# Un-replicated Factorial Design

OR

# A Single Replicate of the $2^k$ Design

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The number of replicates that the experimenter can employ may be restricted. Frequently, available resources only allow a single replicate of the design to be run. A single-replicate strategy is often used in screening experiments when there are relatively many factors under consideration. Because we can never be entirely certain in such cases that the experimental error is small. A single replicate of a $2^k$ design is sometimes called an **Un-replicate factorial**. With only one replicate, there is no internal estimate of error (or "pure error"). One approach to the analysis of an unreplicated factorial is to assume that certain high-order interactions are negligible and combine their mean squares to estimate the error. This is an appeal to the "sparsity of effects principle"; that is, most systems are dominated by some of the main effects and low-order interactions and most high order interactions are negligible.

When analyzing data from unreplicated factorial designs, occasionally real high-order interactions occur. The use of an error mean square obtained by pooling high-order interactions is inappropriate in these cases. To overcome this problem, Daniel (1959) provides a method of analysis which is known as Daniel's Method.

## Daniel's Method:

Daniel suggests examining a normal probability plot of the estimates of the effects. The effects that are negligible are normally distributed with mean zero and variance $\sigma^2$ and will tend to fall along a straight line on this plot, whereas significant effects will have non-zero means and will not lie along the straight line.

Thus the preliminary model will be specified to contain those effects that are apparently non-zero based on the normal probability plot. The apparently negligible effects are combined as an estimate of error.

**Procedure for analysis by using Daniel's Method:**

1- Calculate effects
2- Calculate estimated effects/main effects.
3- Calculate Sum of Square of effects.
4- Calculate percentage contribution by using the following formula:

$$\frac{\text{SS of a factor}}{\text{Total SS}} \times 100$$

5- Draw normal probability plot. taking estimated effects on $x$-axis and percentage contribution on $Y$-axis.
6- Then the effects that are fall along a straight line neglect these effects and combine these as an estimate of error

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Example # 6.2 A Single Replicate of the $2^4$ Design

A chemical product is produced in a pressure vessel. A factorial experiment is carried out in the pilot plant to study the factors thought to influence the filtration rate of this product. The four factors are temperature (A), pressure (B), concentration of formaldehyde (C) and stirring rate (D). Each factor is present at two levels. The design matrix and the response data obtained from a single replicate of the $2^4$ experiment are shown in the following table.

| Run Number | A | B | C | D | Run Label | Filtration Rate (gal/h) | $y^2$ |
|---|---|---|---|---|---|---|---|
| 1 | − | − | − | − | 1 | 45 | 2025 |
| 2 | + | − | − | − | a | 71 | 5041 |
| 3 | − | + | − | − | b | 48 | 2304 |
| 4 | + | + | − | − | ab | 65 | 4225 |
| 5 | − | − | + | − | c | 68 | 4624 |
| 6 | + | − | + | − | ac | 60 | 3600 |
| 7 | − | + | + | − | bc | 80 | 6400 |
| 8 | + | + | + | − | abc | 65 | 4225 |
| 9 | − | − | − | + | d | 43 | 1849 |
| 10 | + | − | − | + | ad | 100 | 10000 |
| 11 | − | + | − | + | bd | 45 | 2025 |
| 12 | + | + | − | + | abd | 104 | 10816 |
| 13 | − | − | + | + | cd | 75 | 5625 |
| 14 | + | − | + | + | acd | 86 | 7396 |
| 15 | − | + | + | + | bcd | 70 | 4900 |
| 16 | + | + | + | + | abcd | 96 | 9216 |
| | | | | | | 1121 | 84271 |

Solution:

Daniel's Method.

effect of A = −1 + a − b + ab − c + ac − bc + abc − d
+ ad − bd + abd − cd + acd − bcd + abcd
= −45 + 71 − 48 + 65 − 68 + 60 − 80 + 65 − 43 + 100
− 45 + 104 − 75 + 86 − 70 + 96.

