2
Linear Interpolation

In order to achieve realism, the many algorithms and techniques employed in computer graphics have to construct mathematical models of curved surfaces, models that are based on curves. It seems that straight line segments and flat surface patches, which are simple geometric figures, cannot play an important role in achieving realism, yet they turn out to be useful in many instances. A smooth curve can be approximated by a set of short straight segments. A smooth, curved surface can similarly be approximated by a set of surface patches, each a small, flat polygon. Thus, this chapter discusses straight lines and flat surfaces that are defined by points. The application of these simple geometric figures to computer graphics is referred to as *linear interpolation*. The chapter also presents two types of surfaces, bilinear and lofted, that are curved, but are partly based on straight lines.

2.1 Straight Segments

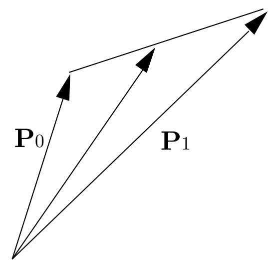
We start with the parametric equation of a straight segment. Given any two points $\mathbf{A}$ and $\mathbf{C}$, the expression $\mathbf{A}+\alpha(\mathbf{C}-\mathbf{A})$ is the sum of a point and a vector, so it is a point (see page 2) that we can denote by $\mathbf{B}$. The vector $\mathbf{C}-\mathbf{A}$ points from $\mathbf{A}$ to $\mathbf{C}$, so adding it to $\mathbf{A}$ results in a point on the line connecting $\mathbf{A}$ to $\mathbf{C}$. Thus, we conclude that the three points $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are collinear. Note that the expression $\mathbf{B}=\mathbf{A}+\alpha(\mathbf{C}-\mathbf{A})$ can be written $\mathbf{B}=(1-\alpha)\mathbf{A}+\alpha\mathbf{C}$, showing that $\mathbf{B}$ is a linear combination of $\mathbf{A}$ and $\mathbf{C}$ with barycentric weights. In general, any of three collinear points can be written as a linear combination of the other two. Such points are not independent.

We therefore conclude that given two arbitrary points $\mathbf{P}_0$ and $\mathbf{P}_1$, the parametric representation of the line segment from $\mathbf{P}_0$ to $\mathbf{P}_1$ is

$$\mathbf{P}(t)=(1-t)\mathbf{P}_0+t\mathbf{P}_1=\mathbf{P}_0+(\mathbf{P}_1-\mathbf{P}_0)t=\mathbf{P}_0+t\mathbf{d}, \quad \text{for} \quad 0\le t\le 1. \tag{2.1}$$

The tangent vector of this line is the constant vector $\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}_1 - \mathbf{P}_0 = \mathbf{d}$, the direction from $\mathbf{P}_0$ to $\mathbf{P}_1$.

If we think of $\mathbf{P}_i$ as the vector from the origin to point $\mathbf{P}_i$, then the figure on the right shows how the straight line is obtained as a linear, barycentric combination of the two vectors $\mathbf{P}_0$ and $\mathbf{P}_1$, with coefficients $(1-t)$ and t. We can think of this combination as a vector that pivots from $\mathbf{P}_0$ to $\mathbf{P}_1$ while varying its magnitude, so its tip always stays on the line.



The expression $\mathbf{P}_0 + t\mathbf{d}$ is also useful. It describes the line as the sum of the point $\mathbf{P}_0$ and the vector $t\mathbf{d}$, a vector pointing from $\mathbf{P}_0$ to $\mathbf{P}_1$, whose magnitude depends on t. This representation is useful in cases where the direction of the line and one point on it are known. Notice that varying t in the interval $[-\infty, +\infty]$ constructs the infinite line that contains $\mathbf{P}_0$ and $\mathbf{P}_1$.

2.1.1 Distance of a Point From a Line

Given a line in parametric form $\mathbf{L}(t) = \mathbf{P}_0 + t\mathbf{v}$ (where $\mathbf{v}$ is a vector in the direction of the line) and a point $\mathbf{P}$, what is the distance between them? Assume that $\mathbf{Q}$ is the point on $\mathbf{L}(t)$ that's the closest to $\mathbf{P}$. Point $\mathbf{Q}$ can be expressed as $\mathbf{Q} = \mathbf{L}(t_0) = \mathbf{P}_0 + t_0\mathbf{v}$ for some t_0. The vector from $\mathbf{Q}$ to $\mathbf{P}$ is $\mathbf{P}-\mathbf{Q}$. Since $\mathbf{Q}$ is the nearest point to $\mathbf{P}$, this vector should be perpendicular to the line. Thus, we end up with the condition $(\mathbf{P}-\mathbf{Q}) \bullet \mathbf{v} = 0$ or $(\mathbf{P} - \mathbf{P}_0 - t_0\mathbf{v}) \bullet \mathbf{v} = 0$, which is satisfied by

$$t_0 = \frac{(\mathbf{P} - \mathbf{P}_0) \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}.$$

Substituting this value of t_0 in the line equation gives

$$\mathbf{Q} = \mathbf{P}_0 + \frac{(\mathbf{P} - \mathbf{P}_0) \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v}. \tag{2.2}$$

The distance between $\mathbf{Q}$ and $\mathbf{P}$ is the magnitude of vector $\mathbf{P} - \mathbf{Q}$.

This method always works since vector $\mathbf{v}$ cannot be zero (otherwise there would be no line).

In the two-dimensional case, the line can be represented explicitly as $y = ax+b$ and the problem can be easily solved with just elementary trigonometry. Figure 2.1 shows a general point $\mathbf{P} = (P_x, P_y)$ at a distance d from a line $y = ax + b$. It is easy to see that the vertical distance e between the line and $\mathbf{P}$ is $|P_y - aP_x - b|$. We also know from trigonometry that

$$1 = \sin^2\alpha + \cos^2\alpha = \tan^2\alpha\cos^2\alpha + \cos^2\alpha = \cos^2\alpha(1 + \tan^2\alpha),$$

implying

$$\cos^2\alpha = \frac{1}{1 + \tan^2\alpha}.$$

We therefore get

$$d = e \cos\alpha = e\sqrt{\cos^2\alpha} = \frac{e}{\sqrt{1+\tan^2\alpha}} = \frac{|P_y - aP_x - b|}{\sqrt{1+a^2}}. \tag{2.3}$$

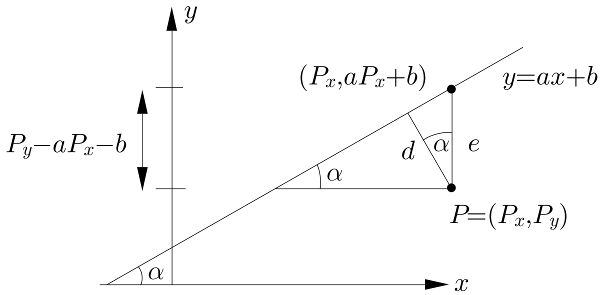


Figure 2.1: Distance Between $\mathbf{P}$ and $y = ax + b$.

◇ **Exercise 2.1:** Many mathematics problems can be solved in more than one way and this problem is a good example. It is easy to solve by approaching it from different directions. Suggest some approaches to the solution.

