# Case Studies: Numerical Integration and Differentiation

The purpose of this chapter is to apply the methods of numerical integration and differentiation discussed in Part Six to practical engineering problems. Two situations are most frequently encountered. In the first case, the function under study can be expressed in analytic form but is too complicated to be readily evaluated using the methods of calculus. Numerical methods are applied to situations of this type by using the analytic expression to generate a table of arguments and function values. In the second case, the function to be evaluated is inherently tabular in nature. This type of function usually represents a series of measurements, observations, or some other empirical information. Data for either case are directly compatible with several schemes discussed in this part of the book.

Section 24.1, which deals with heat calculations from chemical engineering, involves equations. In this application, an analytic function is integrated numerically to determine the heat required to raise the temperature of a material.

Sections 24.2 and 24.3 also involve functions that are available in equation form. Section 24.2, which is taken from civil engineering, uses numerical integration to determine the total wind force acting on the mast of a racing sailboat. Section 24.3 determines the root-mean-square current for an electric circuit. This example is used to demonstrate the utility of Romberg integration and Gauss quadrature.

Section 24.4 focuses on the analysis of tabular information to determine the work required to move a block. Although this application has a direct connection with mechanical engineering, it is germane to all other areas of engineering. Among other things, we use this example to illustrate the integration of unequally spaced data.

## 24.1 INTEGRATION TO DETERMINE THE TOTAL QUANTITY OF HEAT (CHEMICAL/BIO ENGINEERING)

Background. Heat calculations are employed routinely in chemical and bio engineering as well as in many other fields of engineering. This application provides a simple but useful example of such computations.

One problem that is often encountered is the determination of the quantity of heat required to raise the temperature of a material. The characteristic that is needed to carry out this computation is the heat capacity $c$. This parameter represents the quantity of

**673**

heat required to raise a unit mass by a unit temperature. If $c$ is constant over the range of temperatures being examined, the required heat $\Delta H$ (in calories) can be calculated by

$$\Delta H = mc \, \Delta T \tag{24.1}$$

where $c$ has units of cal/(g · °C), $m$ = mass (g), and $\Delta T$ = change in temperature (°C). For example, the amount of heat required to raise 20 g of water from 5 to 10°C is equal to

$$\Delta H = 20(1)(10 - 5) = 100 \text{ cal}$$

where the heat capacity of water is approximately 1 cal/(g · °C). Such a computation is adequate when $\Delta T$ is small. However, for large ranges of temperature, the heat capacity is not constant and, in fact, varies as a function of temperature. For example, the heat capacity of a material could increase with temperature according to a relationship such as

$$c(T) = 0.132 + 1.56 \times 10^{-4}T + 2.64 \times 10^{-7}T^2 \tag{24.2}$$

In this instance you are asked to compute the heat required to raise 1000 g of this material from $-100$ to 200°C.

**Solution.** Equation (PT6.4) provides a way to calculate the average value of $c(T)$:

$$\bar{c}(T) = \frac{\displaystyle\int_{T_1}^{T_2} c(T) \, dT}{T_2 - T_1} \tag{24.3}$$

which can be substituted into Eq. (24.1) to yield

$$\Delta H = m \int_{T_1}^{T_2} c(T) \, dT \tag{24.4}$$

where $\Delta T = T_2 - T_1$. Now because, for the present case, $c(T)$ is a simple quadratic, $\Delta H$ can be determined analytically. Equation (24.2) is substituted into Eq. (24.4) and the result integrated to yield an exact value of $\Delta H = 42{,}732$ cal. It is useful and instructive to compare this result with the numerical methods developed in Chap. 21. To accomplish this, it is necessary to generate a table of values of $c$ for various values of $T$:

| T, °C | c, cal/(g · °C) |
|-------|-----------------|
| -100  | 0.11904         |
| -50   | 0.12486         |
| 0     | 0.13200         |
| 50    | 0.14046         |
| 100   | 0.15024         |
| 150   | 0.16134         |
| 200   | 0.17376         |

These points can be used in conjunction with a six-segment Simpson's 1/3 rule to compute an integral estimate of 42,732. This result can be substituted into Eq. (24.4) to yield a value of $\Delta H = 42{,}732$ cal, which agrees exactly with the analytical solution. This exact agreement would occur no matter how many segments were used. This is to be expected because $c$ is a quadratic function and Simpson's rule is exact for polynomials of the third order or less (see Sec. 21.2.1).

**TABLE 24.1** Results using the trapezoidal rule with various step sizes.

| Step Size, °C | $\Delta H$ | $\varepsilon_t$ (%) |
|---|---|---|
| 300 | 96,048 | 125 |
| 150 | 43,029 | 0.7 |
| 100 | 42,864 | 0.3 |
| 50 | 42,765 | 0.07 |
| 25 | 42,740 | 0.018 |
| 10 | 42,733.3 | <0.01 |
| 5 | 42,732.3 | <0.01 |
| 1 | 42,732.01 | <0.01 |
| 0.05 | 42,732.00003 | <0.01 |

The results using the trapezoidal rule are listed in Table 24.1. It is seen that the trapezoidal rule is also capable of estimating the total heat very accurately. However, a small step ($< 10°C$) is required for five-place accuracy. The same calculation can also be implemented with software. For example, MATLAB software yields

```
>> m=1000;
>> DH=m*quad(@(T) 0.132+1.56e-4*T+2.64e-7*T.^2,-100,200)

DH =
    42732
```

## 24.2 EFFECTIVE FORCE ON THE MAST OF A RACING SAILBOAT (CIVIL/ENVIRONMENTAL ENGINEERING)

Background. A cross section of a racing sailboat is shown in Fig. 24.1a. Wind forces ($f$) exerted per foot of mast from the sails vary as a function of distance above the deck

**FIGURE 24.1**
(a) Cross section of a racing sailboat. (b) Wind forces $f$ exerted per foot of mast as a function of distance $z$ above the deck of the boat.
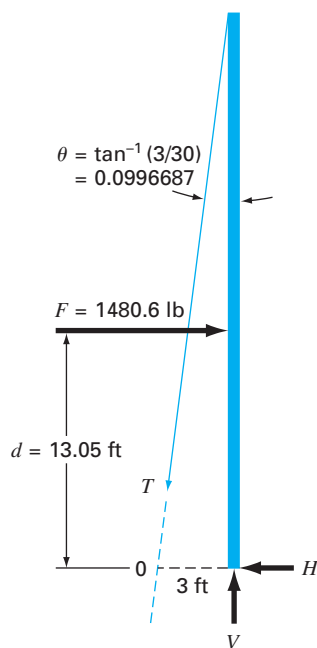
$\theta = \tan^{-1}(3/30)$
$= 0.0996687$

$F = 1480.6$ lb

$d = 13.05$ ft

$T$

0

3 ft

$H$

$V$
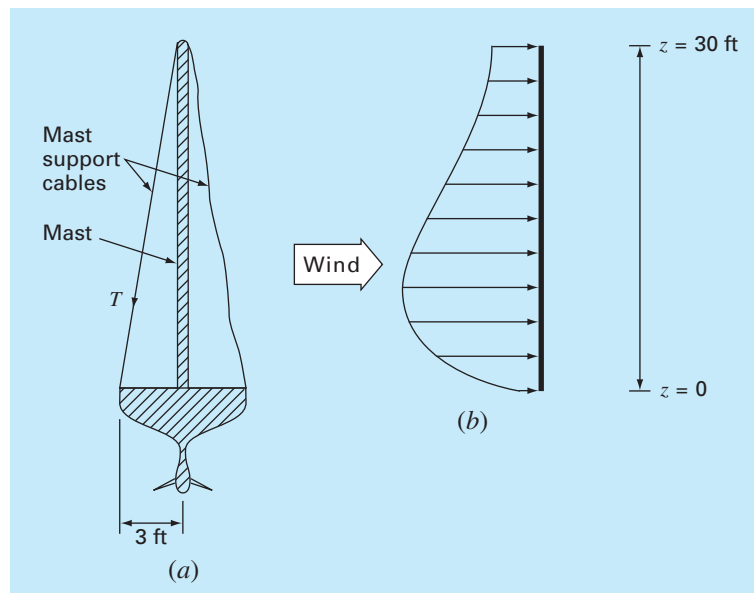
**FIGURE 24.2**
Free-body diagram of the forces exerted on the mast of a sailboat.

of the boat ($z$), as in Fig. 24.1$b$. Calculate the tensile force $T$ in the left mast support cable, assuming that the right support cable is completely slack and the mast joins the deck in a manner that transmits horizontal or vertical forces but no moments. Assume that the mast remains vertical.

Solution.    To proceed with this problem, it is required that the distributed force $f$ be converted to an equivalent total force $F$ and that its effective location above the deck $d$ be calculated (Fig. 24.2). This computation is complicated by the fact that the force exerted per foot of mast varies with the distance above the deck. The total force exerted on the mast can be expressed as the integral of a continuous function:

$$F = \int_0^{30} 200\left(\frac{z}{5+z}\right)e^{-2z/30}\,dz \tag{24.5}$$

This nonlinear integral is difficult to evaluate analytically. Therefore, it is convenient to employ numerical approaches such as Simpson's rule and the trapezoidal rule for this problem. This is accomplished by calculating $f(z)$ for various values of $z$ and then using Eq. (21.10) or (21.18). For example, Table 24.2 has values of $f(z)$ for a step size of 3 ft that provide data for Simpson's 1/3 rule or the trapezoidal rule. Results for several step sizes are given in Table 24.3. It is observed that both methods give a value of $F = 1480.6$ lb as the step size becomes small. In this case, step sizes of 0.05 ft for the trapezoidal rule and 0.5 ft for Simpson's rule provide good results.

