Beta Distribution:

A continuous r.v 'x' is said to have a beta distribution with the parameters α & β if its p.d.f is defined by.

$$f(x) = f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1 \\ 0 & \text{else where} \end{cases}$$

$\alpha,\beta>0$

⇒ If we make transformation $y=1-x$, then the probability density fn of the dist of y is

$$f(y) = f(y;\beta\,\alpha) = \frac{1}{\beta(\beta,\alpha)}(1-y)^{\alpha-1}y^{\beta-1}$$

$0<y<1$
$\alpha,\beta>0$

i-e the symmetrical distribution. and is the beta distribution of 1st kind.

⇒ Beta distribution of 2nd kind:

$$f(x) = \frac{1}{\beta(\alpha,\beta)}\frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} \qquad 0<x<\infty$$

$\alpha,\beta>0.$

It is also called beta prime distribution.

Relationship Beta Distributions with other Dist:

① If $\alpha=\beta=1$ then Beta dist becomes rectangular dist or uniform distribution.

② When $\alpha=1/2$, $\beta=1/2$, then arc ~~sign~~ sine distribution is obtained.

Properties of Beta Distribution:

Mean:

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x \le 1$$

$$\alpha, \beta > 0$$

$$\text{Mean} = E(x) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$= \int_0^1 x \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}\, dx.$$

$$B(\alpha,\beta) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \frac{\Gamma\alpha+1\,\Gamma\beta}{\Gamma\alpha+\beta+}$$

$$= \frac{\Gamma\alpha+\beta}{\cancel{\Gamma\alpha}\,\cancel{\Gamma\beta}} \frac{\alpha\,\cancel{\Gamma\alpha}\,\cancel{\Gamma\beta}}{(\alpha+\beta)\,\cancel{\Gamma\alpha+\beta}}$$

$$E(x) = \frac{\alpha}{\alpha+\beta}$$

$$\text{variance} = E(x^2) - (E(x))^2$$

$$E(x^2) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^2 x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{(\alpha+2)-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} B(\alpha+2, \beta).$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \frac{\Gamma\alpha+2\,\Gamma\beta}{\Gamma\alpha+\beta+2}$$

$$= \frac{\cancel{\Gamma\alpha+\beta}}{\cancel{\Gamma\alpha}\,\cancel{\Gamma\beta}} \frac{(\alpha+1)\,\alpha\,\cancel{\Gamma\alpha}\,\cancel{\Gamma\beta}}{(\alpha+\beta+1)(\alpha+\beta)\,\cancel{\Gamma\alpha+\beta}}$$

$$E(x^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$var(x) = E(x^2) - (E(x))^2.$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}.$$

$$= \frac{\alpha}{(\alpha+\beta)}\left[\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right]$$

$$= \frac{\alpha}{(\alpha+\beta)}\left[\frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}\right]$$

$$= \frac{\alpha}{(\alpha+\beta)}\left[\frac{\cancel{\alpha^2}+\cancel{\alpha\beta}+\cancel{\alpha}+\beta-\cancel{\alpha^2}-\cancel{\alpha\beta}-\cancel{\alpha}}{(\alpha+\beta)(\alpha+\beta+1)}\right]$$

$$var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

r^{th} moment about origin.

$$\mu'_r = E(x^r) = \int_{-\infty}^{+\infty} x^r f(x)\,dx.$$

$$= \int_0^1 x^r \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx.$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{(\alpha+r)-1}(1-x)^{\beta-1}\,dx.$$

$$= \frac{1}{B(\alpha,\beta)}\, B(\alpha+r,\beta).$$

$$\mu'_r = \frac{B(\alpha+r,\beta)}{B(\alpha,\beta)}.$$

Put $r = 1, 2, 3, 4.$

$$\mu_1' = \frac{\alpha}{\alpha+\beta}, \quad \mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Put $r = 3$

$$\mu_3' = \frac{\beta(\alpha+3, \beta)}{\beta(\alpha, \beta)} = \frac{\Gamma\alpha+3\,\cancel{\Gamma\beta}}{\Gamma\alpha+3+\beta} \cdot \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\cancel{\Gamma\beta}}$$

$$= \frac{\cancel{\Gamma\alpha+\beta}\,(\alpha+2)(\alpha+1)\,\alpha\,\cancel{\Gamma\alpha}}{\cancel{\Gamma\alpha}\,(\alpha+\beta+2)(\alpha+\beta+1)(\alpha+\beta)\,\cancel{\Gamma\alpha+\beta}}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)}$$

Put $r=4$.

$$\mu_4' = \frac{\beta(\alpha+4, \beta)}{\beta(\alpha, \beta)}$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\cancel{\Gamma\beta}} \cdot \frac{\Gamma\alpha+4\,\cancel{\Gamma\beta}}{\Gamma\alpha+\beta+4}$$

$$= \frac{\cancel{\Gamma\alpha+\beta}\,(\alpha+3)(\alpha+2)(\alpha+1)\,\alpha\,\cancel{\Gamma\alpha}}{\cancel{\Gamma\alpha}\,(\alpha+\beta+3)(\alpha+\beta+2)(\alpha+\beta+1)(\alpha+\beta)\,\cancel{\Gamma\alpha+\beta}}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+B)(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}$$

Central Moments:

$$\mu_1 = 0$$

$$\mu_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= \frac{2\alpha\beta(\beta-\alpha)}{(\alpha+\beta)^3(\alpha+\beta+1)(\alpha+\beta+2)}$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3(\mu_1')^4$$

$$= \frac{3\alpha\beta\left[\alpha\beta(\alpha+\beta-6)+2(\alpha+\beta)^2\right]}{(\alpha+\beta)^4(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}$$

Mode of Beta Distribution:

$$f(x) = \frac{1}{B(\alpha,B)} x^{\alpha-1} (1-x)^{B-1}$$

Mode is that value of x for which $f'(x)=0$ and $f''(x) < 0$

$$f'(x) = \frac{1}{B(\alpha,B)}\left[(\alpha-1) x^{\alpha-2} (1-x)^{B-1} + (B-1)(1-x)^{B-2}(-1) x^{\alpha-1}\right]$$

$$= \frac{1}{B(\alpha,B)}\left[(\alpha-1) x^{\alpha-2} (1-x)^{B-1} - x^{\alpha-1} (B-1)(1-x)^{B-2}\right]$$

$$\Rightarrow \frac{1}{B(\alpha,B)} x^{\alpha-2} (1-x)^{B-2}\left[(\alpha-1)(1-x) - x(B-1)\right] = 0$$

$$(\alpha-1)(1-x) - x(B-1) = 0$$

$$\alpha - \alpha x - 1 + x - xB + x = 0$$

$$\alpha - \alpha x + 2x - xB - 1 = 0$$

$$\alpha - 1 = \alpha x + xB - 2x$$

$$\alpha - 1 = x(\alpha + B - 2)$$

$$\boxed{\frac{\alpha-1}{\alpha+B-2} = x.}$$

for which it may be shown that

$$[f''(x)] < 0 \qquad \begin{array}{l}\alpha > 1 \\ B > 1\end{array}$$

Hence the mode is $\dfrac{\alpha-1}{\alpha+B-2}$