A man who boasts about never changing his views is a man who's decided always to travel in a straight line—the kind of idiot who believes in absolutes.

—Honoré de Balzac, *Père Goriot*, 1834

2.1.2 Intersection of Lines

Here is a simple, fast algorithm for finding the intersection point(s) of two line segments. Assuming that the two segments $\mathbf{P}_1 + \alpha(\mathbf{P}_2 - \mathbf{P}_1)$ and $\mathbf{P}_3 + \beta(\mathbf{P}_4 - \mathbf{P}_3)$ are given [Equation (2.1)], their intersection point satisfies

$$\mathbf{P}_1 + \alpha(\mathbf{P}_2 - \mathbf{P}_1) = \mathbf{P}_3 + \beta(\mathbf{P}_4 - \mathbf{P}_3),$$

or

$$\alpha(\mathbf{P}_2 - \mathbf{P}_1) - \beta(\mathbf{P}_4 - \mathbf{P}_3) + (\mathbf{P}_1 - \mathbf{P}_3) = 0.$$

This can also be written $\alpha\mathbf{A} + \beta\mathbf{B} + \mathbf{C} = 0$, where $\mathbf{A} = \mathbf{P}_2 - \mathbf{P}_1$, $\mathbf{B} = \mathbf{P}_3 - \mathbf{P}_4$, and $\mathbf{C} = \mathbf{P}_1 - \mathbf{P}_3$. The solutions are

$$\alpha = \frac{B_y C_x - B_x C_y}{A_y B_x - A_x B_y}, \qquad \beta = \frac{A_x C_y - A_y C_x}{A_y B_x - A_x B_y}.$$

The calculation of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ requires six subtractions. The calculation of α and β requires three subtractions, six multiplications (since the denominators are identical), and two divisions.

Example: To calculate the intersection of the line segment from $\mathbf{P}_1 = (-1, 1)$ to $\mathbf{P}_2 = (1, -1)$ with the line segment from $\mathbf{P}_3 = (-1, -1)$ to $\mathbf{P}_4 = (1, 1)$, we first calculate

$$\mathbf{A} = \mathbf{P}_2 - \mathbf{P}_1 = (2, -2), \quad \mathbf{B} = \mathbf{P}_3 - \mathbf{P}_4 = (-2, -2), \quad \mathbf{C} = \mathbf{P}_1 - \mathbf{P}_3 = (0, 2).$$

Then calculate

$$\alpha = \frac{0+4}{4+4} = \frac{1}{2}, \qquad \beta = \frac{4-0}{4+4} = \frac{1}{2}.$$

The lines intersect at their midpoints.

Example: The line segment from $\mathbf{P}_1 = (0, 0)$ to $\mathbf{P}_2 = (1, 0)$ and the line segment from $\mathbf{P}_3 = (2, 0)$ to $\mathbf{P}_4 = (2, 1)$ don't intersect. However, the calculation shows the values of α and β necessary for them to intersect,

$$\mathbf{A} = \mathbf{P}_2 - \mathbf{P}_1 = (1, 0), \quad \mathbf{B} = \mathbf{P}_3 - \mathbf{P}_4 = (0, -1), \quad \mathbf{C} = \mathbf{P}_1 - \mathbf{P}_3 = (-2, 0),$$

yields

$$\alpha = \frac{2-0}{0+1} = 2, \qquad \beta = \frac{0-0}{0+1} = 0.$$

The lines would intersect at $\alpha = 2$ (i.e., if we extend the first segment to twice its length beyond $\mathbf{P}_2$) and $\beta = 0$ (i.e., point $\mathbf{P}_3$).

◇ **Exercise 2.2:** How can we identify overlapping lines (i.e., the case of infinitely many intersection points) and parallel lines (no intersection points)? See Figure 2.2.

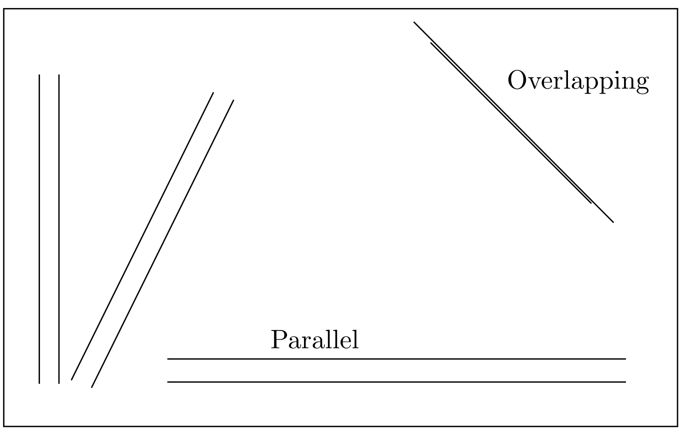


Figure 2.2: Parallel and Overlapped Lines.

The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn.

—Isaac Newton, 1687.

2.2 Polygonal Surfaces

A polygonal surface consists of a number of flat faces, each a polygon. A polygon in such a surface is typically a triangle, because the three points of a triangle are always on the same plane. With higher-order polygons, the surface designer should make sure that all the corners of the polygon are on the same plane.

Each polygon is a collection of vertices (the points defining it) and edges (the lines connecting the points). Such a surface is easy to display, either as a wire frame or as a solid surface. In the former case, the edges of all the polygons should be displayed. In the latter case, all the points in a polygon are assigned the same color and brightness. They are all assumed to reflect the same amount of light, since the polygon is flat and has only one normal vector. As a result, a polygonal surface shaded this way appears angular and unnatural, but there is a simple method, known as Gouraud's algorithm [Gouraud 71], that smooths out the reflections from the individual polygons and makes the entire polygonal surface look curved.

Three methods are described for representing such a surface in memory:

1. Explicit polygons. Each polygon is represented as a list

$$\big(\,(x_1,y_1,z_1),(x_2,y_2,z_2),\ldots,(x_n,y_n,z_n)\,\big)$$

of its vertices, and it is assumed that there is an edge from point 1 to point 2, from 2 to 3, and so on, and also an edge from point n to point 1.

This representation is simple but has two disadvantages:

I. A point may be shared by several polygons, so several copies have to be stored. If the user decides to modify the point, all its copies have to be located and updated. This is a minor problem, because an edge is rarely shared by more than two polygons.

II. An edge may also be shared by several polygons. When displaying the surface, such an edge will be displayed several times, slowing down the entire process.

2. Polygon definition by pointers. There is one list

$$\mathbf{V}=\big(\,(x_1,y_1,z_1),(x_2,y_2,z_2),\ldots,(x_n,y_n,z_n)\,\big)$$

of all the vertices of the surface. A polygon is represented as a list of pointers, each pointing to a vertex in $\mathbf{V}$. Hence, $\mathbf{P}=(3,5,7,10)$ implies that polygon $\mathbf{P}$ consists of vertices 3, 5, 7, and 10 in $\mathbf{V}$. Problem II still exists.

3. Explicit edges. List $\mathbf{V}$ is as before, and there is also an edge list

$$\mathbf{E}=(\,(v_1,v_6,p_3),(v_5,v_7,p_1,p_3,p_6,p_8),\ldots).$$

Each element of $\mathbf{E}$ represents an edge. It contains two pointers to the vertices of the edge followed by pointers to all the polygons that share the edge. Each polygon is represented by a list of pointers to $\mathbf{E}$, for example, $\mathbf{P}_1=(e_1,e_4,e_5)$. Problem II still exists, but it is minor.