effect of A = 173

effect of B = −1 − a + b + ab − c − ac + bc + abc − d − ad
+ bd + abd − cd − acd + bcd + abcd
= −45 − 71 + 48 + 65 − 68 − 60 + 80 + 65 − 43
− 100 + 45 + 104 − 75 − 86 + 70 + 96

effect of B = 25

effect of AB = +1 − a − b + ab + c − ac − bc + abc + d − ad
− bd + abd + cd − acd − bcd + abcd
= 45 − 71 − 48 + 65 + 68 − 60 − 80 + 65 + 43 − 100
− 45 + 104 + 75 − 86 − 70 + 96

effect of AB = 1

effect of C = −1 − a − b − ab + c + ac + bc + abc − d − ad
− bd − abd + cd + acd + bcd + abcd.
= −45 − 71 − 48 − 65 + 68 + 60 + 80 + 65 − 43 − 100
− 45 − 104 + 75 + 86 + 70 + 96

effect of C = 79

effect of AC = +1 − a + b − ab − c + ac − bc + abc + d − ad
+ bd − abd − cd + acd − bcd + abcd
= 45 − 71 + 48 − 65 − 68 + 60 − 80 + 65 + 43 − 100
+ 45 − 104 − 75 + 86 − 70 + 96

effect of AC = −145

effect of BC = +45 + 71 − 48 − 65 − 68 − 60 + 80 + 65 + 43 + 100
− 45 − 104 − 75 − 86 + 70 + 96

effect of BC = 19

effect of ABC = −45 + 71 + 48 − 65 + 68 − 60 − 80 + 65 − 43 + 100
+ 45 − 104 + 75 − 86 − 70 + 96

effect of ABC = 15

effect of D = −45 − 71 − 48 − 65 − 68 − 60 − 80 − 65 + 43 + 100
+ 45 + 104 + 75 + 86 + 70 + 96

effect of D = 117

effect of AD = +45 − 71 + 48 − 65 + 68 − 60 + 80 − 65 − 43 + 100 − 45 + 104 − 75 + 86 − 70 + 96

$\boxed{\text{effect of } AD = 133}$

effect of BD = +45 + 71 − 48 − 65 + 68 + 60 − 80 − 65 − 43 − 100 + 45 + 104 − 75 − 86 + 70 + 96

$\boxed{\text{effect of } BD = -3}$

effect of ABD = −45 + 71 + 48 − 65 − 68 + 60 + 80 − 65 + 43 − 100 − 45 + 104 + 75 − 86 − 70 + 96

$\boxed{\text{effect of } ABD = 33}$

effect of CD = +45 + 71 + 48 + 65 − 68 − 60 − 80 − 65 − 43 − 100 − 45 − 104 + 75 + 86 + 70 + 96

$\boxed{\text{effect of } CD = -9}$

effect of ACD = −45 + 71 − 48 + 65 + 68 − 60 + 80 − 65 − 43 − 100 + 45 − 104 − 75 + 86 − 70 + 96

$\boxed{\text{effect of } ACD = -13}$

effect of BCD = −45 − 71 + 48 + 65 + 68 + 60 − 80 − 65 + 43 + 100 − 45 − 104 − 75 − 86 + 70 + 96

$\boxed{\text{effect of } BCD = -21}$

effect of ABCD = +45 − 71 − 48 + 65 − 68 + 60 + 80 − 65 − 43 + 100 + 45 − 104 + 75 − 86 − 70 + 96

$\boxed{\text{effect of } ABCD = 11}$

Now the estimated effect $= \dfrac{1}{\frac{P^n}{2}(r)}$ (effect of factor)

Here formula will become according question.

estimated effect $= \dfrac{1}{\frac{2^4}{2}(1)}$ (effect of factor) $= \dfrac{1}{8}$ (effect of factor)

estimated effect of A $= \dfrac{173}{8} = 21.63$

estimated effect of B $= \dfrac{25}{8} = 3.13$

estimated effect of AB $= \dfrac{1}{8} = 0.13$

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estimated effect of $C = \frac{79}{8} = 9.88$

estimated effect of $AC = \frac{-145}{8} = -18.13$

estimated effect of $BC = \frac{19}{8} = 2.38$

estimated effect of $ABC = \frac{15}{8} = 1.88$

estimated effect of $D = \frac{117}{8} = 14.63$

estimated effect of $AD = \frac{133}{8} = 16.63$

estimated effect of $BD = \frac{-3}{8} = -0.38$

estimated effect of $ABD = \frac{33}{8} = 4.13$

estimated effect of $CD = \frac{-9}{8} = -1.13$

estimated effect of $ACD = \frac{-13}{8} \doteq -1.63$

estimated effect of $BCD = \frac{-21}{8} = -2.63$

estimated effect of $ABCD = \frac{11}{8} = 1.38$

Now we have to find Sum of Squares by using following formula:

