Blocking a Replicated 2^k Factorial Design

In many situations it is impossible to perform all the runs in a 2^k factorial experiment under homogeneous conditions. For example, a single batch of raw material might not be large enough to make all of the required runs.

In other cases, it might be desirable to deliberately vary the experiment conditions to ensure that the treatments are equally effective (i.e. robust) across many situations that are likely to be encountered in practice. The design technique used in these situation is **blocking**.

Suppose that the 2^k factorial design has been replicated "n" times, then each set of nonhomogeneous conditions defines a block, and each replicate is run in one of the blocks. The runs in each block (or replicate) would be made in random order.

⇒ General ANOVA Table for blocked 2^2 Factorial Design-

Possible combinations $= 2^2 = 4$ $(1, a, b, ab)$ effects $(m) = 2^2 - 1 = 4 - 1 = 3$ (A, B, AB)

SoV	d.f	SS	MS	F
Blocks	$r-1$	$\frac{\sum_{k=1}^{r} B_k^2}{p^n} - \frac{(T_{\dots})^2}{p^n(r)}$	$SS(Block)/r-1$	
A	$P-1$	$(\text{effect of }A)^2/p^n(r)$	$SS(A)/p-1$	MSA/MSE
B	$P-1$	$(\text{effect of }B)^2/p^n(r)$	$SS(B)/p-1$	MSB/MSE
AB	$P-1$	$(\text{effect of }AB)^2/p^n(r)$	$SS(AB)/p-1$	$MSAB/MSE$
Error	By substraction	By Substraction	$SS(Error)/\text{its d.f}$	
Total	$P^n(r)-1$	$\sum_i\sum_j\sum_K Y_{ijk}^2 - \frac{(T_{\dots})^2}{p^n(r)}$		

Number of blocks are equal to replications so the degree of freedom for blocks will be $r-1$

ANOVA Model for this design

$$Y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + \delta_k + \epsilon_{ijk}$$

where

$i = 1, 2$ $\quad j = 1, 2$ $\quad$ & $\quad k = 1, 2, \cdots r$

δ_k is the effect of k^{th} block.

Example:

Consider an investigation into the effect of the concentration of the reactants and the amount of the catalyst on the conversion (Yield) in a chemical process. Let the reactant concentration be factor A and let the two level of interest be 15 and 25 percent. The catalyst is factor B with the high level denoting the use of 2 pounds of the catalyst and the low level denoting the use of 1 pounds of the catalyst. The experimenter use three batches of raw material to ensure that the treatments are equally effective.

Block 1	Block 2	Block 3	run total
1 = 28	1 = 25	1 = 27	80
a = 36	a = 32	a = 32	100
b = 18	b = 19	b = 23	60
ab = 31	ab = 30	ab = 29	90
Block totals $\delta_1 = 113$	$\delta_2 = 106$	$\delta_3 = 111$	

Solution:

① Formulation of hypothesis:

(i) H_0: Main effect A reactant constration(A) is insignificant

H_1: Main/estimated effect of reactant constration(A) is significant

(ii) H_0: Estimated effect of catalyst amount(B) is insignificant

H_1: Estimated effect of catalyst amount(B) is significant

Scanned with CamScanner

(iii) H_0: there is no interaction between reactant concentration and catalyst amount.
H_1: there is interaction between reactant concentration and catalyst amount.

(2) Level of Significance.

$$\alpha = 0.05$$

(3) Test statistic

(i) $F_1 = \frac{MSA}{MSE}$ (ii) $F_2 = \frac{MSB}{MSE}$ (iii) $F_3 = \frac{MS(AB)}{MSE}$

(4) Calculation:

effect of $A = ab + a - b - 1$
$= 90 + 100 - 60 - 80 = 50$

effect of $B = ab - a + b - 1$
$= 90 - 100 + 60 - 80 = -30$

effect of $AB = ab - a - b + 1$
$= 90 - 100 - 60 + 80 = 10$

$$SSA = \frac{(\text{effect of } A)^2}{p^n(r)} = \frac{(50)^2}{2^2(3)} = 208.33$$

$$SSB = \frac{(\text{effect of } B)^2}{p^n(r)} = \frac{(-30)^2}{2^2(3)} = 75$$

$$SS\,AB = \frac{(\text{effect of } AB)^2}{p^n(r)} = \frac{(10)^2}{2^2(3)} = 8.33$$

$$\text{Total } SS = \sum_i \sum_j \sum_k Y_{ijk}^2 - \frac{(T_{...})^2}{p^n(r)}$$

$$= 9398 - \frac{(330)^2}{2^2(3)}$$

$$= 323$$

$$\text{Block } SS = \frac{\sum_k B_k^2}{p^n} - \frac{(T_{...})^2}{p^n(r)} = \frac{(113)^2 + (106)^2 + (111)^2}{2^2} - \frac{(330)^2}{2^2(3)}$$

$$= 6.50$$

Error SS = Total SS - Blocks SS - SSA - SSB - SSAB
$= 323 - 6.50 - 208.33 - 75.\ldots - 8.33$
$= 24.84$

ANOVA Table.

SoV	d.f	SS	MS	F
Blocks	2	6.50	3.25	
A(concentration)	1	208.33	208.33	$50.32 = F_1$
B(Catalyst)	1	75	75	$18.12 = F_2$
AB	1	8.33	8.33	$2.01 = F_3$
Error	6	24.84	4.14	
Total	11	323		

⑤ Critical region:

$$F_{\alpha(v_1, v_2)} = F_{0.05(1, 6)} = 5.99$$

Same for A, B and AB

If $F_{cal} \geq 5.99$ then reject Ho otherwise don't reject Ho.