2.2.1 Polygon Planarity

Given a polygon defined by points $\mathbf{P}_1$, $\mathbf{P}_2$, …, $\mathbf{P}_n$, we use the scalar triple product [Equation (1.7)] to test for polygon planarity (i.e., to check whether all the polygon's vertices $\mathbf{P}_i$ are on the same plane). Such a test is necessary only if $n > 3$. We select $\mathbf{P}_1$ as the "pivot" point and calculate the $n-1$ pivot vectors $\mathbf{v}_i = \mathbf{P}_i - \mathbf{P}_1$ for $i = 2, \ldots, n$. Next, we calculate the $n-3$ scalar triple products $\mathbf{v}_i \bullet (\mathbf{v}_2 \times \mathbf{v}_3)$ for $i = 4, \ldots, n$. If any of these products are nonzero, the polygon is not planar. Note that limited accuracy on some computers may cause an otherwise null triple product to come out as a small floating-point number.

◇ **Exercise 2.3:** Consider the polygon defined by the four points $\mathbf{P}_1 = (1, 0, 0)$, $\mathbf{P}_2 = (0, 1, 0)$, $\mathbf{P}_3 = (1, a, 1)$, and $\mathbf{P}_4 = (0, -a, 0)$. For what values of a will it be planar?

2.2.2 Plane Equations

A polygonal surface consists of flat polygons (often triangles). To calculate the normal to a polygon, we first need to know the polygon's equation. The implicit equation of a flat plane is $Ax + By + Cz + D = 0$. It seems that we need four equations in order to calculate the four unknown coefficients A, B, C, and D, but it turns out that three equations are enough. Assuming that the three points $\mathbf{P}_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, are given, we can write the four equations

$$\begin{aligned} Ax + By + Cz + D &= 0, \\ Ax_1 + By_1 + Cz_1 + D &= 0, \\ Ax_2 + By_2 + Cz_2 + D &= 0, \\ Ax_3 + By_3 + Cz_3 + D &= 0. \end{aligned}$$

The first equation is true for any point (x, y, z) on the plane. We cannot solve this system of four equations in four unknowns, but we know that it has a solution if and only if its determinant is zero. The expression below assumes this and also expands the determinant by its top row:

$$\begin{aligned} 0 &= \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \\ &= x \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} - y \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} + z \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \end{aligned}$$

This expression is of the form $Ax + By + Cz + D = 0$ where

$$A = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \quad B = -\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \quad C = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad D = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \tag{2.4}$$

◇ **Exercise 2.4:** Calculate the expression of the plane containing the z axis and passing through the point $(1, 1, 0)$.

◇ **Exercise 2.5:** In the plane equation $Ax + By + Cz + D = 0$, if $D = 0$, then the plane passes through the origin. Assuming $D \neq 0$, we can write the same equation as $x/a + y/b + z/c = 1$, where $a = -D/A$, $b = -D/B$, and $c = -D/C$. What is the geometrical interpretation of a, b, and c?

> We operate with nothing but things which do not exist, with lines, planes, bodies, atoms, divisible time, divisible space—how should explanation even be possible when we first make everything into an image, into our own image!
>
> —Friedrich Nietzsche

In some practical situations, the normal to the plane as well as one point on the plane, are known. It is easy to derive the plane equation in such a case.

We assume that $\mathbf{N}$ is the (known) normal vector to the plane, $\mathbf{P}_1$ is a known point, and $\mathbf{P}$ is any point in the plane. The vector $\mathbf{P} - \mathbf{P}_1$ is perpendicular to $\mathbf{N}$, so their dot product $\mathbf{N} \bullet (\mathbf{P} - \mathbf{P}_1)$ equals zero. Since the dot product is associative, we can write $\mathbf{N} \bullet \mathbf{P} = \mathbf{N} \bullet \mathbf{P}_1$. The dot product $\mathbf{N} \bullet \mathbf{P}_1$ is just a number, to be denoted by s, so we obtain

$$\mathbf{N} \bullet \mathbf{P} = s \quad \text{or} \quad N_x x + N_y y + N_z z - s = 0. \tag{2.5}$$

Equation (2.5) can now be written as $Ax + By + Cz + D = 0$, where $A = N_x$, $B = N_y$, $C = N_z$, and $D = -s = -\mathbf{N} \bullet \mathbf{P}_1$. The three unknowns A, B, and C are therefore the components of the normal vector and D can be calculated from any known point $\mathbf{P}_1$ on the plane. The expression $\mathbf{N} \bullet \mathbf{P} = s$ is a useful equation of the plane and is used elsewhere in this book.

◇ **Exercise 2.6:** Given $\mathbf{N} = (1, 1, 1)$ and $\mathbf{P}_1 = (1, 1, 1)$, calculate the plane equation.

Note that the direction of the normal in this case is unimportant. Substituting $(-A, -B, -C)$ for (A, B, C) would also change the sign of D, resulting in the same equation. However, the direction of the normal is important when the surface is to be shaded. To be used for the calculation of reflection, the normal has to point *outside* the surface. This has to be verified by the user, since the computer has no idea of the shape of the surface and the meaning of "inside" and "outside." In the case where a plane is defined by three points, the direction of the normal can be specified by arranging the three points (in the data structure in memory) in a certain order.

It is also easy to derive the equation of a plane when three points on the plane, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, are known. In order for the points to define a plane, they should not be collinear. We consider the vectors $\mathbf{r} = \mathbf{P}_2 - \mathbf{P}_1$ and $\mathbf{s} = \mathbf{P}_3 - \mathbf{P}_1$ a local coordinate system on the plane. Any point $\mathbf{P}$ on the plane can be expressed as a linear combination $\mathbf{P} = u\mathbf{r} + w\mathbf{s}$, where u and w are real numbers. Since $\mathbf{r}$ and $\mathbf{s}$ are local coordinates on the plane, the position of point $\mathbf{P}$ relative to the origin is expressed as (Figure 2.3)

$$\mathbf{P}(u, w) = \mathbf{P}_1 + u\mathbf{r} + w\mathbf{s}, \quad -\infty < u, w < \infty. \tag{2.6}$$

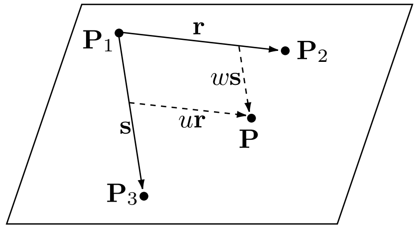


Figure 2.3: Three Points on a Plane.

◇ **Exercise 2.7:** Given the three points $\mathbf{P}_1 = (3, 0, 0)$, $\mathbf{P}_2 = (0, 3, 0)$, and $\mathbf{P}_3 = (0, 0, 3)$, write the equation of the plane defined by them.

