_{12}}{N_1/N_2 - B_{21}/B_{12}}} \quad \text{...(1.59)}$$

It follows from eq. (1.44) that

$$N_1/N_2 = e^{(E_2-E_1)/kT}$$

As $$(E_2 - E_1) = h\nu$$

$$N_1/N_2 = e^{h\nu/hT}$$

$$\therefore \quad \rho(\nu) = \frac{A_{21}}{B_{12}}\left[\frac{1}{e^{h\nu/kT} - B_{21}/B_{12}}\right] \quad \text{...(1.60)}$$

To maintain thermal equilibrium, the system must release energy in the form of electromagnetic radiation. It is required that the radiation be identical with black body radiation and be consistent with Planck's radiation law for any value of T. According to Planck's law

$$\rho(\nu) = (8\pi\,h\nu^3\,\mu^3/c^3)\,\frac{1}{e^{h\nu/kT} - 1} \quad \text{...(1.61)}$$

where μ is the refractive index of the medium and c is the velocity of light in free space.

Energy density ρ(ν) given by Eq. (1.60) will be consistent with Planck's law (Eq. 1.61) only if

$$\boxed{B_{21} = B_{12}} \qquad ...(1.62)$$

and

$$A_{21}/B_{12} = (8\pi h\nu^3 \mu^3/c^3) \qquad ...(1.63)$$

Therefore,

$$\boxed{B_{12} = B_{21} = \frac{c^3}{8\pi h\nu^3\mu^3} A_{21}} \qquad ...(1.64)$$

Equations (1.62) and (1.63) are known as the *Einstein relations.* Equation (1.64) gives the relationship between the A and B coefficients.

The first relation (1.62) shows that the coefficients for both absorption and stimulated emission are numerically equal. The equality implies the following. When an atom with two energy levels is placed in the radiation field, the probability for an upward (absorption) transition is equal to the probability for a downward (stimulated emission) transition.

The second relation (1.63) shows that the ratio of coefficients of spontaneous versus stimulated emission is proportional to the third power of frequency of the radiation. This is why it is difficult to achieve laser action in higher frequency ranges such as x-rays.

Example 1.29. *The wavelength of emission is 6000 Å and the lifetime* τ_{sp} *is* 10^{-6} *s. Determinie the coefficient for the stimulated emission.*

Solution : The coefficient for stimulated emission is given by

$$B_{21} = \frac{c^3 A_{21}}{8\pi \mu^3 h\nu^3} \qquad \text{Eq. (1.64)}$$

But $A_{21} = \frac{1}{\tau_{sp}}$ and $\frac{c^3}{\nu^3} = \lambda^3$. Taking $\mu = 1$, we get

$$B_{21} = \frac{\lambda^3}{8\pi h\tau_{sp}} = \frac{(6000\times10^{-10})^3 \text{ m}^3}{8\times\pi\times6.626\times10^{-34}\text{ Js}\times10^{-6}\text{s}}$$

$$= \frac{216\times10^{-21}}{166.6\times10^{-40}} \cdot \frac{\text{m}^3}{\text{J.s}^2} = \mathbf{1.3\times10^{19}\ m/kg.}$$

1.23 CONDITIONS FOR LARGE STIMULATED EMISSIONS

The key to laser action is the existence of stimulated emission. In practice, the absorption and spontaneous emissions always occur together with stimulated emission. Let us now study the conditions under which the number of stimulated emissions can be made larger than those of the other two processes.

