Examples, Solved Probelms and exercise of Chap. 2

Example 2.6

(a) Discuss the hermiticity of the operators $(\hat{A}+\hat{A}^\dagger)$, $i(\hat{A}+\hat{A}^\dagger)$, and $i(\hat{A}-\hat{A}^\dagger)$.

(b) Find the Hermitian adjoint of $f(\hat{A}) = (1+i\hat{A}+3\hat{A}^2)(1-2i\hat{A}-9\hat{A}^2)/(5+7\hat{A})$.

(c) Show that the expectation value of a Hermitian operator is real and that of an anti-Hermitian operator is imaginary.

Solution: (a) The operator $\hat{B} = \hat{A} + \hat{A}^\dagger$ is Hermitian as

$$\hat{B}^\dagger = (\hat{A}+\hat{A}^\dagger)^\dagger = \hat{A}^\dagger + \hat{A} = \hat{A}+\hat{A}^\dagger$$

and is Hermitian regardless of whether $\hat{A}$ is Hermitian or not.

$\Rightarrow$ $\hat{B} = i(\hat{A}+\hat{A}^\dagger)$

so $\hat{B}^\dagger = -i(\hat{A}+\hat{A}^\dagger)^\dagger = -i(\hat{A}+\hat{A}^\dagger) = -\hat{B}$

so it is anti-Hermitian

$\Rightarrow$ $\hat{B} = i(\hat{A}-\hat{A}^\dagger)$

$$\hat{B}^\dagger = [i(A-\hat{A}^\dagger)]^\dagger = -i(A-\hat{A}^\dagger)^\dagger = -i(\hat{A}^\dagger-\hat{A})$$

$$\hat{B}^\dagger = i(\hat{A}-\hat{A}^\dagger)$$

so $i(\hat{A}-\hat{A}^\dagger)$ is Hermitian.

(b)

Since we are given $f(\hat{A}) = (1+i\hat{A}+3\hat{A}^2)(1-2i\hat{A}-9\hat{A}^2)/(5+7\hat{A})$

Here $f(\hat{A})$ is function of operator, and its Hermitian adjoint is

$(f(\hat{A}))^\dagger = f^*(\hat{A}^\dagger)$ since f may be either real or complex. If it is real then $(f(\hat{A}))^\dagger = f^*(\hat{A}^\dagger) = f(\hat{A}^\dagger)$ and if $\hat{A}$ is Hermitian i.e., $\hat{A}^\dagger = A$ then $f(\hat{A}^\dagger) = f(A)$, so

$$(f(\hat{A}))^\dagger = [(1+i\hat{A}+3\hat{A}^2)(1-2i\hat{A}-9\hat{A}^2)/(5+7\hat{A})]^\dagger$$

$$f^*(\hat{A}^\dagger) = \frac{(1-i\hat{A}^\dagger+3\hat{A}^{\dagger 2})(1+2i\hat{A}^\dagger-9\hat{A}^{\dagger 2})}{(5+7\hat{A}^\dagger)}$$

(c)

The expectation value of a Hermitian operator is real as $\hat{A}^\dagger = \hat{A}$

so $\langle\psi|\hat{A}|\psi\rangle^* = \langle\psi|\hat{A}^\dagger|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle = \text{real}$

since for anti-Hermitian operator $\hat{B}^\dagger = -\hat{B}$

$$\langle\psi|\hat{B}|\psi\rangle^* = \langle\psi|\hat{B}^\dagger|\psi\rangle = \langle\psi|-\hat{B}|\psi\rangle = -\langle\psi|\hat{B}|\psi\rangle$$

So to hold the equality we must have

$$\langle\psi|\hat{B}|\psi\rangle = -\langle\psi|\hat{B}|\psi\rangle^*$$

which means that $\langle\psi|\hat{B}|\psi\rangle$ is purely imaginary. so

* expectation value of Hermitian operator is purely real.
* expectation value of anti-Hermitian operator is purely imaginary.

If we are given by $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$ & $\Delta\hat{B} = \hat{B} - \langle\hat{B}\rangle$

One thing must be remembered that $\langle\hat{A}\rangle$ and $\langle\hat{B}\rangle$ are only numbers either complex or real so their properties or discussed in previous article and above solved example. Let us try the commutation relation

$$[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle] = [\hat{A}, \hat{B}] - [\hat{A}, \underbrace{\langle\hat{B}\rangle}_{\text{scalar}}] - [\underbrace{\langle\hat{A}\rangle}_{\text{scalar}}, \hat{B}] + [\underbrace{\langle\hat{A}\rangle, \langle\hat{B}\rangle}_{\text{scalar}}]$$

since $[\hat{A}, a] = 0$ commutation of An operator with a scalar is zero

so $$[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A}, \hat{B}]$$