2.2.3 Space Division

An infinite plane divides the entire three-dimensional space into two parts. We can call them "outside" and "inside" (or "above" and "below"), and define the outside direction as the direction pointed to by the normal. Using the plane equation, $\mathbf{N} \bullet \mathbf{P} = s$, it is possible to tell if a given point $\mathbf{P}_i$ lies inside, outside, or on the plane. All that's necessary is to examine the sign of the dot product $\mathbf{N} \bullet (\mathbf{P}_i - \mathbf{P})$, where $\mathbf{P}$ is any point on the plane, different from $\mathbf{P}_i$.

This dot product can also be written $|\mathbf{N}|\,|\mathbf{P}_i - \mathbf{P}| \cos\theta$, where θ is the angle between the normal $\mathbf{N}$ and the vector $\mathbf{P}_i - \mathbf{P}$. The sign of the dot product equals the sign of $\cos\theta$, and Figure 2.4a shows that for $-90° < \theta < 90°$, point $\mathbf{P}_i$ lies outside the plane, for $\theta = 90°$, point $\mathbf{P}_i$ lies on the plane, and for $\theta > 90°$, $\mathbf{P}_i$ lies inside the plane.

> The regular division of the plane into congruent figures evoking an association in the observer with a familiar natural object is one of these hobbies or problems.... I have embarked on this geometric problem again and again over the years, trying to throw light on different aspects each time. I cannot imagine what my life would be like if this problem had never occurred to me; one might say that I am head over heels in love with it, and I still don't know why.
>
> —M. C. Escher

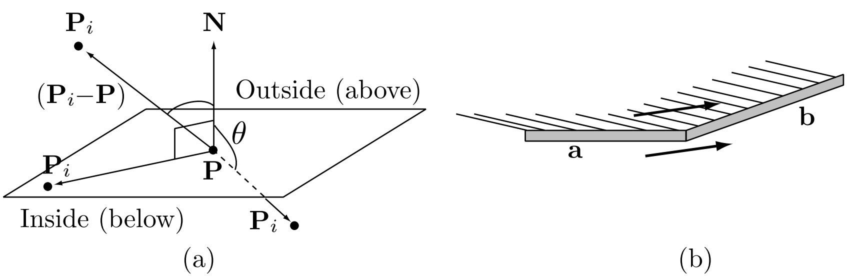


Figure 2.4: (a) Space Division. (b) Turning On a Polygon.

2.2.4 Turning Around on a Polygon

When moving along the edges of a polygon from vertex to vertex, we make a turn at each vertex. Sometimes, the "sense" of the turn (left or right) is important. However, the terms "left" and "right" are relative, depending on the location of the observer, and are therefore ambiguous. Consider Figure 2.4b. It shows two edges, **a** and **b**, of a "thick" polygon, with two arrows pointing from **a** to **b**. Imagine each arrow to be a bug crawling on the polygon. The bug on the top considers the turn from **a** to **b** a left turn, while the bug crawling on the bottom considers the same turn to be a "right" turn.

It is therefore preferable to define terms such as "positive turn" and "negative turn," that depend on the polygon and on the coordinate axes, but not on the position of any observer. To define these terms, consider the plane defined by the vectors **a** and **b** (if they are parallel, they don't define any plane, but then there is no sense talking about turning from **a** to **b**). The cross product $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane. It can point in the direction of the normal **N** to the plane, or in the opposite direction. In the former case, we say that the turn from **a** to **b** is positive; in the latter case, the turn is said to be negative.

To calculate the sense of the turn, simply check the sign of the triple scalar product $\mathbf{N} \bullet (\mathbf{a} \times \mathbf{b})$. A positive sign implies a positive turn.

◇ **Exercise 2.8:** Why?

2.2.5 Convex Polygons

Given a polygon, we select two arbitrary points on its edges and connect them with a straight line. If for any two such points the line is fully contained in the polygon, then the polygon is called convex. Another way to define a convex polygon is to say that a line can intersect such a polygon at only two points (unless the line is identical to one of the edges or it grazes the polygon at one point).

The sense of a turn (positive or negative) can also serve to define a convex polygon. When traveling from vertex to vertex in such a polygon all turns should have the same sense. They should all be positive or all negative. In contrast, when traveling along a concave polygon, both positive and negative turns must be made (Figure 2.5).

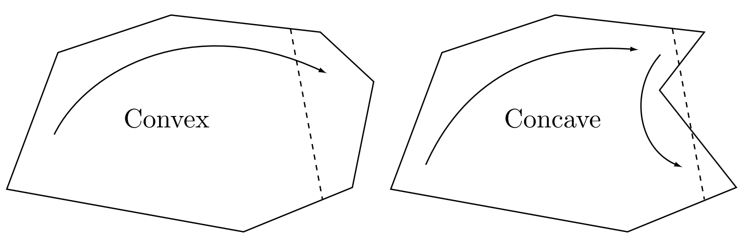


Figure 2.5: Convex and Concave Polygons.

We can think of a polygon as a set of points in two dimensions. The concept of a set of points, however, exists in any number of dimensions. A set of points is convex if it satisfies the definition regardless of the number of dimensions. One important concept associated with a set of points is the *convex hull* of the set. This is the set of "extreme" points that satisfies the following: the set obtained by connecting the points of the convex hull contains all the points of the set. (A simple, two-dimensional analogy is to consider the points nails driven into a board. A rubber band placed around all the nails and stretched will identify the points that constitute the convex hull.)

2.2.6 Line and Plane Intersection

Given a plane $\mathbf{N} \bullet \mathbf{P} = s$ and a line $\mathbf{P} = \mathbf{P}_1 + t\mathbf{d}$ [Equation (2.1)], it is easy to calculate their intersection point. We simply substitute the value of $\mathbf{P}$ in the plane equation to obtain $\mathbf{N} \bullet (\mathbf{P}_1 + t\mathbf{d}) = s$. This results in $t = (s - \mathbf{N} \bullet \mathbf{P}_1)/(\mathbf{N} \bullet \mathbf{d})$. Thus, we compute the value of t and substitute it in the line equation, to get the point of intersection. Such a process is important in ray tracing, an important rendering algorithm where the intersections of light rays and polygons are computed all the time.

◇ **Exercise 2.9:** The intersection of a line parallel to a plane is either the entire line (if the line happens to be in the plane) or is empty. How do we distinguish these cases from the equation above?

2.2.7 Triangles

A polygonal surface is often constructed of triangles. A triangle is flat but finite, whereas the plane equation describes an infinite plane. We therefore need to modify this equation to describe only the area inside a given triangle

Given any three noncollinear points $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ in three dimensions, we first derive the equation of the (infinite) plane defined by them. Following that, we limit ourselves to just that part of the plane that's inside the triangle. We start with the two vectors $(\mathbf{P}_2 - \mathbf{P}_1)$ and $(\mathbf{P}_3 - \mathbf{P}_1)$. They can serve as local coordinate axes on the plane (even though they are not normally perpendicular), with point $\mathbf{P}_1$ as the local origin. The linear combination $u(\mathbf{P}_2 - \mathbf{P}_1) + w(\mathbf{P}_3 - \mathbf{P}_1)$, where both u and w can take any real values, is a vector on the plane. To get the coordinates of an arbitrary point on the plane, we simply add point $\mathbf{P}_1$ to this linear combination (recall that the sum of a point and a vector is a point). The resulting plane equation is

$$\mathbf{P}_1 + u(\mathbf{P}_2 - \mathbf{P}_1) + w(\mathbf{P}_3 - \mathbf{P}_1) = \mathbf{P}_1(1 - u - w) + \mathbf{P}_2 u + \mathbf{P}_3 w. \tag{2.7}$$

To limit the area covered to just the triangle whose corners are $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, we note that Equation (2.7) yields

$$\begin{aligned}&\mathbf{P}_1, \text{ when } u = 0 \text{ and } w = 0,\\ &\mathbf{P}_2, \text{ when } u = 1 \text{ and } w = 0,\\ &\mathbf{P}_3, \text{ when } u = 0 \text{ and } w = 1.\end{aligned}$$

The entire triangle can therefore be obtained by varying u and w under the conditions $u \ge 0, w \ge 0$, and $u + w \le 1$.