(a) From equations (1.51) and (1.56), we can write for the ratio of the stimulated transitions to spontaneous transitions as

$$R_1 = \frac{\text{Stimulated transitions}}{\text{Spontaneous transitions}}$$

$$= B_{21}\,\rho(\nu)\,N_2/A_{21}\,N_2$$

$$= (B_{21}/A_{21})\,\rho(\nu) \qquad ...(1.65)$$

Using eq.(1.61) for ρ(ν), we get

$$R_1 = (B_{21}/A_{21})\left[\frac{8\pi h\nu^3\mu^3}{c^3}\,\frac{1}{e^{h\nu/kT}-1}\right] \qquad ...(1.66)$$

From eq.(1.62, 1.63) and (1.64), we can write

$$\frac{B_{21}}{A_{21}} = \frac{B_{12}}{A_{21}} = \frac{c^3}{8\pi h\nu^3\mu^3} \quad ...(1.67)$$

Using the eq.(1.67) into eq(1.66), we obtain

$$R_1 = (c^3/8\,\pi\, h\nu^3\, \mu^3)\left[(8\pi\, h\nu^3\mu^3/c^3)\frac{1}{e^{h\nu/kT}-1}\right]$$

or

$$\boxed{R_1 = \frac{1}{(e^{h\nu/kT}-1)}} \quad ...(1.68)$$

If we assume $\nu = 5\times10^{14}$ Hz (yellow light) and $T = 2000$ K, the value of $h\nu/kT$ is 11.99.

$$\therefore \quad R_1 = \frac{1}{e^{11.99}-1} = 6\times10^{-6}$$

The above result shows that in the optical region spontaneous emissions dominate over the stimulated emissions.

The equation (1.65) suggests that the light field density $\rho(\nu)$ present within the material is required to be enhanced if we want large number of stimulated emissions.

(*b*) The ratio of stimulated transitions to absorption transitions is given by

$$R_2 = \frac{\text{Stimulated transitions}}{\text{Absorption transitions}}$$

$$= \frac{B_{21}\,\rho(\nu)\,N_2}{B_{12}\,\rho(\nu)\,N_1} \quad ...(1.69)$$

As $$B_{21} = B_{12}$$

$$\boxed{R_2 = N_2/N_1} \quad ...(1.70)$$

At thermodynamic equilibrium,

$$N_2/N_1 << 1$$

Therefore at equilibrium, absorption transitions overwhelm stimulated transitions. A photon of the light field may hit an excited atom leading to stimulated emission, or be absorbed on hitting an atom in the ground state. As $N_2 << N_1$ at thermodynamic equilibrium, a photon has a much higher probability of being absorbed than of stimulating an excited atom. As a result, the absorption process dominates stimulated emission and the medium will absorb the incident light. If, on the other hand, more atoms are in the excited state, *i.e.* $N_2 > N_1$, photons are more likely to cause stimulated emission than absorption. Therefore, in order to achieve more stimulated emissions, the population N_2 of the excited state should be made larger than the population N_1 of the lower energy state.

To sum up, three conditions are to be satisfied to make stimulated transitions overwhelm the other transitions: (*i*) the population at excited level should be greater than that that at the lower energy level, (*ii*) the ratio B_{21}/A_{21} should be large and (*iii*) a very high density of radiation density should be present in the medium. A medium amplifies light only when these three conditions are fulfilled.

1.24 SPONTANEOUS AND STIMULATION EMISSION IN OPTICAL REGION

Rate of spontaneous emission,

$$R_{sp} = A_{21}\,N_2 \qquad (\because \text{Eq. 1.51}) \qquad ...(i)$$

and rate of stimulated emission

$$R_{st} = B_{21}\,\rho(\nu)\,N_2 \qquad (\because \text{Eq. 1.56}) \qquad ...(ii)$$

Dividing Eq. (*ii*) by Eq. (*i*), we get

$$R_1 = \frac{\text{Stimulated transitions}}{\text{Spontaneous transitions}} = \frac{B_{21}\,\rho(\nu)\,N_2}{A_{21}\,N_2}$$

$$= \frac{B_{21}}{A_{21}}\,\rho(\nu) \qquad ...(iii)$$

Applying Planck's radiation law,

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3\,(e^{h\nu/kT} - 1)} \qquad ...(iv)$$

From Eq. (*iii*), we conclude that stimulated transitions will dominate spontaneous transitions, if the radiation density $\rho(\nu)$ is very large and B_{21}/A_{21} is also very large. Using Eq. (*iv*) in Eq. (*iii*), we get

$$R_1 = \left(\frac{B_{21}}{A_{21}}\right)\left[\frac{8\pi h\nu^3}{c^3}\cdot\frac{1}{e^{h\nu/kT} - 1}\right]$$

But $$\frac{B_{21}}{A_{21}} = \frac{c^3}{8\pi h\nu}$$

i.e., $$R_1 = \frac{1}{e^{h\nu/kT} - 1}$$

Considering $\nu = 5 \times 10^{14}$ Hertz and T = 300*k*, then

$$R_1 = 10^{-58}$$

Thus, in optical region, stimulated emission is negligible compared to spontaneous emission.

