Two-Level Factorial Designs

The most important of these special cases is that of K factors, each at only two levels. These levels may be quantitative, such as two values of temperature, pressure or time. or they may be qualitative, such as two machines, two operator, the high and low levels of a factor, or perhaps the presence and absence of a factor. A complete replicate of such a design requires $2 \times 2 \times \cdots \times 2 = 2^k$ observations and is called a 2^k factorial design.

The 2^k design is particularly useful in the early stages of experimental work when many factors are likely to be investigated. It provides the smallest number of runs with which K factors can be studied in a complete factorial design. Consequently, these designs are widely used in factor screening experiments. (an experimental plan that is indended to find the few significant factors from a list of many potential ones).

Keep in mind Here we assume:

(1) the factors are fixed
(2) the designs are completely randomized
(3) the usual normality assumptions are satisfied

The 2^2 Design

The first design in the 2^k series is one with only two factors, say A and B, each run at two levels. This is called a 2^2 factorial design. The levels of the factors may be arbitrarily called low and high.

Example:

Consider an investigation into the effect of the concentration of the reactant and the amount of the catalyst on the conversion (yield) in a chemical process. The objective of the experiment was to determine if adjustments to either of these two factors would increase the yield.

Let the reactant concentration be factor A and let the two levels of interest be 15 and 25 percent. The catalyst is factor B, with the high level denoting the use of 2 pounds of the catalyst and the low level denoting the use of 1 pound. The experiment is replicated three times, so there are 12 runs. The order in which the runs are made is random, so this is a completely randomized experiment. The data obtained are as follows:

Factor A	Factor B	Treatment Combination	Replicate I	Replicate II	Replicate III
−	−	A low, B low	28	25	27
+	−	A high, B low	36	32	32
−	+	A low, B high	18	19	23
+	+	A high, B high	31	30	29

In 2^2 design, the low and high levels of A and B are denoted by "−" and "+" respectively. We denote the effect of a factor by a capital Latin letter. And the four treatment combinations in the design are ~~also~~ represented by lowercase letters.

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a ⇒ represents the treatment combination of A at the high level and B at the low level.
b ⇒ represents A at the low level and B at the high level.
ab ⇒ represents both factors at the high level.
1 ⇒ used to denote both factors at the low level.

So

run lable	Replicate I	Replicate II	Replicate III	Total
1	28	25	27	80
a	36	32	32	100
b	18	19	23	60
ab	31	30	29	90

⇒ Effects of Factors by using Algebric (constrast) method.

$$\text{effect of } A = (a-1)(b+1)$$
$$= ab + a - b - 1$$
$$= 90 + 100 - 60 - 80$$
$$= 50$$

$$\text{effect of } B = (a+1)(b-1)$$
$$= ab - a + b - 1$$
$$= 90 - 100 + 60 - 80$$
$$= -30$$

$$\text{effect of } AB = (a-1)(b-1)$$
$$= ab - a - b + 1$$
$$= 90 - 100 - 60 + 80$$
$$= 10$$

In a two level factorial design, we may define the average effect of a factor as the change in response produced by a change in the level of that factor averaged over the levels of the other factor.

So the main effects.

Main effect of a factor $= \frac{1}{\frac{P^n}{2}(r)}$ [effect of that factor]

Main effect of A $= \frac{1}{\frac{2^2}{2}(3)}(50) = \frac{1}{2(3)}(50)$

$= 8.33$

Main effect of B $= \frac{1}{2^2/2(3)}(-30)$

$= -5.00$

Main effect of AB $= \frac{1}{2^2/2(3)}(10)$

$= 1.67$

The effect of A (reactant concentration) is positive; this sugests the increasing A from the low level (15%) to the high level (25%) will increase the yield. The effect of B (catalyst) is negative; this suggests that increasing the amount of catalyst added to the process will decrease the yield. The interaction effect appears to be small relative to the two main effects.

In experiments involving 2^k designs, it is always important to examine the magnitude and direction of the factor effects to determine which variables are likely to be important. The Analysis of Variance ANOVA) can generally be used to confirm this interpretation.

General ANOVA procedure for 2^2 factorial designs

① Formulation of Hypothesis:

(i) H_0: Main effect of factor A is not statistically significant.
H_1: The main effect of factor A is statistically significant

(ii) H_0: The main effect of factor B is in-significant
H_1: The main effect of factor B is significant.

(iii) H_0: There is no interaction between factors A & B.
H_1: There is interaction between factors A & B.

② Level of Significance (α):

$\alpha = ?$

③ Test Statistic:

(i) $F_1 = \dfrac{MS(A)}{MSE}$ (ii) $F_2 = \dfrac{MS(B)}{MSE}$ (iii) $F_3 = \dfrac{MS(AB)}{MSE}$

④ Critical Region/Rejection Region:

If $F_{cal} \geq F_{\alpha(v_1, v_2)}$ then reject H_0 otherwise do not reject H_0.

$v_1 \rightarrow p-1$
$v_2 \rightarrow$ error df.