◇ **Exercise 2.10:** Given the three points $\mathbf{P}_1 = (10, -5, 4)$, $\mathbf{P}_2 = (8, -4, 3.2)$, and $\mathbf{P}_3 = (8, 4, 3.2)$, derive the equation of the triangle defined by them.

If triangles had a God, He'd have three sides. —Yiddish proverb

◇ **Exercise 2.11:** Given the three points $\mathbf{P}_1 = (10, -5, 4)$, $\mathbf{P}_2 = (8, -4, 3.2)$, and $\mathbf{P}_3 = (12, -6, 4.8)$, calculate the triangle defined by them.

For more information, see [Triangles 04] or [Kimberling 94].

2.3 Bilinear Surfaces

A flat polygon is the simplest type of surface. The bilinear surface is the simplest nonflat (curved) surface because it is fully defined by means of its four corner points. It is discussed here because its four boundary curves are straight lines and because the coordinates of any point on this surface are derived by linear interpolations. Since this patch is completely defined by its four corner points, it cannot have a very complex shape. Nevertheless it may be highly curved. If the four corners are coplanar, the bilinear patch defined by them is flat.

Let the corner points be the four distinct points $\mathbf{P}_{00}$, $\mathbf{P}_{01}$, $\mathbf{P}_{10}$, and $\mathbf{P}_{11}$. The top and bottom boundary curves are straight lines and are easy to calculate (Figure 2.6). They are $\mathbf{P}(u, 0) = \big(\mathbf{P}_{10} - \mathbf{P}_{00}\big)u + \mathbf{P}_{00}$ and $\mathbf{P}(u, 1) = \big(\mathbf{P}_{11} - \mathbf{P}_{01}\big)u + \mathbf{P}_{01}$.

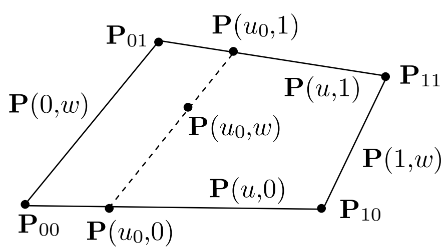


Figure 2.6: A Bilinear Surface.

To linearly interpolate between these boundary curves, we first calculate two corresponding points $\mathbf{P}(u_0, 0)$ and $\mathbf{P}(u_0, 1)$, one on each curve, then connect them with a straight line $\mathbf{P}(u_0, w)$. The two points are

$$\mathbf{P}(u_0, 0) = (\mathbf{P}_{10} - \mathbf{P}_{00})u_0 + \mathbf{P}_{00} \quad \text{and} \quad \mathbf{P}(u_0, 1) = (\mathbf{P}_{11} - \mathbf{P}_{01})u_0 + \mathbf{P}_{01},$$

and the straight segment connecting them is

$$\begin{aligned}\mathbf{P}(u_0, w) &= (\mathbf{P}(u_0, 1) - \mathbf{P}(u_0, 0))\, w + \mathbf{P}(u_0, 0)\\ &= \Big[(\mathbf{P}_{11} - \mathbf{P}_{01})u_0 + \mathbf{P}_{01} - \big((\mathbf{P}_{10} - \mathbf{P}_{00})u_0 + \mathbf{P}_{00}\big)\Big]w\\ &\quad + (\mathbf{P}_{10} - \mathbf{P}_{00})u_0 + \mathbf{P}_{00}.\end{aligned}$$

The expression for the entire surface is obtained when we release the parameter u from its fixed value u_0 and let it vary. The result is:

$$\begin{aligned}\mathbf{P}(u,w) &= \mathbf{P}_{00}(1-u)(1-w) + \mathbf{P}_{01}(1-u)w + \mathbf{P}_{10}u(1-w) + \mathbf{P}_{11}uw \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} B_{1i}(u)\mathbf{P}_{ij}B_{1j}(w), \qquad (2.8)\\ &= [B_{10}(u), B_{11}(u)]\begin{bmatrix}\mathbf{P}_{00} & \mathbf{P}_{01}\\ \mathbf{P}_{10} & \mathbf{P}_{11}\end{bmatrix}\begin{bmatrix}B_{10}(w)\\ B_{11}(w)\end{bmatrix},\end{aligned}$$

where the functions $B_{1i}(t)$ are the Bernstein polynomials of degree 1, introduced in Section 6.16. This implies that the bilinear surface is a special case of the rectangular Bézier surface, introduced in the same section. (The Bernstein polynomials crop up in unexpected places.) Mathematically, the bilinear surface is a hyperbolic paraboloid (see answer to exercise 2.12). Its parametric expression is linear in both u and w.

The expression $\mathbf{P}(t) = (1-t)\mathbf{P}_1 + t\,\mathbf{P}_2$ has already been introduced. This is the straight segment from point $\mathbf{P}_1$ to point $\mathbf{P}_2$ expressed as a blend (or a barycentric sum) of the points with the two weights $(1-t)$ and t. Since $B_{10}(t) = 1-t$ and $B_{11}(t) = t$, this expression can also be written in the form

$$[B_{10}(t), B_{11}(t)]\begin{bmatrix}\mathbf{P}_1\\ \mathbf{P}_2\end{bmatrix}. \qquad (2.9)$$

The reader should notice the similarity between Equations (2.8) and (2.9). The former expression is a direct extension of the latter and is a simple example of the technique of Cartesian product, discussed in Section 1.9, which is used to extend many curves to surfaces.

Figure 2.7 shows a bilinear surface together with the *Mathematica* code that produced it. The coordinates of the four corner points and the final, simplified expression of the surface are also included. The figure illustrates the bilinear nature of this surface. Every line in the u or in the w directions on this surface is straight, but the surface itself is curved.