$$SS = \frac{(\text{effect of factor})^2}{p^n(r)} = \frac{(\text{effect of factor})^2}{2^4(1)}$$

$$SS = \frac{(\text{effect of factor})^2}{16}$$

$SSA = \frac{(173)^2}{16} = 1870.56$

$SSB = \frac{(25)^2}{16} = 39.06$

$SSAB = \frac{(1)^2}{16} = 0.06$

$SSC = \frac{(79)^2}{16} = 390.06$

$SSAC = \frac{(-45)^2}{16} = 1314.06$

$SSBC = \frac{(19)^2}{16} = 22.56$

$SSABC = \frac{(15)^2}{16} = 14.06$

$SSD = \frac{(117)^2}{16} = 855.56$

$SSAD = \frac{(133)^2}{16} = 1105.56$

$SSBD = \frac{(-3)^2}{16} = 0.56$

$SSABD = \frac{(33)^2}{16} = 68.06$

$SSCD = \frac{(-9)^2}{16} = 5.06$

$SSACD = \frac{(-13)^2}{16} = 10.56$

$SSBCD = \frac{(-21)^2}{16} = 27.56$

$$SSABCD = \frac{(11)^2}{16} = 7.56$$

Now we have to find percentage contribution by using following formula:

$$PC = \frac{SS \text{ of factor}}{\text{Total } SS} \times 100$$

for this we have to calculate Total SS.

$$TSS = \Sigma y^2 - \frac{(T_{..})^2}{p^n(r)}$$

$$= 84271 - \frac{(1121)^2}{2^4(1)}$$

$$= 84271 - \frac{1256641}{16}$$

$$TSS = 5730.94$$

Now percentatage contribution.

$PC \text{ of } A = \frac{SSA}{TSS} \times 100 = \frac{1870.56}{5730.94} \times 100 = 32.64\ \%$

$PC \text{ of } B = \frac{SSB}{TSS} \times 100 = \frac{39.06}{5730.94} \times 100 = 0.68\%$

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$$PC \text{ of } AB = \frac{SSAB}{TSS} \times 100 = \frac{0.13}{5730.94} \times 100 = 0.002\%$$

$$PC \text{ of } C = \frac{SSC}{TSS} \times 100 = \frac{390.06}{5730.94} \times 100 = 6.81\%$$

$$PC \text{ of } AC = \frac{SSAC}{TSS} \times 100 = \frac{1314.06}{5730.94} \times 100 = 22.93\%$$

$$PC \text{ of } BC = \frac{SSBC}{TSS} \times 100 = \frac{22.56}{5730.94} \times 100 = 0.39\%$$

$$PC \text{ of } ABC = \frac{SSABC}{TSS} \times 100 = \frac{14.06}{5730.94} \times 100 = 0.25\%$$

$$PC \text{ of } D = \frac{SSD}{TSS} \times 100 = \frac{855.56}{5730.94} \times 100 = 14.93\%$$

$$PC \text{ of } AD = \frac{SSAD}{TSS} \times 100 = \frac{1105.56}{5730.94} \times 100 = 19.29\%$$

$$PC \text{ of } BD = \frac{SSBD}{TSS} \times 100 = \frac{0.56}{5730.94} \times 100 = 0.0098\%$$

$$PC \text{ of } ABD = \frac{SSABD}{TSS} \times 100 = \frac{68.06}{5730.94} \times 100 = 1.19\%$$

$$PC \text{ of } CD = \frac{SSCD}{TSS} \times 100 = \frac{5.06}{5730.94} \times 100 = 0.09\%$$

$$PC \text{ of } ACD = \frac{SSACD}{TSS} \times 100 = \frac{10.56}{5730.94} \times 100 = 0.18\%$$