**TABLE 24.2** Values of $f(z)$ for a step size of 3 ft that provide data for the trapezoidal rule and Simpson's 1/3 rule.

| $z$, ft | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(z)$, lb/ft | 0 | 61.40 | 73.13 | 70.56 | 63.43 | 55.18 | 47.14 | 39.83 | 33.42 | 27.89 | 23.20 |

**TABLE 24.3** Values of $F$ computed on the basis of various versions of the trapezoidal rule and Simpson's 1/3 rule.

| Technique | Step Size, ft | Segments | $F$, lb |
|---|---|---|---|
| Trapezoidal rule | 15 | 2 | 1001.7 |
| | 10 | 3 | 1222.3 |
| | 6 | 5 | 1372.3 |
| | 3 | 10 | 1450.8 |
| | 1 | 30 | 1477.1 |
| | 0.5 | 60 | 1479.7 |
| | 0.25 | 120 | 1480.3 |
| | 0.1 | 300 | 1480.5 |
| | 0.05 | 600 | 1480.6 |
| Simpson's 1/3 rule | 15 | 2 | 1219.6 |
| | 5 | 6 | 1462.9 |
| | 3 | 10 | 1476.9 |
| | 1 | 30 | 1480.5 |
| | 0.5 | 60 | 1480.6 |

The effective line of action of $d$ (Fig. 24.2) can be calculated by evaluation of the integral

$$d = \frac{\displaystyle\int_0^{30} zf(z)\,dz}{\displaystyle\int_0^{30} f(z)\,dz} \tag{24.6}$$

or

$$d = \frac{\displaystyle\int_0^{30} 200z[z/(5 + z)]e^{-2z/30}\,dz}{1480.6} \tag{24.7}$$

This integral can be evaluated using methods similar to the above. For example, Simpson's 1/3 rule with a step size of 0.5 gives $d = 19{,}326.9/1480.6 = 13.05$ ft.

With $F$ and $d$ known from numerical methods, a free-body diagram is used to develop force and moment balance equations. This free-body diagram is shown in Fig. 24.2. Summing forces in the horizontal and vertical direction and taking moments about point 0 gives

$$\sum F_H = 0 = F - T\sin\theta - H \tag{24.8}$$
$$\sum F_V = 0 = V - T\cos\theta \tag{24.9}$$
$$\sum M_0 = 0 = 3V - Fd \tag{24.10}$$

where $T$ = the tension in the cable and $H$ and $V$ = the unknown reactions on the mast transmitted by the deck. The direction, as well as the magnitude, of $H$ and $V$ is unknown. Equation (24.10) can be solved directly for $V$ because $F$ and $d$ are known.

$$V = \frac{Fd}{3} = \frac{(1480.6)(13.05)}{3} = 6440.6 \text{ lb}$$

Therefore, from Eq. (24.9),

$$T = \frac{V}{\cos\theta} = \frac{6440.6}{0.995} = 6473 \text{ lb}$$

and from Eq. (24.8),

$$H = F - T\sin\theta = 1480.6 - (6473)(0.0995) = 836.54 \text{ lb}$$

These forces now enable you to proceed with other aspects of the structural design of the boat such as the cables and the deck support system for the mast. This problem illustrates nicely two uses of numerical integration that may be encountered during the engineering design of structures. It is seen that both the trapezoidal rule and Simpson's 1/3 rule are easy to apply and are practical problem-solving tools. Simpson's 1/3 rule is more accurate than the trapezoidal rule for the same step size and thus may often be preferred.

## 24.3   ROOT-MEAN-SQUARE CURRENT BY NUMERICAL INTEGRATION (ELECTRICAL ENGINEERING)

Background.   The average value of an oscillating electric current over one period may be zero. For example, suppose that the current is described by a simple sinusoid: $i(t) = \sin(2\pi t/T)$, where $T$ is the period. The average value of this function can be determined
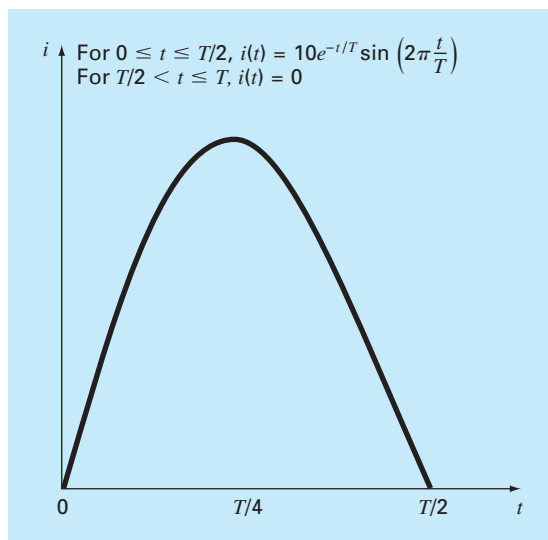
**FIGURE 24.3**
A periodically varying electric current.

by the following equation:

$$i = \frac{\int_0^T \sin\left(\frac{2\pi t}{T}\right) dt}{T - 0} = \frac{-\cos(2\pi) + \cos 0}{T} = 0$$

Despite the fact that the net result is zero, such current is capable of performing work and generating heat. Therefore, electrical engineers often characterize such current by

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\, dt} \tag{24.11}$$

where $i(t)$ = the instantaneous current. Calculate the RMS or root-mean-square current of the waveform shown in Fig. 24.3 using the trapezoidal rule, Simpson's 1/3 rule, Romberg integration, and Gauss quadrature for $T = 1$ s.

Solution.    Integral estimates for various applications of the trapezoidal rule and Simpson's 1/3 rule are listed in Table 24.4. Notice that Simpson's rule is more accurate than the trapezoidal rule.

The exact value for the integral is 15.41261. This result is obtained using a 128-segment trapezoidal rule or a 32-segment Simpson's rule. The same estimate is also determined using Romberg integration (Fig. 24.4).

In addition, Gauss quadrature can be used to make the same estimate. The determination of the root-mean-square current involves the evaluation of the integral ($T = 1$)

$$I = \int_0^{1/2} \left(10e^{-t}\sin 2\pi t\right)^2 dt \tag{24.12}$$

**TABLE 24.4** Values for the integral calculated using various numerical schemes. The percent relative error $\varepsilon_t$ is based on a true value of 15.41261.

| Technique | Segments | Integral | $\varepsilon_t$ (%) |
|---|---|---|---|
| Trapezoidal rule | 1 | 0.0 | 100 |
| | 2 | 15.16327 | 1.62 |
| | 4 | 15.40143 | 0.0725 |
| | 8 | 15.41196 | $4.21 \times 10^{-3}$ |
| | 16 | 15.41257 | $2.59 \times 10^{-4}$ |
| | 32 | 15.41261 | $1.62 \times 10^{-5}$ |
| | 64 | 15.41261 | $1.30 \times 10^{-6}$ |
| | 128 | 15.41261 | 0 |
| Simpson's 1/3 rule | 2 | 20.21769 | −31.2 |
| | 4 | 15.48082 | −0.443 |
| | 8 | 15.41547 | −0.0186 |
| | 16 | 15.41277 | $1.06 \times 10^{-3}$ |
| | 32 | 15.41261 | 0 |

**FIGURE 24.4**
Result of using Romberg integration to estimate the RMS current.

| $O(h^2)$ | $O(h^4)$ | $O(h^6)$ | $O(h^8)$ | $O(h^{10})$ | $O(h^{12})$ |
|---|---|---|---|---|---|
| 0 | 20.21769 | 15.16503 | 15.41502 | 15.41261 | 15.41261 |
| 15.16327 | 15.48082 | 15.41111 | 15.41262 | 15.41261 | |
| 15.40143 | 15.41547 | 15.41225 | 15.41261 | | |
| 15.41196 | 15.41277 | 15.41261 | | | |
| 15.41257 | 15.41262 | | | | |
| 15.41261 | | | | | |

First, a change in variable is performed by applying Eqs. (22.29) and (22.30) to yield

$$t = \frac{1}{4} + \frac{1}{4}t_d \qquad dt = \frac{1}{4}dt_d$$

These relationships can be substituted into Eq. (24.12) to yield

$$I = \int_{-1}^{1} \left[ 10e^{-[1/4 + (1/4)t_d]} \sin 2\pi \left( \frac{1}{4} + \frac{1}{4}t_d \right) \right]^2 \frac{1}{4} \, dt \qquad (24.13)$$

For the two-point Gauss-Legendre formula, this function is evaluated at $t_d = -1/\sqrt{3}$ and $1/\sqrt{3}$, with the results being 7.684096 and 4.313728, respectively. These values can be substituted into Eq. (22.23) to yield an integral estimate of 11.99782, which represents an error of $\varepsilon_t = 22.1\%$.