Example 1.30. *(a) At what temperature are the rates of spontaneous and stimulated emission equal ? Assume λ = 5000 Å.*

(b) At what wavelength are they equal at 300 K ?

Solution : If the rates of spontaneous and stimulated emission are equal, then

$$R = \left[e^{h\nu/kT} - 1\right]^{-1}$$

$$\lambda = 5000 \text{ Å},\ \nu = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{5000 \times 10^{-10}\text{ m}}$$

$$= 6 \times 10^{14} \text{ Hz}$$

$$\therefore \quad \frac{h\nu}{kT} = \frac{(6.626 \times 10^{-34}\text{ J.s})\,(6 \times 10^{14}\text{ s}^{-1})}{(1.38 \times 10^{-23}\text{ J/K})T}$$

$$= \frac{28.81 \times 10^3}{T}\text{K}$$

(*a*) $$\therefore \exp\left[\frac{28.81 \times 10^3}{T}\text{K}\right] = 2$$

or $$\frac{28.81 \times 10^3}{T}K = ln2 = 0.693$$

$$\therefore \quad T = \frac{28.81 \times 10^3}{T} K = \mathbf{41{,}573\ K}$$

(b) $\therefore$

$$\frac{h\nu}{kT} = \frac{(6.626 \times 10^{-34}\ \text{J.s})\ \nu}{(1.38 \times 10^{-23}\ \text{J/K})\ (300\ \text{K})}$$

$$= (1.6 \times 10^{-13}\ \text{s})\nu$$

$$\therefore \quad \exp[(1.6 \times 10^{-13}\ \text{s})\nu] = 2$$

or

$$(1.6 \times 10^{-13}\ \text{s})\nu] = 0.693$$

$$\therefore \quad \nu = \frac{0.693}{1.6 \times 10^{-13}}\ \text{Hz} = 4.3 \times 10^{12}\ \text{Hz}$$

$$\therefore \quad \lambda = \frac{c}{\nu} = \frac{3 \times 10^8\ \text{m/s}}{43 \times 10^{12}\ \text{s}^{-1}} = \mathbf{69.8\ \mu m}$$

Example 1.31. *Find the ratio of spontaneous emission to stimulated emission for a cavity of temperature 50K and wavelength 10^{-5} m.*

Solution :

$$\frac{\text{Rate of spontaneous emission}}{\text{Rate of Stimulated emission}} = \frac{A_{21}}{B_{21}} = e^{\left(\frac{h\nu}{kT}\right)} - 1$$

$$= e^{\left(\frac{hc}{\lambda kT}\right)} - 1$$

$$= e^{\left[\frac{6.625 \times 10^{-34} \times 3 \times 10^8}{10^{-5} \times 1.38 \times 10^{-23} \times 50}\right]} - 1$$

$$= e^{(28.8)} - 1$$

$$= e^{28.8} \qquad (\because e^{28.8} >> 1)$$

1.25 CONDITION FOR LIGHT AMPLIFICATION

Let us consider a beam of light propagating through a material medium. If the photons strike lower state (unexcited) atoms, they may be absorbed and removed from the stream of photons which therefore loses energy. However, if the photons strike excited atoms, more photons can be produced which are added to the light beam and increase its energy. Since the probabilities of absorption and stimulated emission are the same, both attenuation and amplification of the light beam occur simultaneously. We will now show that amplification can predominate only if there are more atoms in the higher level than in the lower level.