⑤ Calculation:

ANOVA Table

SoV	df	SS	MS	F
A	$P-1$	$(\text{effects of } A)^2/p^n(r)$	$SS(A)/P-1$	$MS(A)/MSE$
B	$P-1$	$(\text{effect of } B)^2/p^n(r)$	$SS(B)/P-1$	$MS(B)/MSE$
AB	$P-1$	$(\text{effect of } AB)^2/p^n(r)$	$SS(AB)/P-1$	$MS(AB)/MSE$.
Error	By Substration	By Substrations	$SSE/\text{its d.f}$	
Total	$p^n(r)-1$	$\sum_i\sum_j Y_{ij}^2 - \dfrac{(T..)^2}{P^n(r)}$		

⑥ Decision:

Using a 2^2 design and estimating all the effects that can be estimated (A, B, AB) implies a tentative ANOVA model of the form:

$$Y_{ij} = \mu + A_i + B_j + (AB)_{ij}$$

where

$i = 1, 2$ & $j = 1, 2$

Y_{ij} denoting the response when the ith level of factor A is used in combination with the jth level of factor B and 1 & 2 are denoting the low and high level respectively.

Note that there is no error term in this model because there is an exact fit when this model is used and the design is not replicated. There is an error term when the design is replicated. So then the model is

$$Y_{ij} = \mu + A_i + B_j + (AB)_{ij} + \epsilon_{ij}$$

: All 2^k designs are orthogonal designs:

The concept of orthogonality is important in design of experiments because it says something about independence. Experimental analysis of an orthogonal design is usually straightforward because you can estimate each main effect and interaction independently.

Orthogonality: Two vectors are orthogonal if the sum of the products of their corresponding elements is zero.

Consider 2^2 design (How it is an orthogonal design?)

Sign table:

	1	a	b	ab
A	−	+	−	+
B	−	−	+	+
AB	+	−	−	+

$$A = \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \end{bmatrix}$$

$A \cdot B = (-1)(-1) + (+1)(-1) + (-1)(+1) + (+1)(+1)$
$= +1 - 1 - 1 + 1$
$= 0$

This implies that the design is orthogonal which means that the main effect of each factor and interaction can be estimated independently of each other.

The 2^3 Design

Suppose that three factors, A, B, and C, each at two levels are of interest. The design is called a 2^3 factorial design.

$2^3 = 2 \times 2 \times 2 = 8$

8 possible combinations

$1, a, b, ab, c, ac, bc, abc$

effect of A $= (a-1)(b+1)(c+1)$
$= (ab + a - b - 1)(c+1)$
$= abc + ac - bc - c + ab + a - b - 1$

effect of B $= (a+1)(b-1)(c+1)$
$= (ab - a + b - 1)(c+1)$
$= abc - ac + bc - c + ab - a + b - 1$

effect of AB $= (a-1)(b-1)(c+1)$
$= (ab - a - b + 1)(c+1)$
$= abc - ac - bc + c + ab - a - b + 1$

effect of C $= (a+1)(b+1)(c-1)$
$= (ab + a + b + 1)(c-1)$
$= abc + ac + bc + c - ab - a - b - 1$

effect of BC $= (a+1)(b-1)(c-1)$
$= (ab - a + b - 1)(c-1)$
$= abc - ac + bc - c - ab + a - b + 1$

effect of AC $= (a-1)(b+1)(c-1)$
$= (ab + a - b - 1)(c-1)$
$= abc + ac - bc - c - ab - a + b + 1$

$$\text{effect of } ABC = (a-1)(b-1)(c-1)$$
$$= (ab-a-b+1)(c-1)$$
$$= abc - ac - bc + c - ab + a + b - 1$$

Main effect $= \dfrac{1}{p^n/2\,(r)}$ (effect of corresponding)

here it will be 2^3

General ANOVA Table for 2^3 factorial Design.

⇒ Calculation:

ANOVA Table

SOV	d.f	SS	MS	F
A	$P-1$	$(\text{effect of } A)^2 / p^n(r)$	$SS(A)/p-1$	$MSA/MSE = F_1$
B	$P-1$	$(\text{effect of } B)^2 / p^n(r)$	$SS(B)/p-1$	$MSB/MSE = F_2$
AB	$P-1$	$(\text{effect of } AB)^2 / p^n(r)$	$SS(AB)/p-1$	$MS(AB)/MSE = F_3$
C	$P-1$	$(\text{effect of } C)^2 / p^n(r)$	$SS(C)/p-1$	$MSC/MSE = F_4$
AC	$P-1$	$(\text{effect of } AC)^2 / p^n(r)$	$SS(AC)/p-1$	$MS(AC)/MSE = F_5$
BC	$P-1$	$(\text{effect of } BC)^2 / p^n(r)$	$SS(BC)/p-1$	$MS(BC)/MSE = F_6$
ABC	$P-1$	$(\text{effect of } ABC)^2 / p^n(r)$	$SS(ABC)/p-1$	$MS(ABC)/MSE = F_7$
Error	By subtraction	By subtraction	$SSE/\text{its d.f}$	
Total	$p^n(r)-1$	$\sum\sum Y_{ij}^2 - \dfrac{(T_{..})^2}{p^n(r)}$		

here $p^n = 2^3$

$p=2 \quad n=3$

The 2^4 factorial Design

Suppose that four factors A, B, C & D, each at two levels of interest. The design is called a 2^4 factorial design.