The three-point formula is (Table 22.1)

$$I = 0.5555556(1.237449) + 0.8888889(15.16327) + 0.5555556(2.684915)$$

$$= 15.65755 \qquad |\varepsilon_t| = 1.6$$

The results of using the higher-point formulas are summarized in Table 24.5.

**TABLE 24.5** Results of using various-point Gauss quadrature formulas to approximate the integral.

| Points | Estimate | $\varepsilon_t$ (%) |
|--------|----------|-----------|
| 2 | 11.9978243 | 22.1 |
| 3 | 15.6575502 | −1.59 |
| 4 | 15.4058023 | $4.42 \times 10^{-2}$ |
| 5 | 15.4126391 | $-2.01 \times 10^{-4}$ |
| 6 | 15.4126109 | $-1.82 \times 10^{-5}$ |

The integral estimate of 15.41261 can be substituted into Eq. (24.12) to compute an $I_{RMS}$ of 3.925890 A. This result could then be employed to guide other aspects of the design and operation of the circuit.

## 24.4 NUMERICAL INTEGRATION TO COMPUTE WORK (MECHANICAL/AEROSPACE ENGINEERING)

Background.    Many engineering problems involve the calculation of work. The general formula is

Work = force × distance

When you were introduced to this concept in high school physics, simple applications were presented using forces that remained constant throughout the displacement. For example, if a force of 10 lb was used to pull a block a distance of 15 ft, the work would be calculated as 150 ft · lb.

Although such a simple computation is useful for introducing the concept, realistic problem settings are usually more complex. For example, suppose that the force varies during the course of the calculation. In such cases, the work equation is reexpressed as

$$W = \int_{x_0}^{x_n} F(x)\, dx \tag{24.14}$$

where $W$ = work (ft · lb), $x_0$ and $x_n$ = the initial and final positions, respectively, and $F(x)$ a force that varies as a function of position. If $F(x)$ is easy to integrate, Eq. (24.14) can be evaluated analytically. However, in a realistic problem setting, the force might not be expressed in such a manner. In fact, when analyzing measured data, the force might be available only in tabular form. For such cases, numerical integration is the only viable option for the evaluation.

Further complexity is introduced if the angle between the force and the direction of movement also varies as a function of position (Fig. 24.5). The work equation can be modified further to account for this effect, as in

$$W = \int_{x_0}^{x_n} F(x)\cos\left[\theta(x)\right]\, dx \tag{24.15}$$

Again, if $F(x)$ and $\theta(x)$ are simple functions, Eq. (24.15) might be solved analytically. However, as in Fig. 24.5, it is more likely that the functional relationship is complicated. For this situation, numerical methods provide the only alternative for determining the integral.
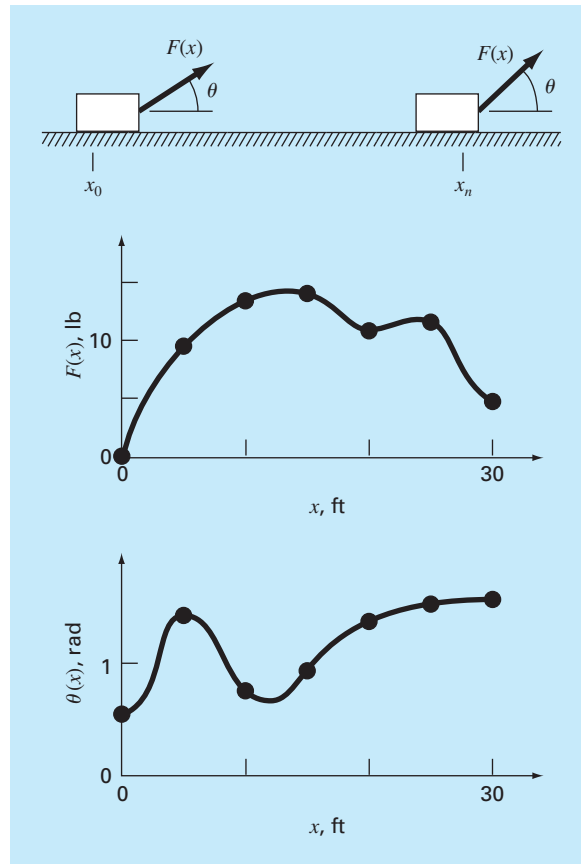
**FIGURE 24.5**
The case of a variable force acting on a block. For this case, the angle, as well as the magnitude, of the force varies.

**TABLE 24.6** Data for force $F(x)$ and angle $\theta(x)$ as a function of position $x$.

| x, ft | F(x), lb | θ, rad | F(x) cos θ |
|-------|----------|--------|------------|
| 0     | 0.0      | 0.50   | 0.0000     |
| 5     | 9.0      | 1.40   | 1.5297     |
| 10    | 13.0     | 0.75   | 9.5120     |
| 15    | 14.0     | 0.90   | 8.7025     |
| 20    | 10.5     | 1.30   | 2.8087     |
| 25    | 12.0     | 1.48   | 1.0881     |
| 30    | 5.0      | 1.50   | 0.3537     |

Suppose that you have to perform the computation for the situation depicted in Fig. 24.5. Although the figure shows the continuous values for $F(x)$ and $\theta(x)$, assume that, because of experimental constraints, you are provided with only discrete measurements at $x = 5$-ft intervals (Table 24.6). Use single- and multiple-application versions of the trapezoidal rule and Simpson's 1/3 and 3/8 rules to compute work for this data.

**Solution.** The results of the analysis are summarized in Table 24.7. A percent relative error $\varepsilon_t$ was computed in reference to a true value of the integral of 129.52 that was estimated on the basis of values taken from Fig. 24.5 at 1-ft intervals.

The results are interesting because the most accurate outcome occurs for the simple two-segment trapezoidal rule. More refined estimates using more segments, as well as Simpson's rules, yield less accurate results.

The reason for this apparently counterintuitive result is that the coarse spacing of the points is not adequate to capture the variations of the forces and angles. This is particularly evident in Fig. 24.6, where we have plotted the continuous curve for the product of $F(x)$ and cos $[\theta(x)]$. Notice how the use of seven points to characterize the continuously varying function misses the two peaks at $x = 2.5$ and 12.5 ft. The omission of these two points effectively limits the accuracy of the numerical integration estimates in Table 24.7. The fact that the two-segment trapezoidal rule yields the most accurate result is due to the chance positioning of the points for this particular problem (Fig. 24.7).

The conclusion to be drawn from Fig. 24.6 is that an adequate number of measurements must be made to accurately compute integrals. For the present case, if data were

**TABLE 24.7** Estimates of work calculated using the trapezoidal rule and Simpson's rules. The percent relative error $\varepsilon_t$ as computed in reference to a true value of the integral (129.52 ft · lb) that was estimated on the basis of values at 1-ft intervals.

| Technique | Segments | Work | $\varepsilon_t$, % |
|---|---|---|---|
| Trapezoidal | 1 | 5.31 | 95.9 |
| | 2 | 133.19 | 2.84 |
| | 3 | 124.98 | 3.51 |
| | 6 | 119.09 | 8.05 |
| Simpson's 1/3 rule | 2 | 175.82 | −35.75 |
| | 6 | 117.13 | 9.57 |
| Simpson's 3/8 rule | 3 | 139.93 | −8.04 |

