STAT-6207
DESIGN AND ANALYSIS OF EXPERIMENT-II

Class: Msc Statistics 2^{nd} semester

Instructor: Miss Summera Kinat

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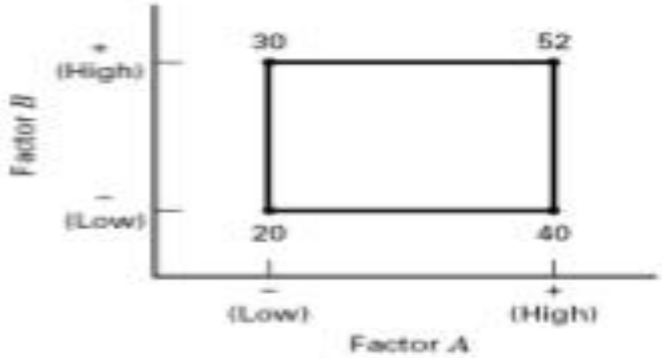
- Factorial experiments
- Informal way to find effects in Factorial Design
- Formal ways to find effects in Factorial Design
  1. Sign method
  2. Yates method
  3. Algebraic method

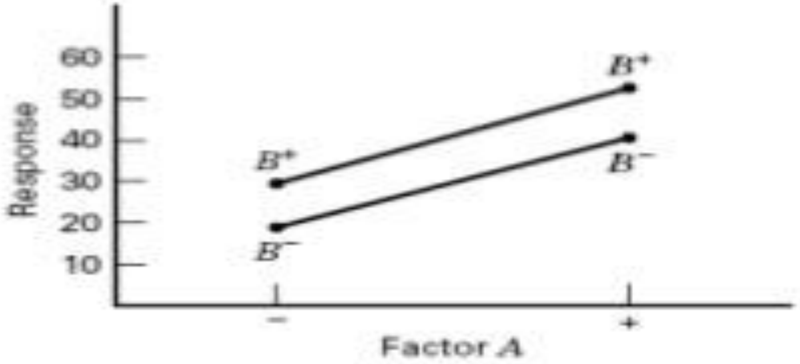
Basic Definition

- Many experiments involve the study of the effects of two or more factors.
- Factorial designs most efficient for this type of experiment.
- By a factorial design, we mean that in each complete trial or replicate of the experiment all possible combinations of the levels of the factors are investigated
- If there are a levels of factor A and b levels of factor B, each replicate contains all ab treatment combinations.

Informal way to find effects in Factorial Design



■ FIGURE 5.1 A two-factor factorial experiment, with the response (y) shown at the corners



■ FIGURE 5.3 A factorial experiment without interaction

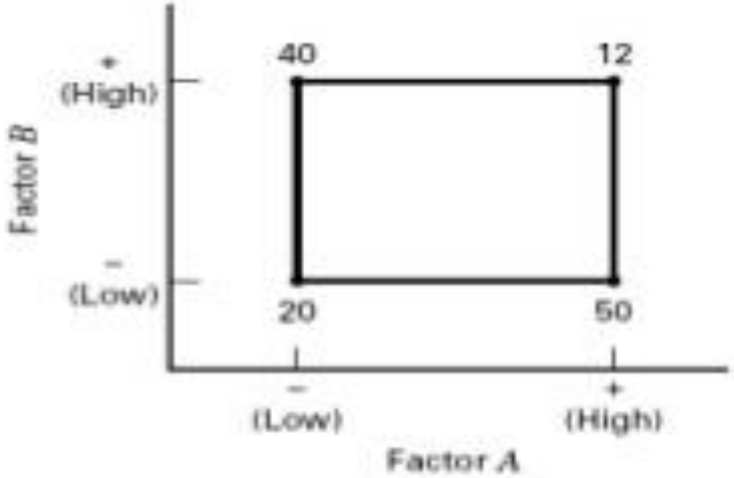
Definition of a factor effect: The change in the mean response when the factor is changed from low to high

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{40+52}{2} - \frac{20+30}{2} = 21$$