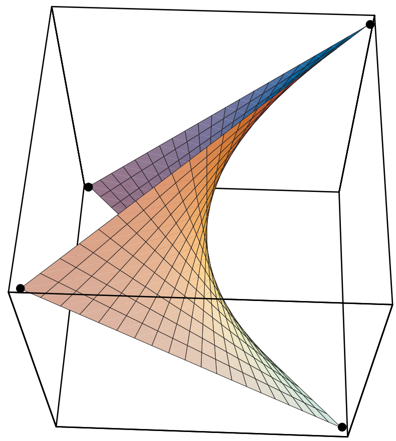
Example: We select the four points $\mathbf{P}_{00} = (0,0,1)$, $\mathbf{P}_{10} = (1,0,0)$, $\mathbf{P}_{01} = (1,1,1)$, and $\mathbf{P}_{11} = (0,1,0)$ (Figure 2.7) and apply Equation (2.8). The resulting surface patch is

$$\begin{aligned}P(u,w) &= (0,0,1)(1-u)(1-w) + (1,1,1)(1-u)w + (1,0,0)u(1-w) + (0,1,0)uw\\ &= \bigl(u+w-2uw, w, 1-u\bigr). \qquad (2.10)\end{aligned}$$

It is easy to check the expression by substituting $u = 0,1$ and $w = 0,1$, which reduces the expression to the four corner points. The tangent vectors can easily be calculated. They are

$$\frac{\partial\mathbf{P}(u,w)}{\partial u} = (1-2w, 0, -1), \qquad \frac{\partial\mathbf{P}(u,w)}{\partial w} = (1-2u, 1, 0).$$

The first vector lies in the xz plane, and the second lies in the xy plane.



```
(* a bilinear surface patch *)
Clear[bilinear,pnts,u,w];
<<:Graphics:ParametricPlot3D.m;
pnts=ReadList["Points",{Number,Number,Number}, RecordLists->True];
bilinear[u_,w_]:=pnts[[1,1]](1-u)(1-w)+pnts[[1,2]]u(1-w) \
 +pnts[[2,1]]w(1-u)+pnts[[2,2]]u w;
Simplify[bilinear[u,w]]
g1=Graphics3D[{AbsolutePointSize[5], Table[Point[pnts[[i,j]]],{i,1,2},{j,1,2}]}];
g2=ParametricPlot3D[bilinear[u,w],{u,0,1,.05},{w,0,1,.05}, Compiled->False,
 DisplayFunction->Identity];
Show[g1,g2, ViewPoint->{0.063, -1.734, 2.905}];
{{0, 0, 1}, {1, 1, 1}, {1, 0, 0}, {0, 1, 0}}
{u + w - 2 u w, u, 1 - w}
```

Figure 2.7: A Bilinear Surface.

Example: The four points $\mathbf{P}_{00} = (0, 0, 1)$, $\mathbf{P}_{10} = (1, 0, 0)$, $\mathbf{P}_{01} = (0.5, 1, 0)$, and $\mathbf{P}_{11} = (1, 1, 0)$ are selected and Equation (2.8) is applied to them. The resulting surface patch is (Figure 2.8)

$$\begin{aligned} P(u, w) &= (0, 0, 1)(1-u)(1-w) + (0.5, 1, 0)(1-u)w + (1, 0, 0)u(1-w) + (1, 1, 0)uw \\ &= \big(0.5(1-u)w + u, w, (1-u)(1-w)\big). \end{aligned} \tag{2.11}$$

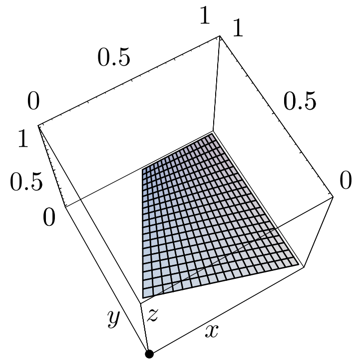
Note that the y coordinate is simply w. This means that points with the same w value, such as $\mathbf{P}(0.1, w)$ and $\mathbf{P}(0.5, w)$ have the same y coordinate and are therefore located on the same horizontal line. Also, the z coordinate is a simple function of u and w, varying from 1 (when $u = w = 0$) to 0 as we move toward $u = 1$ or $w = 1$.

The boundary curves are very easy to calculate from Equation (2.11). Here are two of them

$$\mathbf{P}(0, w) = (0.5w, w, 1 - w), \qquad \mathbf{P}(u, 1) = (0.5(1-u) + u, 1, 0).$$

The tangent vectors can also be obtained from Equation (2.11)

$$\frac{\partial \mathbf{P}(u, w)}{\partial u} = (-0.5w + 1, 0, w - 1), \quad \frac{\partial \mathbf{P}(u, w)}{\partial w} = (0.5(1-u), 1, u - 1). \tag{2.12}$$




```
(* Another bilinear surface example *)
ParametricPlot3D[{0.5(1-u)w+u,w,(1-u)(1-w)}, {u,0,1},{w,0,1}, Compiled->False,
ViewPoint->{-0.846, -1.464, 3.997}, DefaultFont->{"cmr10", 10}];
```

Figure 2.8: A Bilinear Surface.

The first is a vector in the xz plane, while the second is a vector in the $y=1$ plane. The following two tangent values are especially simple: $\frac{\partial\mathbf{P}(u,1)}{\partial u}=(0.5,0,0)$ and $\frac{\partial\mathbf{P}(1,w)}{\partial w}=(0,1,0)$. The first is a vector in the x direction and the second is a vector in the y direction.

Finally, we compute the *normal vector* to the surface. This vector is normal to the surface at any point, so it is perpendicular to the two tangent vectors $\partial\mathbf{P}(u,w)/\partial u$ and $\partial\mathbf{P}(u,w)/\partial w$ and is therefore the cross-product [Equation (1.5)] of these vectors. The calculation is straightforward:

$$\mathbf{N}(u,w)=\frac{\partial\mathbf{P}}{\partial u}\times\frac{\partial\mathbf{P}}{\partial w}=(1-w,0.5(1-u),1-0.5w). \tag{2.13}$$

There are two ways of satisfying ourselves that Equation (2.13) is the correct expression for the normal:

1. It is easy to prove, by directly calculating the dot products, that the normal vector of Equation (2.13) is perpendicular to both tangents of Equation (2.12).

2. A closer look at the coordinates of our points shows that three of them have a z coordinate of zero and only $\mathbf{P}_{00}$ has $z=1$. This means that the surface approaches a flat xy surface as one moves away from point $\mathbf{P}_{00}$. It also means that the normal should approach the z direction when u and w move away from zero, and it should move away from that direction when u and w approach zero. It is, in fact, easy to confirm the following limits:

$$\lim_{u,w\to 1}\mathbf{N}(u,w)=(0,0,0.5),\qquad \lim_{u,w\to 0}\mathbf{N}(u,w)=(1,0.5,1).$$

◇ **Exercise 2.12:** (1) Calculate the bilinear surface for the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,1)$. (2) Guess the explicit representation $z=F(x,y)$ of this surface. (3) What curve results from the intersection of this surface with the plane $z=k$ (parallel to the

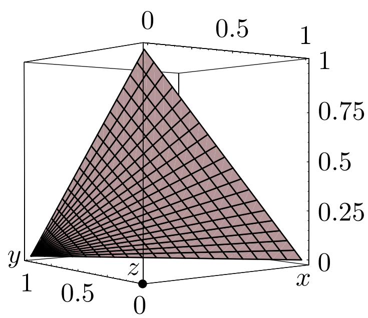
xy plane). (4) What curve results from the intersection of this surface with a plane containing the z axis?