Let there be n photons per unit volume in the light beam. As the beam travels through the medium, some photons are absorbed due to absorption transitions and some photons are generated due to emission transitions. We will not take into account the photons generated by spontaneous emission as these photons go in random directions and do not contribute to the light beam propagating through the medium. Thus, we consider only the photons generated by stimulated emissions.

Let $-(dn/dt)$ be the net rate of loss of photons from the beam as it travels through an elemental volume of the medium having a thickness Δx and an area of unity.

The net rate of loss of photons from the light beam must be equal to the difference between the net rates of absorption and stimulated emission transitions. Thus,

$$-\frac{dn}{dt} = B_{12}\,\rho(\nu)\,N_1 - B_{21}\,\rho(\nu)\,N_2$$

or

$$-\frac{dn}{dt} = (N_1 - N_2)\,\rho(\nu)\,B_{12} \qquad \text{...(1.71)}$$

If the energy density of the light field in the medium is $\rho(\nu)$, then the intensity I is

$$I = \rho(\nu)\, \upsilon \qquad ...(1.72)$$

where υ $(= c/\mu)$ is the velocity of light in the medium.

As $\rho(\nu) = nh\nu$,

$$I = nh\, \nu\upsilon \qquad ...(1.73)$$

The loss of photons, $-dn$ in a small thickness dx of medium may be written as,

$$-dn = \frac{dI\,(x)}{dx} \cdot \frac{dx}{h\,\nu\upsilon}$$

The net rate of loss during a time interval dt is given by,

$$-\frac{dn}{dt} = \frac{dI\,(x)}{dx} \cdot \frac{1}{h\nu} \qquad ...(1.74)$$

$dx/dt = \upsilon$ is used in obtaining the above equation.

Using eq.(1.2) into eq.(1.74), we get

$$-\frac{dn}{dt} = +\alpha I\,(x) \cdot \frac{1}{h\nu}$$

Using Eq. (1.40) into the above, we get

$$\therefore \qquad -\frac{dn}{dt} = \alpha\rho\,(\nu)\,\upsilon \cdot \frac{1}{h\nu} \qquad ...(1.75)$$

Comparing equations (1.71) and (1.75), we obtain

$$\alpha\rho\,(\nu)\,\upsilon\frac{1}{h\nu} = (N_1 - N_2)\,\rho(\nu)\,B_{12}$$

$$\therefore \qquad \alpha = (N_1 - N_2)\,\frac{B_{12}\,h\nu}{\upsilon} \qquad ...(1.76)$$

Eq. (1.76) relates the absorption coefficient α to the difference in populations $(N_1 - N_2)$ of the two energy levels. For a material in thermal equilibrium, $N_1 > N_2$ and α is positive.

If N_2 is somehow made greater than N_1, then α becomes a negative quantity and the relation (1.4) takes the following form.

$$I = I_0 e^{(-\alpha)x}$$

or

$$I = I_0 e^{\gamma x} \qquad ...(1.77)$$

where γ $(= -\alpha)$ is referred to as the *gain coefficient* per unit length. As the gain coefficient γ is a positive quantity, the equation (1.77) implies that the intensity of light grows exponentially as the light beam travels through the medium. This is clearly amplification of light. Incorporating γ, we rewrite eq.(1.76) as,

$$\boxed{\gamma = (N_2 - N_1)\,\frac{B_{12}\,h\nu}{\upsilon} \quad \textit{Condition for amplification}} \qquad ...(1.78)$$

γ will be positive if $(N_2 - N_1) > 0$, that is,

$$\boxed{N_2 > N_1} \qquad ...(1.79)$$

The condition (1.79) is known as *population inversion*, because it is the inverse of the normal situation. Eq. (1.78) thus indicates that population inversion is a necessary condition to be satisfied for causing the amplification of incident light.

Using the relations (1.64) for B_{12}, (1.52) for A_{21} and $c = \mu\,\upsilon$, we can rewrite equation (1.78) as

$$\gamma = (N_2 - N_1).\frac{\nu^2}{8\,\pi\,\nu^2\tau_{sp}} \qquad ...(1.80)$$