⑥ Conclusion:

On the basis of critical region, we conclude that the reactant concentration and the amount of catalyst have a significant effect but there is no interaction between these two factors.

Confounding in the 2^k Factorial Design

In many problems it is impossible to perform a complete replicate of a factorial design in one block. **Confounding** is a design technique for arranging a complete factorial experiment in blocks, where the block size is smaller than the number of treatment combinations in one replicate. The technique causes information about certain treatment effects (usually high order interactions) to be indistinguishable from or confounded with blocks. Note that even though the designs presented are incomplete block designs because each block does not contain all the treatments or treatment combinations, the special structure of the 2^k factorial system allows a simplified method of analysis.

We consider the construction and analysis of the 2^k factorial design 2^c incomplete blocks, where $c<K$. Consequently, these designs can be run in two blocks ($c=1$), four blocks ($c=2$), eight blocks ($c=3$) and so on.

Confunding the 2^k factorial design in two blocks/ 2^c incomplete blocks when $c=1$.

$\Rightarrow 2^2$ design:

Suppose that we wish to run a single replicate of the 2^2 design. Each of the $2^2=4$ treatment combinations requires a quantity of raw material for example and each batch of raw material is only large enough for two treatment combinations to be tested. Thus two batches of raw material are required. If batches of raw material are considered as blocks, then we must assign two of the four treatment combinations to each block.

For confunding the 2^k factorial design in two blocks we can use sign table. Suppose we wish to confound AB with blocks for

2^2 design- then we assign the treatment combinations that are plus on AB to block 1 and those that are minus on AB to block 2. We emphasize that the treatment combinations within a block are run in random order. This approach can be used to confound any effect (A, B, AB) with blocks

Sign table:

		treatment combinations			
		1	a	b	ab
Factorial effect	A	−	+	−	+
	B	−	−	+	+
	AB	+	−	−	+

→ AB is confounded with blocks.

block 1	block 2
1	a
ab	b

→ A is confounded with blocks.

block 1	block 2
1	a
b	ab

→ B is confounded with blocks

block 1	block 2
1	b
a	ab

This scheme can be used to confound any 2^k design in two blocks. As second example, consider a 2^3 design run in two blocks. Suppose we wish to confound the three-factor interaction ABC with blocks. From the sign table, we assign the treatment combinations that are minus on ABC to block 1 and those that are plus on ABC to block 2. Once again we emphasize that the treatment combinations within a block are run in random order. The resulting design is shown below:

Sign table for 2^3 design

Factorial effect	treatment combinations							
	1	a	b	ab	c	ac	bc	abc
A	−	+	−	+	−	+	−	+
B	−	−	+	+	−	−	+	+
AB	+	−	−	+	+	−	−	+
C	−	−	−	−	+	+	+	+
AC	+	−	+	−	−	+	−	+
BC	+	+	−	−	−	−	+	+
ABC	−	+	+	−	+	−	−	+

$\Rightarrow$ ABC is confounded with blocks.

block 1	block 2
1	a
ab	b
ac	c
bc	abc

Scanned with CamScanner

Other Methods for Constructing the Blocks

There is another method for constructing these designs. The method uses the linear combination

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_K x_K \Rightarrow$$ this equation called a defining contrast

where x_i is the level of the ith factor appearing in a particular treatment combination and α_i is the exponent appearing on the ith factor in the effect to be confounded.

For the 2^k system we have $\alpha_i = 0$ or 1 and $x_i = 0$ (low level) or $x_i = 1$ (high level). Treatment combinations that produce the same value of $L \pmod 2$ will be placed in the same block.

To illustrate the approach, consider a 2^2 design with AB confounded with blocks. Here x_1 corresponds to A, x_2 corresponds to B and $\alpha_1 = \alpha_2 = 1$. Thus, the defining contrast corresponding to AB is

$$L = x_1 + x_2$$

For the treatment combinations

(1): $L = 0 + 0 = 0 = 0 \pmod 2$

a: $L = 1 + 0 = 1 = 1 \pmod 2$

b: $L = 0 + 1 = 1 = 1 \pmod 2$

ab: $L = 1 + 1 = 2 = 0 \pmod 2$

$$2 \overline{)2}\ \ 1,\quad 2,\quad \boxed{0}$$

Thus "(1)" and "ab" are run in block 1 and "a" and "b" are run in block 2. This is the same design which was generated from the sign table.

Block 1	Block 2
(1)	a
ab	b

Now consider 2^3 design with ABC confounded with blocks. Thus the defining contrast corresponding to ABC is

$$L = x_1 + x_2 + x_3$$

where x_1 corresponds to A, x_2 corresponds to B, x_3 corresponds to C and $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

$(1): L = 0+0+0 = 0 = 0 \pmod 2$
$a : L = 1+0+0 = 1 = 1 \pmod 2$
$b : L = 0+1+0 = 1 = 1 \pmod 2$
$ab : L = 1+1+0 = 2 = 0 \pmod 2$
$c : L = 0+0+1 = 1 = 1 \pmod 2$
$ac : L = 1+0+1 = 2 = 0 \pmod 2$
$bc : L = 0+1+1 = 2 = 0 \pmod 2$
$abc : L = 1+1+1 = 3 = 1 \pmod 2$

Thus (1), ab, ac and bc are run in block 1 and a, b, c and abc are run in block 2.

Block 1	Block 2
(1)	a
ab	b
ac	c
bc	abc

* Confound 2^4 design with ABCD in two blocks. (By using sign table and defining contrast).