Treatment combinations $= 2^4 = 16$

1, a, b, ab, c, ac, bc, abc, d, ad, bd, abd, cd, acd, bcd, abcd

The complete model would contain $2^k - 1$ effects for a 2^k design so for 2^4 design it contain $2^4 - 1 = 15$ effects.

⇒ You can use any method to find effects.

effect of $A = (a-1)(b+1)(c+1)(d+1)$

effect of $B = (a+1)(b-1)(c+1)(d+1)$

effect of $AB = (a-1)(b-1)(c+1)(d+1)$

effect of $C = (a+1)(b+1)(c-1)(d+1)$

effect of $AC = (a-1)(b+1)(c-1)(d+1)$

effect of $BC = (a+1)(b-1)(c-1)(d+1)$

effect of $ABC = (a-1)(b-1)(c-1)(d+1)$

effect of $D = (a+1)(b+1)(c+1)(d-1)$

effect of $AD = (a-1)(b+1)(c+1)(d-1)$

effect of $BD = (a+1)(b-1)(c+1)(d-1)$

effect of $ABD = (a-1)(b-1)(c+1)(d-1)$

effect of $CD = (a+1)(b+1)(c-1)(d-1)$

effect of $ACD = (a-1)(b+1)(c-1)(d-1)$

effect of $BCD = (a+1)(b-1)(c-1)(d-1)$

effect of $ABCD = (a-1)(b-1)(c-1)(d-1)$

We prefer sign table to find out the effect, if we have more effects to find out.

In paper, students are allowed to bring sign table of 2^5 with them and by using this they can solve questions

⇒ Main effect is also Known as estimated effect.

$$\text{Main/estimated effect of a factor} = \frac{1}{\frac{p^n}{2}(r)}(\text{effect of that factor}).$$

Here $p^n = 2^4$

General ANOVA table for 2^4 Design

S.o.V	d.f	SS	MS	F
A	$p-1$	$(\text{effect of } A)^2 / p^n(r)$	$SS(A)/p-1$	$MS(A)/MSE = F_1$
B	$p-1$	$(\text{effect of } B)^2 / p^n(r)$	$SS(B)/p-1$	$MS(B)/MSE = F_2$
AB	$p-1$	$(\text{effect of } AB)^2 / p^n(r)$	$SS(AB)/p-1$	$MS(AB)/MSE = F_3$
C	$p-1$	$(\text{effect of } C)^2 / p^n(r)$	$SS(C)/p-1$	$MS(C)/MSE = F_4$
AC	$p-1$	$(\text{effect of } AC)^2 / p^n(r)$	$SS(AC)/p-1$	$MS(AC)/MSE = F_5$
BC	$p-1$	$(\text{effect of } BC)^2 / p^n(r)$	$SS(BC)/p-1$	$MS(BC)/MSE = F_6$
ABC	$p-1$	$(\text{effect of } ABC)^2 / p^n(r)$	$SS(ABC)/p-1$	$MS(ABC)/MSE = F_7$
D	$p-1$	$(\text{effect of } D)^2 / p^n(r)$	$SS(D)/p-1$	$MS(D)/MSE = F_8$
AD	$p-1$	$(\text{effect of } AD)^2 / p^n(r)$	$SS(AD)/p-1$	$MS(AD)/MSE = F_9$
BD	$p-1$	$(\text{effect of } BD)^2 / p^n(r)$	$SS(BD)/p-1$	$MS(BD)/MSE = F_{10}$
ABD	$p-1$	$(\text{effect of } ABD)^2 / p^n(r)$	$SS(ABD)/p-1$	$MS(ABD)/MSE = F_{11}$
CD	$p-1$	$(\text{effect of } CD)^2 / p^n(r)$	$SS(CD)/p-1$	$MS(CD)/MSE = F_{12}$
ACD	$p-1$	$(\text{effect of } ACD)^2 / p^n(r)$	$SS(ACD)/p-1$	$MS(ACD)/MSE = F_{13}$
BCD	$p-1$	$(\text{effect of } BCD)^2 / p^n(r)$	$SS(BCD)/p-1$	$MS(BCD)/MSE = F_{14}$
ABCD	$p-1$	$(\text{effect of } ABCD)^2 / p^n(r)$	$SS(ABCD)/p-1$	$MS(ABCD)/MSE = F_{15}$
Error	By substraction	By Substraction	$SSE/\text{its d.f}$	
Total	$p^n(r)-1$	$\sum\sum Y_{ij}^2 - \frac{(T_{..})^2}{p^n(r)}$		

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