> The scale, properly speaking, does not permit the measure of the intelligence, because intellectual qualities are not superposable, and therefore cannot be measured as linear surfaces are measured.
>
> —Alfred Binet (on his new IQ test)

Example: This is the third example of a bilinear surface. The four points $\mathbf{P}_{00} = (0,0,1)$, $\mathbf{P}_{10} = (1,0,0)$, and $\mathbf{P}_{01} = \mathbf{P}_{11} = (0,1,0)$ create a *triangular surface patch* (Figure 2.9) because two of them are identical. The surface expression is

$$\begin{aligned} P(u,w) &= (0,0,1)(1-u)(1-w) + (0,1,0)(1-u)w + (1,0,0)u(1-w) + (0,1,0)uw \\ &= \big(u(1-w), w, (1-u)(1-w)\big). \end{aligned}$$

Notice that the boundary curve $\mathbf{P}(u,1)$ degenerates to the single point $(0,1,0)$, i.e., it does not depend on u.




```
(* A Triangular bilinear surface example *)
ParametricPlot3D[{u(1-w),w,(1-u)(1-w)}, {u,0,1},{w,0,1}, Compiled->False,
ViewPoint->{-2.673, -3.418, 0.046}, DefaultFont->{"cmr10", 10}];
```

Figure 2.9: A Triangular Bilinear Surface.

◇ **Exercise 2.13:** Calculate the tangent vectors and the normal vector of this surface.

◇ **Exercise 2.14:** Given the two points $\mathbf{P}_{00} = (-1,-1,0)$ and $\mathbf{P}_{10} = (1,-1,0)$, consider them the endpoints of a straight segment $\mathbf{L}_1$.

(1) Construct the endpoints of the three straight segments $\mathbf{L}_2$, $\mathbf{L}_3$, and $\mathbf{L}_4$. Each should be translated one unit above its predecessor on the y axis and should be rotated 60° about the y axis, as shown in Figure 2.10. Denote the four pairs of endpoints by $\mathbf{P}_{00}\mathbf{P}_{10}$, $\mathbf{P}_{01}\mathbf{P}_{11}$, $\mathbf{P}_{02}\mathbf{P}_{12}$ and $\mathbf{P}_{03}\mathbf{P}_{13}$.

(2) Calculate the three bilinear surface patches

$$\begin{aligned}
\mathbf{P}_1(u,w) =& \mathbf{P}_{00}(1-u)(1-w) + \mathbf{P}_{01}(1-u)w + \mathbf{P}_{10}u(1-w) + \mathbf{P}_{11}uw,\\
\mathbf{P}_2(u,w) =& \mathbf{P}_{01}(1-u)(1-w) + \mathbf{P}_{02}(1-u)w + \mathbf{P}_{11}u(1-w) + \mathbf{P}_{12}uw,\\
\mathbf{P}_3(u,w) =& \mathbf{P}_{02}(1-u)(1-w) + \mathbf{P}_{03}(1-u)w + \mathbf{P}_{12}u(1-w) + \mathbf{P}_{13}uw.
\end{aligned}$$

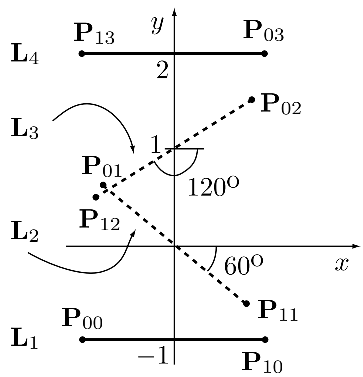


Figure 2.10: Four Straight Segments for Exercise 2.14.

2.4 Lofted Surfaces

This kind of surface patch is curved, but it belongs in this chapter because it is linear in one direction. It is bounded by two arbitrary curves [that we denote by $\mathbf{P}(u,0)$ and $\mathbf{P}(u,1)$] and by two straight segments $\mathbf{P}(0,w)$ and $\mathbf{P}(1,w)$ connecting them. Surface lines in the w direction are therefore straight, whereas each line in the u direction is a blend of $\mathbf{P}(u,0)$ and $\mathbf{P}(u,1)$. The blend of the two curves is simply $(1-w)\mathbf{P}(u,0)+w\mathbf{P}(u,1)$, and this blend, which is linear in w, constitutes the expression of the surface

$$\mathbf{P}(u,w) = (1-w)\mathbf{P}(u,0) + w\mathbf{P}(u,1). \tag{2.14}$$

This expression is linear in w, implying straight lines in the w direction. Moving in the u direction, we travel on a curve whose shape depends on the value of w. For $w_0 \approx 0$, the curve $\mathbf{P}(u,w_0)$ is close to the boundary curve $\mathbf{P}(u,0)$. For $w_0 \approx 1$, it is close to the boundary curve $\mathbf{P}(u,1)$. For $w_0 = 0.5$, it is $0.5\mathbf{P}(u,0)+0.5\mathbf{P}(u,1)$, an equal mixture of the two.

Note that this kind of surface is fully defined by specifying the two boundary curves. The four corner points are implicit in these curves. These surfaces are sometimes called *ruled*, because straight lines are an important part of their description. This is also the reason why this type of surface is sometimes defined as follows: a surface is a lofted surface if and only if through every point on it there is a straight line that lies completely on the surface.

This definition implies that any cylinder is a lofted surface, but a little thinking shows that even a bilinear surface is lofted.

Example: We start with the six points $\mathbf{P}_1 = (-1,0,0)$, $\mathbf{P}_2 = (0,-1,0)$, $\mathbf{P}_3 = (1,0,0)$, $\mathbf{P}_4 = (-1,0,1)$, $\mathbf{P}_5 = (0,-1,1)$, and $\mathbf{P}_6 = (1,0,1)$. Because of the special coordinates of the points (and because of the way we will compute the boundary curves), the surface is easy to visualize (Figure 2.11). This helps to intuitively make sense of the expressions for the tangent vectors and the normal. Note especially that the left and right edges of the surface are in the xz plane, whereas we will see that all the other lines in the w direction have a small negative y component.

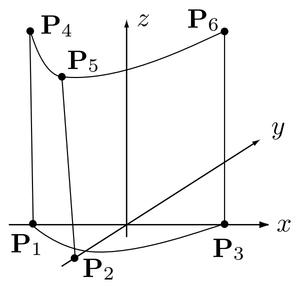


Figure 2.11: A Lofted Surface.

We proceed in six steps as follows:

1. As the top boundary curve, $\mathbf{P}(u,1)$, we select the quadratic polynomial passing through the top three points $\mathbf{P}_4$, $\mathbf{P}_5$, and $\mathbf{P}_6$. There is only one such curve and it has the form $\mathbf{P}(u,1) = \mathbf{A} + \mathbf{B}u + \mathbf{C}u^2$, where the coefficients $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ have to be calculated. We use the fact that the curve passes through the three points to set up the three equations $\mathbf{P}(0,1) = \mathbf{P}_4$, $\mathbf{P}(0.5,1) = \mathbf{P}_5$, and $\mathbf{P}(1,1) = \mathbf{P}_6$, that are written explicitly as

$$\begin{aligned}
\mathbf{A} + \mathbf{B}\times 0 + \mathbf{C}\times 0^2 &= (-1,0,1),\\
\mathbf{A} + \mathbf{B}\times 0.5 + \mathbf{C}\times 0.5^2 &= (0,-1,1),\\
\mathbf{A} + \mathbf{B}\times 1 + \mathbf{C}\times 1^2 &= (1,0,1).
\end{aligned}$$

These are easy to solve and result in $\mathbf{A} = (-1,0,1)$, $\mathbf{B} = (2,-4,0)$, and $\mathbf{C} = (0,4,0)$. The top boundary curve is therefore $\mathbf{P}(u,1) = \big(2u-1, 4u(u-1), 1\big)$.