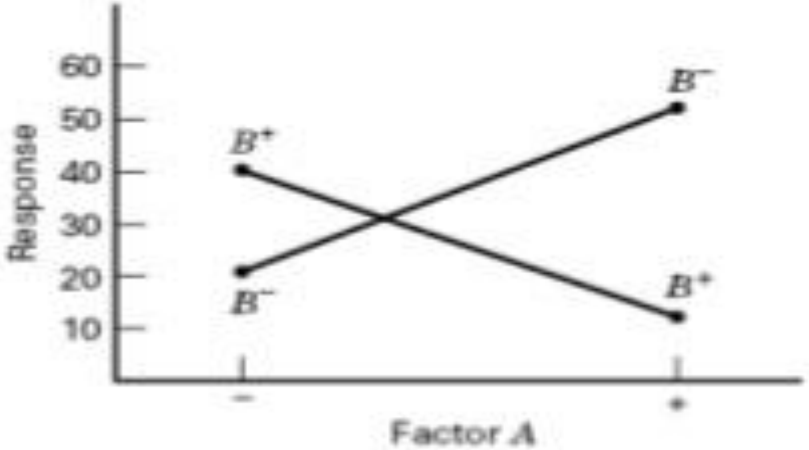
$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{30+52}{2} - \frac{20+40}{2} = 11$$

$$AB = \frac{52+20}{2} - \frac{30+40}{2} = -1$$

The Case of Interaction:



■ **FIGURE 5.2** A two-factor factorial experiment with interaction



■ **FIGURE 5.4** A factorial experiment with interaction

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{50+12}{2} - \frac{20+40}{2} = 1$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{40+12}{2} - \frac{20+50}{2} = -9$$

$$AB = \frac{12+20}{2} - \frac{40+50}{2} = -29$$

Formal ways to find effects in Factorial Design

- **Sign Method:**

2^2 Factorial Experiment

Possible combinations = 1, a, b, ab

$Effect\ of\ factor\ A = -1 + a - \mathrm{b} + \mathrm{ab}$

$Effect\ of\ factor\ B = -1 - a + \mathrm{b} + \mathrm{ab}$

$Interaction\ effect = +1 - a - \mathrm{b} + \mathrm{ab}$

Sign Table				
Effect	**1**	**a**	**b**	**ab**
A	-	+	-	+
B	-	-	+	+
AB	+	-	-	+

Sign Table for 2^3 Factorial Experiment								
Effect	**1**	**a**	**b**	**ab**	**c**	**ac**	**bc**	**abc**
A	-	+	-	+	-	+	-	+
B	-	-	+	+	-	-	+	+
AB	+	-	-	+	+	-	-	+
C	-	-	-	-	+	+	+	+
AC	+	-	+	-	-	+	-	+
BC	+	+	-	-	-	-	+	+
ABC	-	+	+	-	+	-	-	+

- **Yates Method**

2^2 **Factorial Experiment**			
1	a+1	ab+b+a+1	Grand effect
a	ab+b	ab-b+a-1	Effect of factor A
b	a-1	ab+b-a-1	Effect of factor B
ab	ab-b	ab-b-a+1	Interaction effect

2^3 **Factorial Experiment**			
1	a+1	ab+b+a+1	abc+bc+ac+c+ab+b+a+1
a	ab+b	abc+bc+ac+c	abc-bc+ac-c+ab-b+a-1
b	ac+c	ab-b+a-1	abc+bc-ac-c+ab+b-a-1
ab	abc+bc	abc-bc+ac-c	abc-bc-ac+c+ab-b-a+1
c	a-1	ab+b-a-1	abc+bc+ac+c-ab-b-a-1
ac	ab-b	abc+bc-ac-c	abc-bc+ac-c-ab+b-a+1
bc	ac-c	ab-b-a+1	abc+bc-ac-c-ab-b+a+1
abc	abc-bc	abc-bc-ac+c	abc-bc-ac+c-ab+b+a-1

