Chi-Square Distribution:

Let $z_1, z_2 \ldots z_n$ be normally and independently distributed variables with zero means and unit variances. Then the r.v expressed by the quantity

$$\chi^2 = \sum_{i=1}^{n} z_i^2$$ is defined as

a chi-square r.v with 'n' degrees of freedom that is χ^2 r.v is defined as the sum of squares of 'n' independent standard normal r.v's. It density function has following form.

$$f(\chi^2) = \frac{1}{(2)^{n/2}\Gamma n/2} (\chi^2)^{\frac{n-2}{2}} e^{-\chi^2/2} \qquad 0 \le \chi^2 \le \infty$$

or $f(\chi^2) = \frac{1}{(2)^{n/2}\Gamma n/2} (\chi^2)^{n/2-1} e^{-\chi^2/2}$

Random variable χ^2 having above P.d.f is a χ^2 distribution with n degree's of freedom denoted by $\chi^2(n)$ :: n is +ve integer.

Q#1 Show that χ^2 has a complete p.d.f.

Proof:

$$f(\chi^2) = \int_0^\infty \frac{1}{(2)^{n/2}\Gamma n/2} (\chi^2)^{n/2-1} e^{-\chi^2/2}\, d\chi^2$$

let $\chi^2 = y$ $\qquad dy = d\chi^2$

$$f(y) = \int_0^\infty \frac{1}{(2)^{n/2}\Gamma n/2} (y)^{n/2-1} e^{-y/2}\, dy$$

Let $y/2 = x$ $\qquad y = 2x$

$\cancel{y = x/2}$ $\qquad dy = 2dx$

when $y = \infty \qquad x = \infty$

$y = 0 \qquad x = 0$

$$f(x) = \int_0^{\infty} \frac{1}{(2)^{n/2}\Gamma n/2} (2x)^{n/2-1} e^{-x} 2\,dx$$

$$f(x) = \frac{1}{\cancel{(2)^{n/2}}\Gamma n/2} \cancel{(2)^{n/2-1+1}} \int_0^{\infty} x^{n/2-1} e^{-x} dx$$

$$= \frac{1}{\Gamma n/2} \int_0^{\infty} x^{n/2-1} e^{-x} dx$$

$$= \frac{1}{\Gamma n/2} \Gamma n/2 \boxed{= 1}$$

Hence Proved.

Q# Find the moment generating function of Chi-square distribution.

$$E(e^{tx}) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} e^{t\chi^2} \frac{1}{(2)^{n/2}\Gamma n/2} (\chi^2)^{n/2-1} e^{-\chi^2/2} d\chi^2$$

Let $\chi^2 = y$

$d\chi^2 = dy$

$$= \int_0^{\infty} e^{ty} \frac{1}{(2)^{n/2}\Gamma n/2} (y)^{n/2-1} e^{-y/2} dy$$

$$= \frac{1}{(2)^{n/2}\Gamma n/2} \int_0^{\infty} e^{ty-y/2} (y)^{n/2-1} dy$$

Chi-Square Distribution:

let $z_1, z_2 \dots z_n$ be normally and independently distributed variable

$$= \frac{1}{(2)^{n/2}\Gamma n/2}\int_0^\infty e^{-y/2(-2t+1)}(y)^{n/2-1}\,dy$$

$$x = \frac{y}{2}(1-2t)$$

$$y = \frac{2x}{1-2t} \quad \Big]\ \text{limits same}$$

$$dy = \frac{2}{1-2t}\,dx$$

$$= \frac{1}{(2)^{n/2}\Gamma n/2}\int_0^\infty e^{-x}\left(\frac{2x}{1-2t}\right)^{n/2-1}\frac{2}{1-2t}\,dx$$

$$= \frac{1}{2^{n/2}\Gamma n/2}\left(\frac{2}{1-2t}\right)^{n/2-\cancel{1}+\cancel{1}}\int_0^\infty e^{-x}x^{n/2-1}\,dx.$$

$$= \frac{1}{\cancel{2^{n/2}}\Gamma n/2}\cancel{2^{n/2}}\left(\frac{1}{1-2t}\right)^{n/2}\int_0^\infty e^{-x}x^{n/2-1}\,dx$$

$$= \frac{1}{\cancel{\Gamma n/2}}\left(\frac{1}{1-2t}\right)^{n/2}\cancel{\Gamma n/2}$$

$$\boxed{M_x(t) = \frac{1}{(1-2t)^{n/2}}}$$

or

$$M_x(t) = \cancel{1}(1-2t)^{-n/2}$$

Q#3 Find the mean & variance of χ^2 by using m.g.f.

$M_x(t) = (1-2t)^{-n/2}$

$M'_x(t) = \frac{-n}{\cancel{2}}(1-2t)^{-n/2-1}(-\cancel{2})$

$= n(1-2t)^{-n/2-1}$

$t=0$

$M'_x(0) = n(1-0)^{-n/2-1}$

$\boxed{\mu_1' = n}$

Taking 2nd derivative.

$M''_x(t) = n\left(\frac{-n}{2}-1\right)(1-2t)^{n/2-1-1}(-2)$

$= 2n\left(\frac{n}{2}+1\right)(1-2t)^{n/2-2}$

$= 2n\left(\frac{n+2}{2}\right)(1-2t)^{n/2-2}$

$t=0$

$M''_x(0) = \cancel{2}n\left(\frac{n+2}{\cancel{2}}\right) \Rightarrow \mu_2' = n^2+2n$

$var(X) = \mu_2' - (\mu_1')^2$

$= \cancel{n^2}+2n-\cancel{n^2}$

$var(X) = \mu_2 \boxed{= 2n}$