**FIGURE 24.6**
A continuous plot of $F(x)$ cos $[\theta(x)]$ versus position with the seven discrete points used to develop the numerical integration estimates in Table 24.7. Notice how the use of seven points to characterize this continuously varying function misses two peaks at $x = 2.5$ and 12.5 ft.
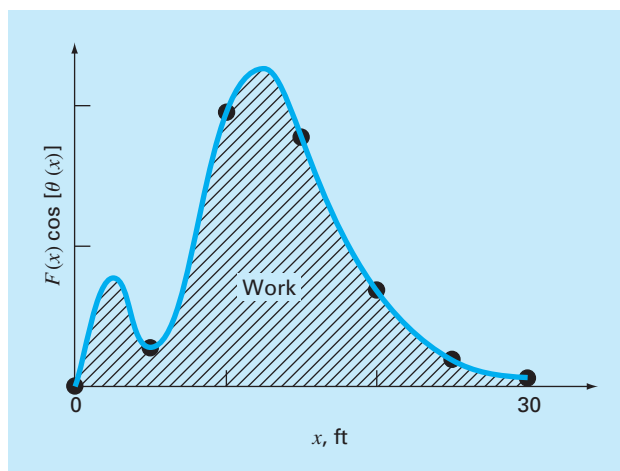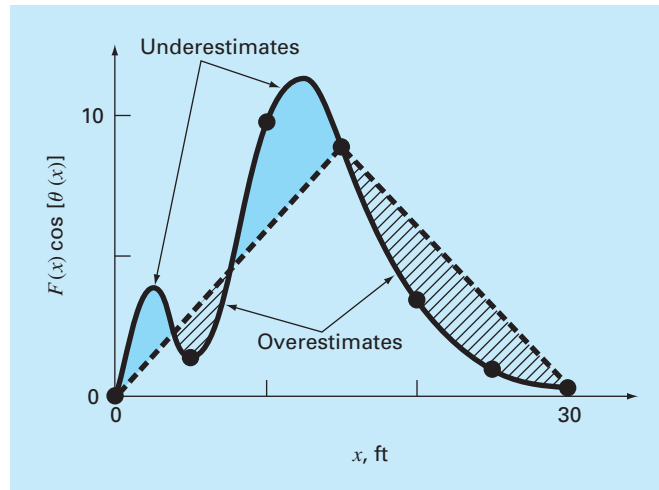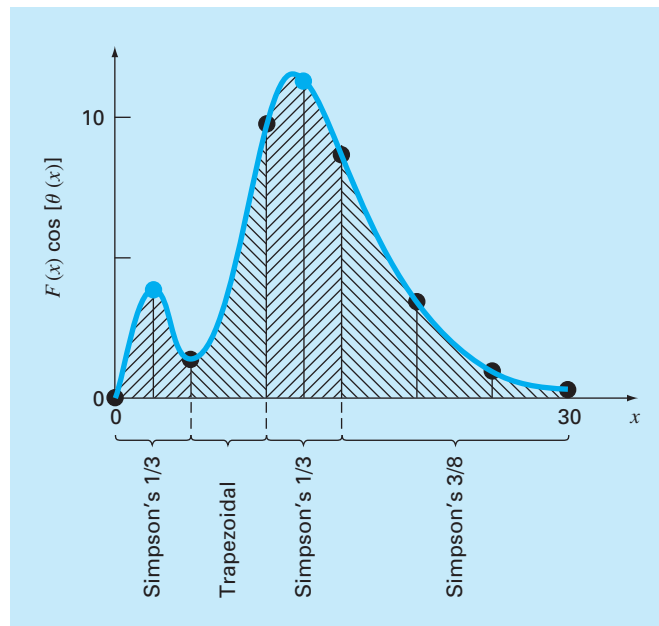
**FIGURE 24.7**
Graphical depiction of why the two-segment trapezoidal rule yields a good estimate of the integral for this particular case. By chance, the use of two trapezoids happens to lead to an even balance between positive and negative errors.



**FIGURE 24.8**
The unequal segmentation scheme that results from the inclusion of two additional points at $x = 2.5$ and $12.5$ in the data in Table 24.6. The numerical integration formulas applied to each set of segments are shown.



available at $F(2.5) \cos [\theta(2.5)] = 4.3500$ and $F(12.5) \cos [\theta(12.5)] = 11.3600$, we could determine an integral estimate using the algorithm for unequally spaced data described previously in Sec. 21.3. Figure 24.8 illustrates the unequal segmentation for this case. Including the two additional points yields an improved integral estimate of 126.9 ($\varepsilon_t = 2.02\%$). Thus, the inclusion of the additional data would incorporate the peaks that were missed previously and, as a consequence, lead to better results.

# PROBLEMS

**Chemical/Bio Engineering**

**24.1** Perform the same computation as Sec. 24.1, but compute the amount of heat required to raise the temperature of 1200 g of the material from $-150$ to $100°C$. Use Simpson's rule for your computation, with values of $T$ at 50°C increments.

**24.2** Repeat Prob. 24.1, but use Romberg integration to $\varepsilon_s = 0.01\%$.

**24.3** Repeat Prob. 24.1, but use a two- and a three-point Gauss-Legendre formula. Interpret your results.

**24.4** Integration provides a means to compute how much mass enters or leaves a reactor over a specified time period, as in

$$M = \int_{t_1}^{t_2} Qc \, dt$$

where $t_1$ and $t_2 =$ the initial and final times, respectively. This formula makes intuitive sense if you recall the analogy between integration and summation. Thus, the integral represents the summation of the product of flow times concentration to give the total mass entering or leaving from $t_1$ to $t_2$. If the flow rate is constant, $Q$ can be moved outside the integral:

$$M = Q \int_{t_1}^{t_2} c \, dt \tag{P24.4.1}$$

Use numerical integration to evaluate this equation for the data listed below. Note that $Q = 4 \text{ m}^3/\text{min}$.

| $t$, min | 0 | 10 | 20 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|
| $c$, mg/m$^3$ | 10 | 35 | 55 | 52 | 40 | 37 | 32 | 34 |

**24.5** Use numerical integration to compute how much mass leaves a reactor based on the following measurements.

| $t$, min | 0 | 10 | 20 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|
| $Q$, m$^3$/min | 4 | 4.8 | 5.2 | 5.0 | 4.6 | 4.3 | 4.3 | 5.0 |
| $c$, mg/m$^3$ | 10 | 35 | 55 | 52 | 40 | 37 | 32 | 34 |

**24.6** *Fick's first diffusion law* states that

$$\text{Mass flux} = -D\frac{dc}{dx} \tag{P24.6.1}$$

where mass flux $=$ the quantity of mass that passes across a unit area per unit time (g/cm$^2$/s), $D =$ a diffusion coefficient (cm$^2$/s), $c =$ concentration, and $x =$ distance (cm). An environmental engineer measures the following concentration of a pollutant in the sediments underlying a lake ($x = 0$ at the sediment-water interface and increases downward):

| $x$, cm | 0 | 1 | 3 |
|---|---|---|---|
| $c$, $10^{-6}$ g/cm$^3$ | 0.06 | 0.32 | 0.6 |

Use the best numerical differentiation technique available to estimate the derivative at $x = 0$. Employ this estimate in conjunction with Eq. (P24.6.1) to compute the mass flux of pollutant out of the sediments and into the overlying waters ($D = 1.52 \times 10^{-6}$ cm$^2$/s). For a lake with $3.6 \times 10^6$ m$^2$ of sediments, how much pollutant would be transported into the lake over a year's time?

**24.7** The following data were collected when a large oil tanker was loading:

| $t$, min | 0 | 10 | 20 | 30 | 45 | 60 | 75 |
|---|---|---|---|---|---|---|---|
| $V$, $10^6$ barrels | 0.4 | 0.7 | 0.77 | 0.88 | 1.05 | 1.17 | 1.35 |

Calculate the flow rate $Q$ (that is, $dV/dt$) for each time to the order of $h^2$.

**24.8** You are interested in measuring the fluid velocity in a narrow rectangular open channel carrying petroleum waste between locations in an oil refinery. You know that, because of bottom friction, the velocity varies with depth in the channel. If your technician has time to perform only two velocity measurements, at what depths would you take them to obtain the best estimate of the average velocity? State your recommendation in terms of the percent of total depth $d$ measured from the fluid surface. For example, measuring at the top would be $0\%d$, whereas at the very bottom would be $100\%d$.

**24.9** Soft tissue follows an exponential deformation behavior in uniaxial tension while it is in the physiologic or normal range of elongation. This can be expressed as

$$\sigma = \frac{E_o}{a}(e^{a\varepsilon} - 1)$$

where $\sigma =$ stress, $\varepsilon =$ strain, and $E_o$ and $a$ are material constants that are determined experimentally. To evaluate the two material constants, the above equation is differentiated with respect to $\varepsilon$, which is a fundamental relationship for soft tissue

$$\frac{d\sigma}{d\varepsilon} = E_o + a\sigma$$

To evaluate $E_o$ and $a$, stress-strain data are used to plot $d\sigma/d\varepsilon$ versus $\sigma$ and the slope and intercept of this plot are the two material constants, respectively. The table contains stress-strain data for heart chordae tendineae (small tendons use to hold heart valves closed during contraction of the heart muscle). This is data from loading the tissue; different curves are produced on unloading.

| $\sigma \times 10^3$ N/m² | 87.8 | 96.6 | 176 | 263 | 350 | 569 | 833 | 1227 | 1623 | 2105 | 2677 | 3378 | 4257 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon \times 10^{-3}$ m/m | 153 | 198 | 270 | 320 | 355 | 410 | 460 | 512 | 562 | 614 | 664 | 716 | 766 |

**(a)** Calculate the derivative $d\sigma/d\varepsilon$ using finite differences that are second-order accurate. Plot the data and eliminate the data points near the zero points that appear not to follow the straight-line relationship. The error in these data comes from the inability of the instrumentation to read the small values in this region. Perform a regression analysis of the remaining data points to determine the values of $E_o$ and $a$. Plot the stress versus strain data points along with the analytic curve expressed by the first equation. This will indicate how well the analytic curve matches these data.

**(b)** Often the previous analysis does not work well because the value of $E_o$ is difficult to evaluate. To solve this problem, $E_o$ is not used. A data point is selected $(\overline{\sigma}, \overline{\varepsilon})$ that is in the middle of the range used for the regression analysis. These values are substituted into the first equation, and a value for $E_o/a$ is determined and substituted into the first equation:

$$\sigma = \left( \frac{\overline{\sigma}}{e^{a\overline{\varepsilon}} - 1} \right)(e^{a\varepsilon} - 1)$$

Using this approach, experimental data that are well defined will produce a good match of the data points and the analytic curve. Use this new relationship and again plot the stress versus the strain data points and the new analytic curve.