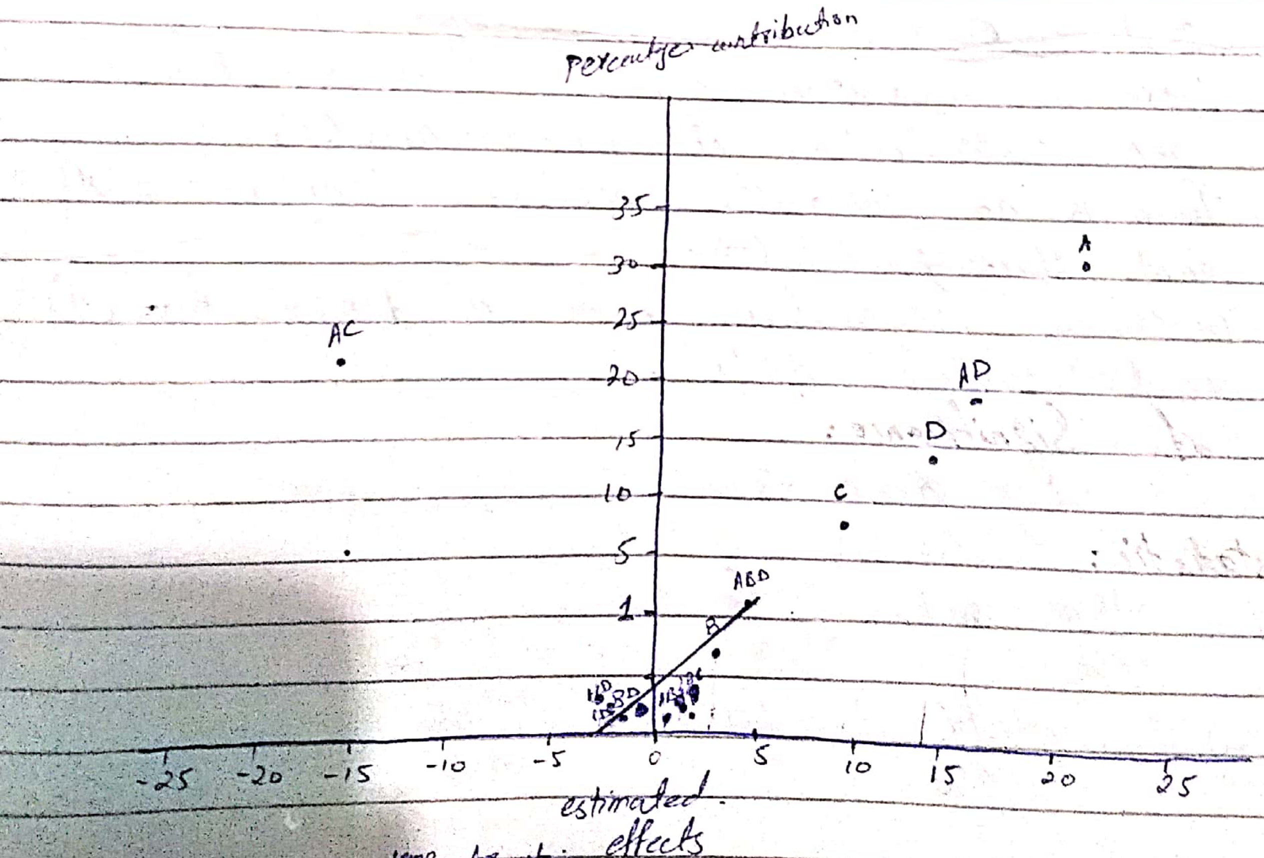
$$PC \text{ of } BCD = \frac{SSBCD}{TSS} \times 100 = \frac{27.56}{5730.94} \times 100 = 0.48\%$$

$$PC \text{ of } ABCD = \frac{SSABCD}{TSS} \times 100 = \frac{7.56}{5730.94} \times 100 = 0.13\%$$

Now draw a plot by taking estimated effects on x-axis and percentage contribution on Y-axis.

For this firstly we can arrange findings in table form for our ease -

| Model Term | Estimated effect | Percentage Contribution |
|---|---|---|
| A | 21.63 | 32.64 |
| B | 3.13 | 0.68 |
| C | 9.88 | 6.81 |
| D | 14.63 | 14.93 |
| AB | 0.13 | 0.002 |
| AC | -18.13 | 22.93 |
| AD | 16.63 | 19.29 |
| BC | 2.38 | 0.39 |
| BD | -0.38 | 0.0098 |
| CD | -1.13 | 0.09 |
| ABC | 1.88 | 0.25 |
| ABD | 4.13 | 1.19 |
| ACD | -1.63 | 0.18 |
| BCD | -2.63 | 0.48 |
| ABCD | 1.38 | 0.13 |

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The effect of A, C, D is positive this suggest the increasing from low to high level will increase the filtration rates. Interaction of tempratture and formaldehyde concentration show negative effect & the interaction of temperature and stirring rate show positive effect.