- **Algebraic Method**

2^2 Factorial Experiment

Effect of factor A $= (a-1)(b+1) =$ ab−b+a−1

Effect of factor B $= (a+1)(b-1)$ =ab+b-a-1

Interaction effect(AB) $= (a-1)(b-1) =$ ab−b−a+1

2^3 Factorial Experiment

Effect of factor A $= (a-1)(b+1)(c+1) =$ abc−bc+ac−c+ab−b+a−1

Effect of factor B $= (a+1)(b-1)(c+1) =$ abc+bc−ac−c+ab+b−a−1

Effect of AB $= (a-1)(b-1)(c+1) =$ abc−bc−ac+c+ab−b−a+1

Effect of factor C $= (a+1)(b+1)(c-1) =$ abc+bc+ac+c−ab−b−a−1

Effect of AC $= (a-1)(b+1)(c-1) =$ abc−bc+ac−c−ab+b−a+1

Effect of BC $= (a+1)(b-1)(c-1) =$ abc+bc−ac−c−ab−b+a+1

Effect of ABC $= (a-1)(b-1)(c-1) =$ abc−bc−ac+c−ab+b+a−1

Example 5.1 The Battery Life Experiment
Text reference pg. 187

■ **TABLE 5.1**

Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F)					
	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

A = Material type; B = Temperature (A **quantitative** variable)

1. What **effects** do material type & temperature have on life?
2. Is there a choice of material that would give long life ***regardless of temperature*** (a **robust** product)?

The General Two-Factor Factorial Experiment

■ **TABLE 5.2**
General Arrangement for a Two-Factor Factorial Design

		Factor *B*			
		1	2	...	*b*
Factor *A*	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

a levels of factor *A*; *b* levels of factor *B*; *n* replicates

This is a **completely randomized design**

■ **TABLE 5.2**

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

Statistical (effects) model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

Other models (means model, regression models) can be useful

Extension of the ANOVA to Factorials (Fixed Effects Case) – pg. 189

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2 + an\sum_{j=1}^{b}(\bar{y}_{.j.}-\bar{y}_{...})^2$$

$$+n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2$$

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

df breakdown:

$$abn-1 = a-1+b-1+(a-1)(b-1)+ab(n-1)$$

ANOVA Table – Fixed Effects Case

▪ **TABLE 5.3**

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B treatments	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

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Second level

- Third level
  - Fourth level
    - Fifth level

The sums of squares for the main effects are

$$SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn} \tag{5.7}$$

and

$$SS_B = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{...}^2}{abn} \tag{5.8}$$

It is convenient to obtain the SS_{AB} in two stages. First we compute the sum of squares between the *ab* cell totals, which is called the sum of squares due to "subtotals":

$$SS_{\text{Subtotals}} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{...}^2}{abn}$$

This sum of squares also contains SS_A and SS_B. Therefore, the second step is to compute SS_{AB} as

$$SS_{AB} = SS_{\text{Subtotals}} - SS_A - SS_B \tag{5.9}$$

We may compute SS_E by subtraction as

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B \tag{5.10}$$

or

$$SS_E = SS_T - SS_{\text{Subtotals}}$$

■ **TABLE 5.4**

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155	(539)	34	40	(229)	20	70	(230)	
1	74	180		80	75		82	58		998
	150	188	(623)	136	122	(479)	25	70	(198)	
2	159	126		106	115		58	45		1300
	138	110	(576)	174	120	(583)	96	104	(342)	
3	168	160		150	139		82	60		1501
$y_{.j.}$		1738			1291			770		$3799 = y_{...}$

Table 5.4

■ **TABLE 5.5**

Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	< 0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