**24.10** The standard technique for determining cardiac output is the indicator dilution method developed by Hamilton. One end of a small catheter is inserted into the radial artery and the other end is connected to a densitometer, which can automatically record the concentration of the dye in the blood. A known amount of dye, 5.6 mg, is injected rapidly, and the following data are obtained:

| Time, s | Concentration, mg/L | Time, s | Concentration, mg/L |
|---|---|---|---|
| 5 | 0 | 21 | 2.3 |
| 7 | 0.1 | 23 | 1.1 |
| 9 | 0.11 | 25 | 0.9 |
| 11 | 0.4 | 27 | 1.75 |
| 13 | 4.1 | 29 | 2.06 |
| 15 | 9.1 | 31 | 2.25 |
| 17 | 8 | 33 | 2.32 |
| 19 | 4.2 | 35 | 2.43 |

Plotting the above data results in the dye dilution curve in Fig. P24.10a. The concentration reaches a maximum value at about 15 seconds and then falls off, followed by a rise due to the recirculation of dye. The curve is replotted on a semilog graph in Fig. P24.10b. Notice that a straight line approximates the descending limb of the

**FIGURE P24.10**



(a)

(b)
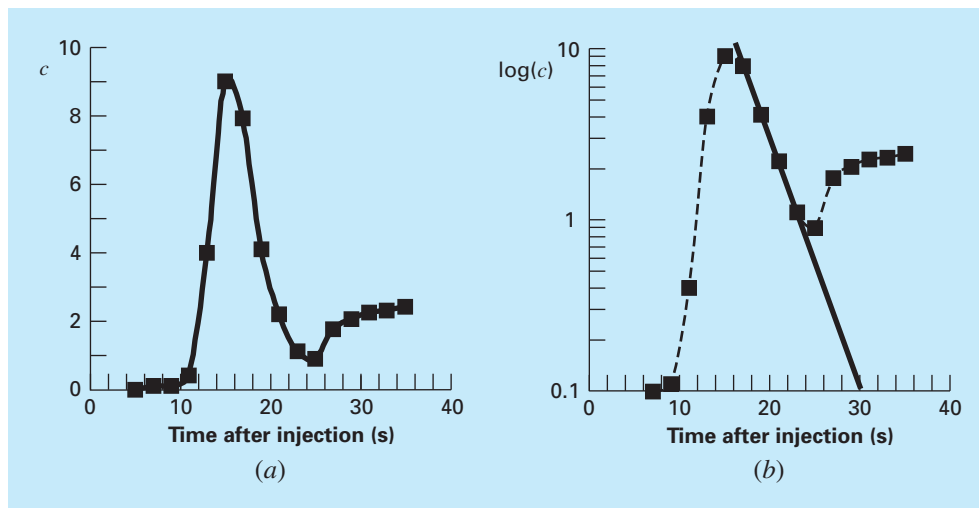
dilution curve. In order to separate out the recirculation effect, analysts extend the straight-line portion. The cardiac output can then be calculated from

$$C = \frac{M}{A} \times 60 \text{ s/min}$$

where $C$ = cardiac output [L/min], $M$ = amount of injected dye (mg), and $A$ = area under the curve with the linear correction. Calculate the cardiac output of this patient using the trapezoidal rule with a step size of 2 s.

**24.11** Glaucoma is the second leading cause of vision loss worldwide. High intraocular pressure (pressure inside the eye) almost always accompanies vision loss. It is postulated that the high pressure damages a subset of cells in the eye that are responsible for vision. One investigator theorizes that the relationship between vision loss and pressure can be described as

$$VL = A \exp\left( k \int_{25}^{t} (P - 13)\, dt \right)$$

where $VL$ is percent vision loss, $P$ is intraocular pressure (mm Hg), $t$ is time (years), and $k$ and $A$ are constants. Using the data below from three patients, estimate the constants $k$ and $A$.

collected the following data on the mass flux of insulin being delivered through the patch (and skin) as a function of time:

| Flux, mg/cm²/h | Time, h | Flux, mg/cm²/h | Time, h |
|---|---|---|---|
| 15 | 0 | 8 | 5 |
| 14 | 1 | 5 | 10 |
| 12 | 2 | 2.5 | 15 |
| 11 | 3 | 2 | 20 |
| 9 | 4 | 1 | 24 |

Remember that mass flux is flow rate through an area or $(1/A)$ $dm/dt$. Provide your best possible estimate for the amount of drug delivered through the skin in 24 hours using a 12 cm² patch.

**24.13** *Videoangiography* is used to measure blood flow and determine the status of circulatory function. In order to quantify the videoangiograms, blood vessel diameter and blood velocity are needed such that total blood flow is determined. Below is the densitometric profile taken from a videoangiogram of a blood vessel. One way to determine consistently where the edge of the blood vessel is from the angiogram is to determine where the first derivative of the profile is an extreme value. Using the data provided, find

| Patient | A | | B | | C | |
|---|---|---|---|---|---|---|
| Age at diagnosis | 65 | | 43 | | 80 | |
| VL | 60 | | 40 | | 30 | |
| | Age, years | P, mm Hg | Age, years | P, mm Hg | Age, years | P, mm Hg |
| | 25 | 13 | 25 | 11 | 25 | 13 |
| | 40 | 15 | 40 | 30 | 40 | 14 |
| | 50 | 22 | 41 | 32 | 50 | 15 |
| | 60 | 23 | 42 | 33 | 60 | 17 |
| | 65 | 24 | 43 | 35 | 80 | 19 |

**24.12** One of your colleagues has designed a new transdermal patch to deliver insulin through the skin to diabetic patients in a controlled way, eliminating the need for painful injections. She has

the boundaries of the blood vessel and estimate the blood vessel diameter. Use both $O(h^2)$ and $O(h^4)$ centered difference formulas and compare the results.

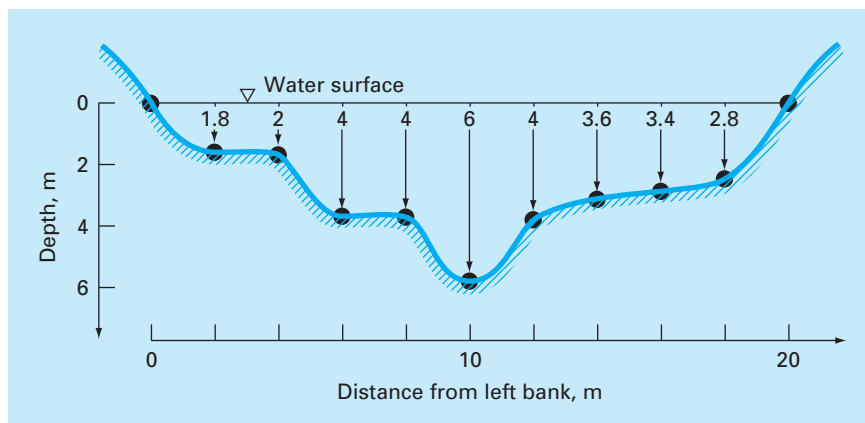| Distance | Density | Distance | Density | Distance | Density | Distance | Density |
|---|---|---|---|---|---|---|---|
| 0 | 26.013 | 28 | 38.273 | 56 | 39.124 | 84 | 37.331 |
| 4 | 26.955 | 32 | 39.103 | 60 | 38.813 | 88 | 35.980 |
| 8 | 26.351 | 36 | 39.025 | 64 | 38.925 | 92 | 31.936 |
| 12 | 28.343 | 40 | 39.432 | 68 | 38.804 | 96 | 28.843 |
| 16 | 31.100 | 44 | 39.163 | 72 | 38.806 | 100 | 26.309 |
| 20 | 34.667 | 48 | 38.920 | 76 | 38.666 | 104 | 26.146 |
| 24 | 37.251 | 52 | 38.631 | 80 | 38.658 | | |

**FIGURE P24.17**
A stream cross section.

### Civil/Environmental Engineering

**24.14** Perform the same computation as in Sec. 24.2, but use $O(h^8)$ Romberg integration to evaluate the integral.

**24.15** Perform the same computation as in Sec. 24.2, but use Gauss quadrature to evaluate the integral.

**24.16** As in Sec. 24.2, compute $F$ using the trapezoidal rule and Simpson's 1/3 and Simpson's 3/8 rules but use the following force. Divide the mast into 5-ft intervals.

$$F = \int_0^{30} \frac{250z}{6+z} e^{-z/10} \, dz$$

**24.17** Stream cross-sectional areas ($A$) are required for a number of tasks in water resources engineering, including flood forecasting and reservoir designing. Unless electronic sounding devices are available to obtain continuous profiles of the channel bottom, the engineer must rely on discrete depth measurements to compute $A$. An example of a typical stream cross section is shown in Fig. P24.17. The data points represent locations where a boat was anchored and depth readings taken. Use two trapezoidal rule applications ($h = 4$ and 2 m) and Simpson's 1/3 rule ($h = 2$ m) to estimate the cross-sectional area from these data.