So the effect of B; AB, BC, BD, CD, ABC, ABD, ACD, BCD and ABCD are neglect and combine These as an estimate of error.
Thus the model is specified to contain the effect of A, C, D, AC & AD.
So

① **Formulation of Hypothesis:**

(i) $H_0$: The main effect of temperature (A) is insignificant.
$H_1$: The main effect of temperature (A) is significant

(ii) $H_0$: The main effect of concentration of formaldehyde (C) is insignificant
$H_1$: The main effect of concetration of formaldehyde (C) is significant.

(iii) $H_0$: The main effect of stirring rate (D) is insignificant
$H_1$: The main effect of stirring rate (D) is significant.

(iv) $H_0$: There is no interaction between temperature (A) and concentration of formaldehyde (C).
$H_1$: There is interaction between temperature (A) and concentration of formaldehyde (C).

(v) $H_0$: There is no interaction between temperature (A) and stirring rate (D).
$H_1$: There is interaction between temperature (A) and stirring rate (D).

② **Level of Significance:**

$$\alpha = 0.05$$

③ **Test statistic:**

(i) $F_1 = \dfrac{MSA}{MSE}$ (ii) $F_2 = \dfrac{MSC}{MSE}$ (iii) $F_3 = \dfrac{MSD}{MSE}$

(iv) $F_4 = \dfrac{MS(AC)}{MSE}$ (v) $F_5 = \dfrac{MS(AD)}{MSE}$

## Design Projection:

If we have a single replicate of a $2^k$ design and if $h(h<k)$ factors are negligible and can be dropped then the the original data correspond to a full two-level factorial in the remaining $k-h$ factors with $2^h$ replicates.
In the following example B(pressure) is not significant and all interaction involving B are negligible, we may discard B from the experiment so that the design becomes a $2^3$ factorial in A, C and D with two replicates.
So

## ④ Calculation:

| SOV | d.f | SS | MS | F |
|---|---|---|---|---|
| A | 2-1=1 | 1870.56 | 1870.56 | 83.36 |
| C | 2-1=1 | 390.06 | 390.06 | 17.38 |
| D | 2-1=1 | 855.56 | 855.56 | 38.13 |
| AC | 2-1=1 | 1314.06 | 1314.06 | 58.56 |
| AD | 2-1=1 | 1105.56 | 1105.56 | 49.27 |
| CD | 2-1=1 | 5.06 | 5.06 | |
| ACD | 2-1=1 | 10.56 | 10.56 | |
| Error | 15-7=8 | 179.521 | 22.44 | |
| Total | $2^4(1)-1=15$ | 5730.94 | | |

## ⑤ Critical Region:

If $F_{cal} \geq F_{\alpha(v_1,v_2)}$ then reject $H_0$ otherwise don't reject $H_0$.
where

$$F_{\alpha(v_1,v_2)} = F_{0.05(1,8)} = 5.32$$

## ⑥ Conclusion:

As $F_1, F_2, F_3$ and $F_4 > 5.32$ so. reject $H_0$ and concluded that The effect of temperature (A), concentration of formaldehyde (c) and stirring rate (D) is significant. And there is interaction between A & C and also A & D.

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## Another Method for Analyzing Unreplicated Factorials.

The standard analysis procedure for an unreplicated two-level factorial design is the normal (or half-normal) plot of the estimated factor effects. However, unreplicated designs are so widely used in practice so many formal analysis procedures have been proposed to overcome the subjectivity of the normal probability plot. Lenth (1989) proposed a method that has good power to detect significant effects. It is known as **Lenth's Method**. It is also easy to implement and as a result it appears in several software packages for analyzing data from unreplicated factorials.

### Procedure of Lenth's Method:

Suppose that we have $m$ contrast of interest say $c_1, c_2, \ldots c_m$. If the design in an unreplicated $2^k$ factorial design, these contrasts correspond to the $m = 2^k - 1$ factor effect estimates. The basis of Lenth's method is to estimate the variance of a contrast from the smallest (in absolute value) contrast estimates. Let

$$S_0 = 1.5 \times \text{median}(|c_j|)$$

modulus value of estimated effects.

and

$$PSE = 1.5 \times \text{median}\left(|c_j| : |c_j| < 2.5 S_0\right)$$

~~median of that~~ modulus estimated effects which are less than $2.5 S_0$

PSE is called the "Peudostandard error". and Lenth shows that it is a reasonable estimator of the contrast variance when there are not many active (Significant) effects. The PSE is used to judge the significance of contrasts.