Table 5.5

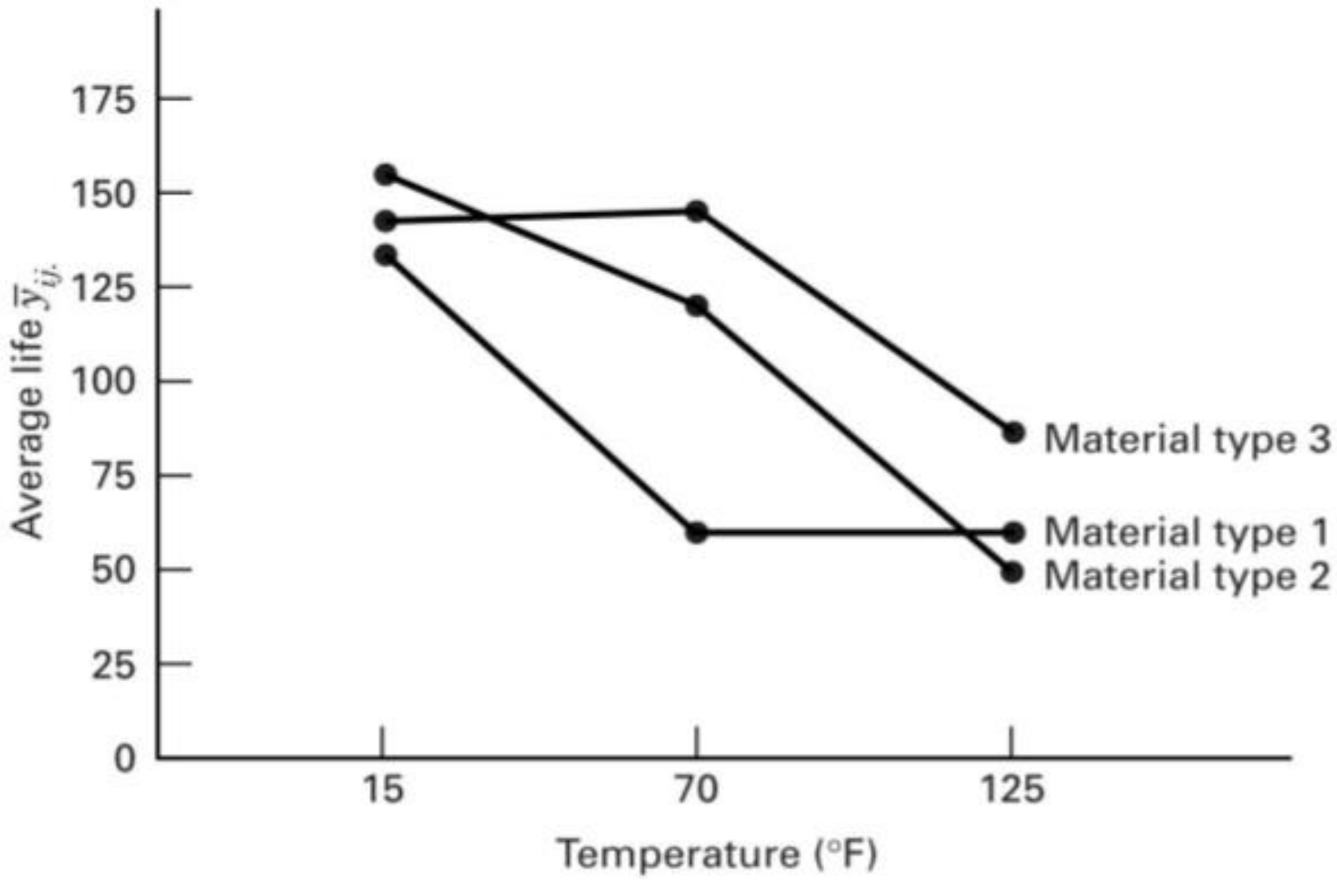
175
150
125
100
75
50
25
0
Average life $\bar{y}_{ij.}$
Material type 3
Material type 1
Material type 2
15
70
125
Temperature (°F)

Figure 5.9

Multiple Comparison

- When ANOVA indicates that row or column means are differ, it is usually of interest to make comparisons between the individual row or column means to discover specific differences
- Turkey's test
- When interaction is significant, comparisons between the means of one factor (e.g. A) may be obscured by the AB interaction
- Fix factor B at a specific level and apply Turkey's test toe means of factor A at that level