**24.18** As described in Prob. 24.17, the cross-sectional area of a channel can be computed as

$$A_c = \int_0^B H(y) \, dy$$

where $B = $ the total channel width (m), $H = $ the depth (m), and $y = $ distance from the bank (m). In a similar fashion, the average flow $Q$ (m³/s) can be computed as

$$Q = \int_0^B U(y)H(y) \, dy$$

where $U = $ water velocity (m/s). Use these relationships and a numerical method to determine $A_c$ and $Q$ for the following data:

| $y$, m | 0 | 2 | 4 | 5 | 6 | 9 |
|---|---|---|---|---|---|---|
| $H$, m | 0.5 | 1.3 | 1.25 | 1.7 | 1 | 0.25 |
| $U$, m/s | 0.03 | 0.06 | 0.05 | 0.12 | 0.11 | 0.02 |

**24.19** The following relationships can be used to analyze uniform beams subject to distributed loads,

$$\frac{dy}{dx} = \theta(x) \quad \frac{d\theta}{dx} = \frac{M(x)}{EI} \quad \frac{dM}{dx} = V(x) \quad \frac{dV}{dx} = -w(x)$$

where $x = $ distance along beam (m), $y = $ deflection (m), $\theta(x) = $ slope (m/m), $E = $ modulus of elasticity (Pa = N/m²), $I = $ moment of inertia (m⁴), $M(x) = $ moment (N m), $V(x) = $ shear (N), and $w(x) = $ distributed load (N/m). For the case of a linearly increasing load (recall Fig. P8.18), the slope can be computed analytically as

$$\theta(x) = \frac{w_0}{120EIL}(-5x^4 + 6L^2x^2 - L^4) \tag{P24.19.1}$$

Employ (a) numerical integration to compute the deflection (in m) and (b) numerical differentiation to compute the moment (in N m) and shear (in N). Base your numerical calculations on values of the slope computed with Eq. P24.19 at equally-spaced intervals of $\Delta x = 0.125$ m along a 3-m beam. Use the following parameter values in your computation: $E = 200$ GPa, $I = 0.0003$ m⁴, and $w_0 = 2.5$ kN/cm. In addition, the deflections at the ends of the beam are set at $y(0) = y(L) = 0$. Be careful of units.

**24.20** You measure the following deflections along the length of a simply-supported uniform beam (see Prob. 24.19)

| $x$, m | 0 | 0.375 | 0.75 | 1.125 | 1.5 | 1.875 | 2.25 | 2.625 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $y$, cm | 0 | −0.2571 | −0.9484 | −1.9689 | −3.2262 | −4.6414 | −6.1503 | −7.7051 | −9.275 |

**TABLE P24.21** Traffic flow rate (cars/min) for an intersection measured at various times within a 24-h period.

| Time | Rate | Time | Rate | Time | Rate |
|---|---|---|---|---|---|
| 12:00 midnight | 2 | 9:00 A.M. | 11 | 6:00 P.M. | 20 |
| 2:00 A.M. | 2 | 10:30 A.M. | 4 | 7:00 P.M. | 10 |
| 4:00 A.M. | 0 | 11:30 A.M. | 11 | 8:00 P.M. | 8 |
| 5:00 A.M. | 2 | 12:30 P.M. | 12 | 9:00 P.M. | 10 |
| 6:00 A.M. | 6 | 2:00 P.M. | 8 | 10:00 P.M. | 8 |
| 7:00 A.M. | 7 | 4:00 P.M. | 7 | 11:00 P.M. | 7 |
| 8:00 A.M. | 23 | 5:00 P.M. | 26 | 12:00 midnight | 3 |

Employ numerical differentiation to compute the slope, the moment (in N m), the shear (in N) and the distributed load (in N/m). Use the following parameter values in your computation: $E = 200$ GPa, and $I = 0.0003$ m$^4$.

**24.21** A transportation engineering study requires the calculation of the total number of cars that pass through an intersection over a 24-h period. An individual visits the intersection at various times during the course of a day and counts the number of cars that pass through the intersection in a minute. Utilize the data summarized in Table P24.21, to estimate the total number of cars that pass through the intersection per day. (Be careful of units.)

**24.22** A wind force distributed against the side of a skyscraper is measured as

| Height, $l$, m | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 |
|---|---|---|---|---|---|---|---|---|---|
| Force, $F(l)$, N/m | 0 | 340 | 1200 | 1600 | 2700 | 3100 | 3200 | 3500 | 3800 |

Compute the net force and the line of action due to this distributed wind.

**24.23** Water exerts pressure on the upstream face of a dam as shown in Fig. P24.23. The pressure can be characterized by

$$p(z) = \rho g(D - z) \tag{P24.23.1}$$

where $p(z)$ = pressure in pascals (or N/m$^2$) exerted at an elevation $z$ meters above the reservoir bottom; $\rho$ = density of water, which for this problem is assumed to be a constant $10^3$ kg/m$^3$; $g$ = acceleration due to gravity (9.8 m/s$^2$); and $D$ = elevation (in m) of the water surface above the reservoir bottom. According to Eq. (P24.23.1), pressure increases linearly with depth, as depicted in Fig. P24.23a. Omitting atmospheric pressure (because it works against both sides of the dam face and essentially cancels out), the total force $f_t$ can be determined by multiplying pressure times the area of the dam face (as shown in Fig. P24.23b). Because both pressure and area vary with elevation, the total force is obtained by evaluating

$$f_t = \int_0^D \rho g w(z)(D - z)\, dz$$

where $w(z)$ = width of the dam face (m) at elevation $z$ (Fig. P24.23b). The line of action can also be obtained by evaluating

**FIGURE P24.23**
Water exerting pressure on the upstream face of a dam: (a) side view showing force increasing linearly with depth; (b) front view showing width of dam in meters.
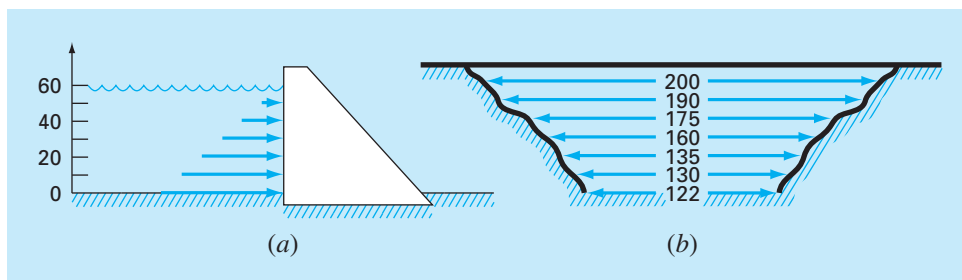
$$d = \frac{\int_0^D \rho g z w(z)(D - z)\, dz}{\int_0^D \rho g w(z)(D - z)\, dz}$$

Use Simpson's rule to compute $f_t$ and $d$. Check the results with your computer program for the trapezoidal rule.

**24.24** To estimate the size of a new dam, you have to determine the total volume of water (m³) that flows down a river in a year's time. You have available the following long-term average data for the river:

| Date | Mid-Jan. | Mid-Feb. | Mid-Mar. | Mid-Apr. | Mid-June | Mid-Sept. | Mid-Oct. | Mid-Nov. | Mid-Dec. |
|---|---|---|---|---|---|---|---|---|---|
| Flow, m³/s | 30 | 38 | 82 | 125 | 95 | 20 | 22 | 24 | 35 |

Determine the volume. Be careful of units, and take care to make a proper estimate of flow at the end points.

**24.25** The data listed in the following table gives hourly measurements of heat flux $q$ (cal/cm²/h) at the surface of a solar collector. As an architectural engineer, you must estimate the total heat absorbed by a 150,000-cm² collector panel during a 14-h period. The panel has an absorption efficiency $e_{ab}$ of 45%. The total heat absorbed is given by

$$h = e_{ab} \int_0^t q A\, dt$$

where $A$ = area and $q$ = heat flux.

| t | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|---|
| q | 0.10 | 5.32 | 7.80 | 8.00 | 8.03 | 6.27 | 3.54 | 0.20 |

**24.26** The heat flux $q$ is the quantity of heat flowing through a unit area of a material per unit time. It can be computed with Fourier's law,

$$J = -k\frac{dT}{dx}$$

where $J$ has units of J/m²/s or W/m² and $k$ is a coefficient of thermal conductivity that parameterizes the heat-conducting properties of the material and has units of W/(°C · m). $T$ = temperature (°C); and $x$ = distance (m) along the path of heat flow. Fourier's law is used routinely by architectural engineers to determine heat flow through walls. The following temperatures are measured from the surface ($x = 0$) into a stone wall:

| x, cm | 0 | 0.08 | 0.16 |
|---|---|---|---|
| T, °C | 20 | 17 | 15 |

If the flux at $x = 0$ is 60 W/m², compute $k$.

**24.27** The horizontal surface area $A_s$ (m²) of a lake at a particular depth can be computed from volume by differentiation,

$$A_s(z) = -\frac{dV}{dz}(z)$$

where $V$ = volume (m³) and $z$ = depth (m) as measured from the surface down to the bottom. The average concentration of a substance that varies with depth $\bar{c}$ (g/m³) can be computed by integration

$$\bar{c} = \frac{\int_0^Z c(z)A_s(z)\, dz}{\int_0^Z A_s(z)\, dz}$$

where $Z$ = the total depth (m). Determine the average concentration based on the following data:

| z, m | 0 | 4 | 8 | 12 | 16 |
|---|---|---|---|---|---|
| V, 10⁶ m³ | 9.8175 | 5.1051 | 1.9635 | 0.3927 | 0.0000 |
| c, g/m³ | 10.2 | 8.5 | 7.4 | 5.2 | 4.1 |

**Electrical Engineering**

**24.28** Perform the same computation as in Sec. 24.3, but for the current as specified by

$$i(t) = 5e^{-1.25t}\sin 2\pi t \qquad \text{for } 0 \le t \le T/2$$
$$i(t) = 0 \qquad \text{for } T/2 < t \le T$$

where $T = 1$ s. Use five-point Gauss quadrature to estimate the integral.