An individual contrast can be compared to the margin of error (ME)

$$ME = t_{0.025,d} \times PSE$$

where the degrees of freedom are defined as $d = m/3$.

For inference on a group of contrasts, Lenth suggests using the simultaneous margin of error (SME):

$$SME = t_{\gamma,d} \times PSE$$

where the percentage point of the t distribution used is $\gamma = 1-(1+0.95^{1/m})/2$.

The estimated effects, which are less than SME. Neglect these effects and combine these as an estimate of error. And which are greater than SME, consider these as significant effect.

Also look, if any estimated effects is not greater than SME but it is greater than ME then this estimated effect will be include in the list of significant effects.

| | $2^3$ | $2^4$ | $2^5$ |
|---|---|---|---|
| $t_{\gamma,d}$ | 9.008 | 5.219 | 4.218 |

In SME the value for $\gamma$ has computation difficulties so use the above table for using value of $t_{\gamma,d}$.

## Numerical by using Lenth's Method:

In previous example we calculate estimated effects that are
21.63, 3.13, 9.88, 14.63, 0.13, −18.13, 16.63, 2.38, −0.38, −1.13, 1.88, 4.13, −1.63, −2.63, 1.38

Taking $|c_j|$ and also arranging in assending order.
0.13, 0.38, 1.13, 1.38, 1.63, 1.88, 2.38, 2.63, 3.13, 4.13, 9.88, 14.63, 16.63, 18.13, 21.63

$\text{Median}|c_j|$ = the value of $\left(\frac{n+1}{2}\right)$th estimated effect

= the value of $\left(\frac{15+1}{2}\right)$th estimated effect

= the value of 8th estimated effect

= 2.63

$$S_o = 1.5 \times \text{Median}|c_j|$$
$$= 1.5 \times 2.63$$
$$\boxed{S_o = 3.945}$$

$$2.5\, S_o = 2.5(3.945)$$
$$= 9.8625$$

$|c_j| < 9.8625$ are:
0.13, 0.38, 1.13, 1.38, 1.63, 1.88, 2.38, 2.63, 3.13, 4.13

$\text{median}(|c_j| : |c_j| < 9.8625) = \frac{1}{2}\left[\text{value of } \left(\frac{n}{2}\right)\text{th estimated effect} + \text{value of } \left(\frac{n+1}{2}\right)\text{th estimated effect}\right]$

$= \frac{1}{2}\left[\text{value of } \left(\frac{10}{2}\right)\text{th estimated effect} + \text{value of } \left(\frac{10+1}{2}\right)\text{th estimated effect}\right]$

$= \frac{1}{2}\left[\text{value of 5th estimated effect} + \text{value of 5.5 th estimated effect}\right]$

$= \frac{1}{2}\left[\text{value of 5th estimated effect} + \text{value of 6th estimated effect}\right]$

$= \frac{1}{2}\left[1.63 + 1.88\right]$

$= 1.755$

$PSE = 1.5 \times \text{median}(|C_j| : |C_j| < 9.8625)$

$= 1.5 \times 1.755$

$\boxed{PSE = 2.63}$

$ME = t_{0.025,d} \times PSE$

In this question $d = m/3 = 15/3 = 5$

So

$ME = t_{0.025,5} \times PSE$

$= 2.571 \times 2.63$

$\boxed{ME = 6.76}$

$SME = t_{\gamma,d} \times PSE$

It is $2^4$ factorial experiment question so.

$= 5.219 \times 2.63$

$\boxed{SME = 13.73}$

The SME criterion would indicate that the four largest effects (A, D, AC, AD) are significant because their effect estimates exceeds SME. The main effect of C is significant according to the ME criterion, but not with respect to SME. However, because AC interaction is clearly important, we would probably include C in the list of significant effects.

Notice that in this example, Lenth's method has produced the same answer that we obtained previously from examination of the normal probability plot of effects. In general, the Lenth's Method is a clever and very useful procedure. However, we recommend using it as a supplement to the usual normal probability plot of effects, not as a replacement for it.