**24.29** Repeat Prob. 24.28, but use five applications of Simpson's 1/3 rule.

**24.30** Repeat Prob. 24.28, but use Romberg integration to $\varepsilon_s = 1\%$.

**24.31** Faraday's law characterizes the voltage drop across an inductor as

$$V_L = L\frac{di}{dt}$$

where $V_L$ = voltage drop (V), $L$ = inductance (in henrys; 1 H = 1 V · s/A), $i$ = current (A), and $t$ = time (s). Determine the voltage drop as a function of time from the following data for an inductance of 4 H.

| t | 0 | 0.1 | 0.2 | 0.3 | 0.5 | 0.7 |
|---|---|---|---|---|---|---|
| i | 0 | 0.16 | 0.32 | 0.56 | 0.84 | 2.0 |

**24.32** Based on Faraday's law (Prob. 24.31), use the following voltage data to estimate the inductance in henrys if a current of 2 A is passed through the inductor over 400 milliseconds.

| t, ms | 0 | 10 | 20 | 40 | 60 | 80 | 120 | 180 | 280 | 400 |
|---|---|---|---|---|---|---|---|---|---|---|
| V, volts | 0 | 18 | 29 | 44 | 49 | 46 | 35 | 26 | 15 | 7 |

**24.33** Suppose that the current through a resistor is described by the function

$$i(t) = (60 - t)^2 + (60 - t)\sin(\sqrt{t})$$

and the resistance is a function of the current,

$$R = 10i + 2i^{2/3}$$

Compute the average voltage over $t = 0$ to 60 using the multiple-segment Simpson's 1/3 rule.

**24.34** If a capacitor initially holds no charge, the voltage across it as a function of time can be computed as

$$V(t) = \frac{1}{C} \int_0^t i(t)\, dt$$

If $C = 10^{-5}$ farad, use the following current data to develop a plot of voltage versus time:

| $t$, s | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.2 |
|---|---|---|---|---|---|---|---|
| $i$, $10^{-3}$ A | 0.2 | 0.3683 | 0.3819 | 0.2282 | 0.0486 | 0.0082 | 0.1441 |

**Mechanical/Aerospace Engineering**

**24.35** Perform the same computation as in Sec. 24.4, but use the following equations:

$$F(x) = 1.6x - 0.045x^2$$
$$\theta(x) = 0.8 + 0.125x - 0.009x^2 + 0.0002x^3$$

Use 4-, 8-, and 16-segment trapezoidal rules to compute the integral.

**24.36** Repeat Prob. 24.35, but use **(a)** Simpson's 1/3 rule, **(b)** Romberg integration to $\varepsilon_s = 0.5\%$, and **(c)** Gauss quadrature.

**24.37** Compute work as described in Sec. 24.4, but use the following equations for $F(x)$ and $\theta(x)$:

$$F(x) = 1.6x - 0.045x^2$$
$$\theta(x) = -0.00055x^3 + 0.0123x^2 + 0.13x$$

The force is in newtons and the angle is in radians. Perform the integration from $x = 0$ to 30 m.

**24.38** As was done in Sec. 24.4, determine the work performed if a constant force of 1 N applied at an angle $\theta$ results in the following displacements. Use the MATLAB function `cumtrapz` to determine the cumulative work and plot the result versus $\theta$.

| $x$, m | 0 | 1 | 2.7 | 3.8 | 3.7 | 3 | 1.4 |
|---|---|---|---|---|---|---|---|
| $\theta$, deg | 0 | 30 | 60 | 90 | 120 | 150 | 180 |

**24.39** The work done on an object is equal to the force times the distance moved in the direction of the force. The velocity of an object in the direction of a force is given by

$$v = 4t \qquad\qquad 0 \le t \le 4$$
$$v = 16 + (4 - t)^2 \qquad 4 \le t \le 14$$

where $v =$ m/s. Employ the multiple-application Simpson's rule to determine the work if a constant force of 200 N is applied for all $t$.
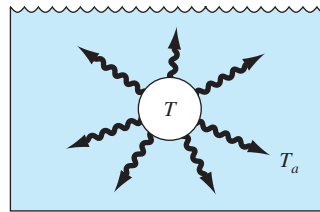


**FIGURE P24.40**

**24.40** The rate of cooling of a body (Fig. P24.40) can be expressed as

$$\frac{dT}{dt} = -k(T - T_a)$$

where $T =$ temperature of the body (°C), $T_a =$ temperature of the surrounding medium (°C), and $k =$ a proportionality constant (per minute). Thus, this equation (called *Newton's law of cooling*) specifies that the rate of cooling is proportional to the difference in the temperatures of the body and of the surrounding medium. If a metal ball heated to 80°C is dropped into water that is held constant at $T_a = 20$°C, the temperature of the ball changes, as in

| Time, min | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $T$, °C | 80 | 44.5 | 30.0 | 24.1 | 21.7 | 20.7 |

Utilize numerical differentiation to determine $dT/dt$ at each value of time. Plot $dT/dt$ versus $T - T_a$ and employ linear regression to evaluate $k$.

**24.41** A rod subject to an axial load (Fig. P24.41*a*) will be deformed, as shown in the stress-strain curve in Fig. P24.41*b*. The area under the curve from zero stress out to the point of rupture is called the *modulus of toughness* of the material. It provides a measure of the energy per unit volume required to cause the material to rupture. As such, it is representative of the material's ability to withstand an impact load. Use numerical integration to compute the modulus of toughness for the stress-strain curve seen in Fig. P24.41*b*.

**24.42** If the velocity distribution of a fluid flowing through a pipe is known (Fig. P24.42), the flow rate $Q$ (that is, the volume of water passing through the pipe per unit time) can be computed by $Q = \int v\, dA$, where $v$ is the velocity and $A$ is the pipe's cross-sectional area. (To grasp the meaning of this relationship physically, recall the close connection between summation and integration.) For a circular pipe, $A = \pi r^2$ and $dA = 2\pi r\, dr$. Therefore,
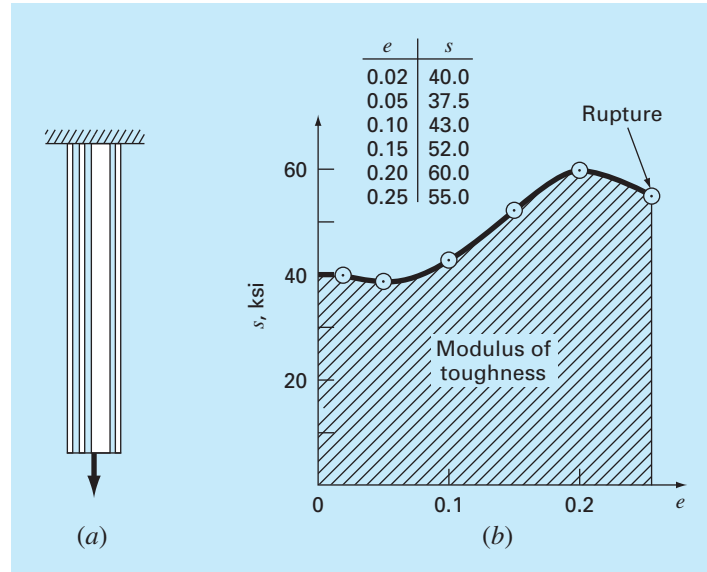
$$Q = \int_0^r v(2\pi r)\, dr$$

| e | s |
|---|---|
| 0.02 | 40.0 |
| 0.05 | 37.5 |
| 0.10 | 43.0 |
| 0.15 | 52.0 |
| 0.20 | 60.0 |
| 0.25 | 55.0 |



**FIGURE P24.41**
(a) A rod under axial loading and (b) the resulting stress-strain curve where stress is in kips per square inch ($10^3$ lb/in$^2$) and strain is dimensionless.
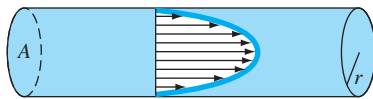


**FIGURE P24.42**

where $r$ is the radial distance measured outward from the center of the pipe. If the velocity distribution is given by

$$v = 2\left(1 - \frac{r}{r_0}\right)^{1/6}$$

where $r_0$ is the total radius (in this case, 3 cm), compute $Q$ using the multiple-application trapezoidal rule. Discuss the results.

**24.43** Using the following data, calculate the work done by stretching a spring that has a spring constant of $k = 300$ N/m to $x = 0.35$ m:

| F, $10^3$N | 0 | 0.01 | 0.028 | 0.046 | 0.063 | 0.082 | 0.11 | 0.13 |
|---|---|---|---|---|---|---|---|---|
| x, m | 0 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 |

**24.44** A jet fighter's position on an aircraft carrier's runway was timed during landing:

| t, s | 0 | 0.52 | 1.04 | 1.75 | 2.37 | 3.25 | 3.83 |
|---|---|---|---|---|---|---|---|
| x, m | 153 | 185 | 208 | 249 | 261 | 271 | 273 |

where $x$ is the distance from the end of the carrier. Estimate **(a)** velocity $(dx/dt)$ and **(b)** acceleration $(dv/dt)$ using numerical differentiation.

**24.45** Employ the multiple-application Simpson's rule to evaluate the vertical distance traveled by a rocket if the vertical velocity is given by

$$v = 11t^2 - 5t \qquad\qquad 0 \le t \le 10$$
$$v = 1100 - 5t \qquad\qquad 10 \le t \le 20$$
$$v = 50t + 2(t - 20)^2 \qquad 20 \le t \le 30$$

In addition, use numerical differentiation to develop graphs of the acceleration $(dv/dt)$ and the *jerk* $(d^2v/dt^2)$ versus time for $t = 0$ to 30. Note that the jerk is very important because it is highly correlated with injuries such as whiplash.

**24.46** The upward velocity of a rocket can be computed by the following formula:

$$v = u \ln\left(\frac{m_0}{m_0 - qt}\right) - gt$$

where $v$ = upward velocity, $u$ = velocity at which fuel is expelled relative to the rocket, $m_0$ = initial mass of the rocket at time $t = 0$,

$q$ = fuel consumption rate, and $g$ = downward acceleration of gravity (assumed constant = 9.8 m/s²). If $u$ = 1800 m/s, $m_0$ = 160,000 kg, and $q$ = 2500 kg/s, use six-segment trapezoidal and Simpson's 1/3 rule, six-point Gauss quadrature, and $O(h^8)$ Romberg methods to determine how high the rocket will fly in 30 s. In addition, use numerical differentiation to generate a graph of acceleration as a function of time.

**24.47** Referring to the data from Problem 20.61, find the strain rate using finite difference methods. Use second-order accurate derivative approximations and plot your results. Looking at the graph, it is apparent that there is some experimental startup error. Find the mean and standard deviation of the strain rate after eliminating the data points representing the experimental startup error.

**24.48** Fully developed flow moving through a 40-cm diameter pipe has the following velocity profile:

| Radius, $r$, cm | 0.0 | 2.5 | 5.0 | 7.5 | 10.0 | 12.5 | 15.0 | 17.5 | 20.0 |
|---|---|---|---|---|---|---|---|---|---|
| Velocity, $v$, m/s | 0.914 | 0.890 | 0.847 | 0.795 | 0.719 | 0.543 | 0.427 | 0.204 | 0 |

Find the volume flow rate $Q$ using the relationship $Q = \int_0^R 2\pi rv \, dr$, where $r$ is the radial axis of the pipe, $R$ is the radius of the pipe, and $v$ is the velocity. Solve the problem using two different approaches.
(a) Fit a polynomial curve to the velocity data and integrate analytically.
(b) Use multiple-application Simpson's 1/3 rule to integrate.
(c) Find the percent error using the integral of the polynomial fit as the more correct value.

**24.49** Fully developed flow of a Bingham plastic fluid moving through a 12-in diameter pipe has the given velocity profile. The flow of a Bingham fluid does not shear the center core, producing plug flow in the region around the centerline.

| Radius, $r$, in | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Velocity, $v$, ft/s | 5.00 | 5.00 | 4.62 | 4.01 | 3.42 | 1.69 | 0.00 |

Find the total volume flow rate $Q$ using the relationship $Q = \int_{r_1}^{r_2} 2\pi rv \, dr + v_c A_c$, where $r$ is the radial axis of the pipe, $v$ is the velocity, $v_c$ is the velocity at the core, and $A_c$ is the cross-sectional area of the plug. Solve the problem using two different approaches.
(a) Fit a polynomial curve to the noncore data and integrate.
(b) Use multiple-application Simpson's rule to integrate.
(c) Find the percent error using the integral of the polynomial fit as the more correct value.

**24.50** The enthalpy of a real gas is a function of pressure as described below. These data were taken for a real fluid. Estimate the enthalpy of the fluid at 400 K and 50 atm (evaluate the integral from 0.1 atm to 50 atm).

$$H \int_0^P \left( V - T\left(\frac{\partial V}{\partial T}\right)_P \right) dP$$

| P, atm | T = 350 K | T = 400 K | T = 450 K |
|---|---|---|---|
| | **V, L** | | |
| 0.1 | 220 | 250 | 282.5 |
| 5 | 4.1 | 4.7 | 5.23 |
| 10 | 2.2 | 2.5 | 2.7 |
| 20 | 1.35 | 1.49 | 1.55 |
| 25 | 1.1 | 1.2 | 1.24 |
| 30 | 0.90 | 0.99 | 1.03 |
| 40 | 0.68 | 0.75 | 0.78 |
| 45 | 0.61 | 0.675 | 0.7 |
| 50 | 0.54 | 0.6 | 0.62 |

**24.51** Given the data below, find the isothermal work done on the gas as it is compressed from 23 L to 3 L (remember that $W = -\int_{V_1}^{V_2} P \, dV$).

| V, L | 3 | 8 | 13 | 18 | 23 |
|---|---|---|---|---|---|
| P, atm | 12.5 | 3.5 | 1.8 | 1.4 | 1.2 |

(a) Find the work performed on the gas numerically, using the 1-, 2-, and 4-segment trapezoidal rule.
(b) Compute the ratios of the errors in these estimates and relate them to the error analysis of the multiple-application trapezoidal rule discussed in Chap. 21.

**24.52** The Rosin-Rammler-Bennet (RRB) equation is used to describe size distribution in fine dust. $F(x)$ represents the cumulative mass of dust particles of diameter $x$ and smaller. $x'$ and $n'$ are constants equal to 30 $\mu$m and 1.44, respectively. The mass density distribution $f(x)$ or the mass of dust particles of a diameter $x$ is found by taking the derivative of the cumulative distribution

$$F(x) = 1 - e^{-(x/x')^{n'}} \qquad f(x) = \frac{dF(x)}{dx}$$

(a) Numerically calculate the mass density distribution $f(x)$ and graph both $f(x)$ and the cumulative distribution $F(x)$.
(b) Using your results from part (a), calculate the mode size of the mass density distribution—that is, the size at which the derivative of $f(x)$ is equal to zero.
(c) Find the surface area per mass of the dust $S_m$ (cm²/g) using

$$S_m = \frac{6}{\rho} \int_{d_{min}}^{\infty} \frac{f(x)}{x} dx$$

The equation is valid only for spherical particles. Assume a density $\rho = 1$ g cm$^{-3}$ and a minimum diameter of dust included in the distribution $d_{min}$ of 1 $\mu$m.

**24.53** For fluid flow over a surface, the heat flux to the surface can be computed as

$$J = -k\frac{dT}{dy}$$

where $J$ = heat flux (W/m$^2$), $k$ = thermal conductivity (W/m · K), $T$ = temperature (K), and $y$ = distance normal to the surface (m). The following measurements are made for air flowing over a flat plate that is 200 cm long and 50 cm wide:

| y, cm | 0 | 1 | 3 | 5 |
|-------|-----|-----|-----|-----|
| T, K | 900 | 480 | 270 | 200 |

If $k = 0.028$ J/s · m · K, **(a)** determine the flux at the surface and **(b)** the heat transfer in watts. Note that 1 J = 1 W · s.

**24.54** The pressure gradient for laminar flow through a constant radius tube is given by

$$\frac{dp}{dx} = -\frac{8\mu\,Q}{\pi r^4}$$

where $p$ = pressure (N/m$^2$), $x$ = distance along the tube's centerline (m), $\mu$ = dynamic viscosity (N · s/m$^2$), $Q$ = flow (m$^3$/s), and $r$ = radius (m).

**(a)** Determine the pressure drop for a 10-cm length tube for a viscous liquid ($\mu = 0.005$ N · s/m$^2$, density $= \rho = 1 \times 10^3$ kg/m$^3$) with a flow of $10 \times 10^{-6}$ m$^3$/s and the following varying radii along its length,

| x, cm | 0 | 2 | 4 | 5 | 6 | 7 | 10 |
|-------|-----|------|------|-----|------|------|-----|
| r, mm | 2 | 1.35 | 1.34 | 1.6 | 1.58 | 1.42 | 2 |

**(b)** Compare your result with the pressure drop that would have occurred if the tube had a constant radius equal to the average radius.

**(c)** Determine the average Reynolds number for the tube to verify that flow is truly laminar (Re $= \rho v D/\mu < 2100$ where $v$ = velocity).

**24.55** Velocity data for air are collected at different radii from the centerline of a circular 16-cm-diameter pipe as tabulated below:

| r, cm | 0 | 1.60 | 3.20 | 4.80 | 6.40 | 7.47 | 7.87 | 7.95 | 8 |
|-------|----|------|------|------|------|------|------|------|---|
| v, m/s | 10 | 9.69 | 9.30 | 8.77 | 7.95 | 6.79 | 5.57 | 4.89 | 0 |

Use numerical integration to determine the mass flow rate, which can be computed as

$$\int_0^R \rho v 2\pi r \, dr$$

where $\rho$ = density ($= 1.2$ kg/m$^3$). Express your results in kg/s.