2. As the bottom boundary curve, we select the quadratic Bézier curve [Equation (6.6)] defined by the three points $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. The curve is

$$\begin{aligned}
\mathbf{P}(u,0) &= \sum_{i=0}^{2} B_{2i}(u)\mathbf{P}_{i+1}\\
&= (1-u)^2(-1,0,0) + 2u(1-u)(0,-1,0) + u^2(1,0,0)\\
&= \big(2u-1, -2u(1-u), 0\big).
\end{aligned}$$

3. The expression of the surface is immediately obtained

$$\mathbf{P}(u,w) = \mathbf{P}(u,0)(1-w) + \mathbf{P}(u,1)w = \big(2u-1, 2u(u-1)(1+w), w\big).$$

(Notice that it does not pass through $\mathbf{P}_2$.)

4. The two tangent vectors are also easy to compute

$$\frac{\partial \mathbf{P}}{\partial u} = \bigl(2, 2(2u-1)(1+w), 0\bigr), \quad \frac{\partial \mathbf{P}}{\partial w} = \bigl(0, 2u(u-1), 1\bigr).$$

5. The normal, as usual, is the cross-product of the tangents and is given by $\mathbf{N}(u,w) = \bigl(2(2u-1)(1+w), -2, 4u(u-1)\bigr)$.

6. The most important feature of this example is the ease with which the expressions of the tangents and the normal can be visualized. This is possible because of the simple shape and orientation of the surface (again, see Figure 2.11). The reader should examine the expressions and make sure the following points are clear:

- The two boundary curves are very similar. One difference between them is, of course, the x and z coordinates. However, the *only* important difference is in the y coordinate. Both curves are quadratic polynomials in u, but although $\mathbf{P}(u,1)$ passes through the three top points, $\mathbf{P}(u,0)$ passes only through the first and last points.

- The tangent in the u direction, $\partial\mathbf{P}/\partial u$, features $z = 0$; it is a vector in the xy plane. At the bottom of the surface, where $w = 0$, it changes direction from $(2,-2,0)$ (when $u = 0$) to $(2,2,0)$ (when $u = 1$), both $45°$ directions in the xy plane. However, at the top, where $w = 1$, the tangent changes direction from $(2,-4,0)$ to $(2,4,0)$, both $63°$ directions. This is because the top boundary curve goes deeper in the y direction.

- The tangent in the w direction, $\partial\mathbf{P}/\partial w$ features $x = 0$; it is a vector in the yz plane. Its z coordinate is a constant 1, and its y coordinate varies from 0 (on the left, where $u = 0$), to -0.5 (in the middle, where $u = 0.5$), and back to 0 (on the right, where $u = 1$). On the left and right edges of the surface, this vector is therefore vertical $(0,0,1)$. In the middle, it is $(0,-0.5,1)$, making a negative half-step in y for each step in z.

- The normal vector features $y = -2$ with a small z component. It therefore points mostly in the negative y direction, and a little in x. At the bottom ($w = 0$), it varies from $(-2,-2,0)$, to $(0,-2,-1)$,* and ends in $(2,-2,0)$. At the top ($w = 1$), it varies from $(-4,-2,0)$, to $(0,-2,-1)$, and ends in $(4,-2,0)$. The top boundary curve is deeper, causing the tangent to be more in the y direction and the normal to be more in the x direction, than on the bottom boundary curve.

◇ **Exercise 2.15:** (a) Given the two three-dimensional points $\mathbf{P}_1 = (-1,-1,0)$ and $\mathbf{P}_2 = (1,-1,0)$, calculate the straight line from $\mathbf{P}_1$ to $\mathbf{P}_2$. This will become the bottom boundary curve of a lofted surface.

(b) Given the three three-dimensional points $\mathbf{P}_4 = (-1,1,0)$, $\mathbf{P}_5 = (0,1,1)$, and $\mathbf{P}_6 = (1,1,0)$, calculate the quadratic polynomial $\mathbf{P}(t) = \mathbf{A}t^2 + \mathbf{B}t + \mathbf{C}$ that passes through them. This will become the top boundary curve of the surface.

(c) Calculate the expression of the lofted surface patch and the coordinates of its center point $\mathbf{P}(0.5, 0.5)$.

---

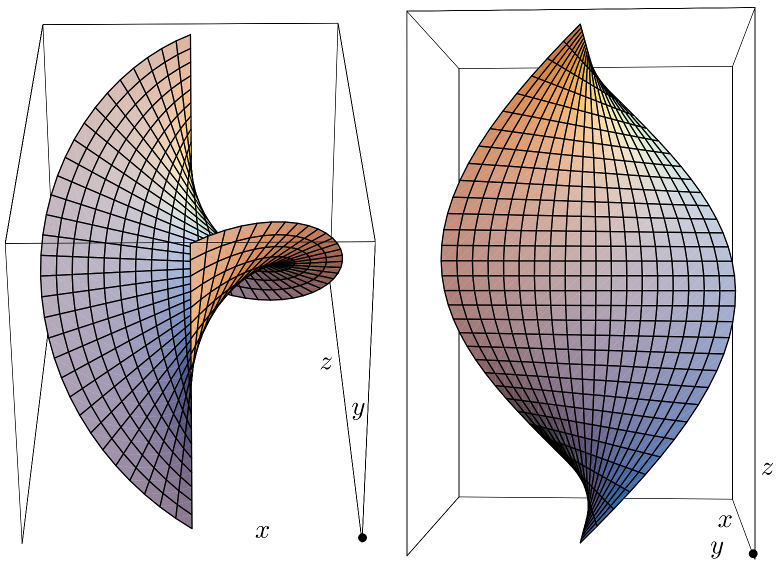
* it has a small z component, reflecting the fact that the surface is not completely vertical at $u = 0.5$.

2.4.1 A Double Helix

This example illustrates how the well-known double helix can be derived as a lofted surface. The two-dimensional parametric curve $(\cos t, \sin t)$ is, of course, a circle (of radius one unit, centered on the origin). As a result, the three-dimensional curve $(\cos t, \sin t, t)$ is a helix spiraling around the z axis upward from the origin. The similar curve $(\cos(t+\pi), \sin(t+\pi), t)$ is another helix, at a 180° phase difference with the first. We consider these the two boundary curves of a lofted surface and create the entire surface as a linear interpolation of the two curves. Hence,

$$\mathbf{P}(u,w) = (\cos u, \sin u, u)(1-w) + (\cos(u+\pi), \sin(u+\pi), u)w,$$

where $0 \le w \le 1$, and u can vary in any range. The two curves form a double helix, so the surface looks like a twisted ribbon. Figure 2.12 shows such a surface, together with the code that generated it.




```
Clear[loftedSurf]; (* double helix as a lofted surface *)
<<:Graphics:ParametricPlot3D.m;
loftedSurf:={Cos[u],Sin[u],u}(1-w)+{Cos[u+Pi],Sin[u+Pi],u}w;
ParametricPlot3D[loftedSurf, {u,0,Pi,.1},{w,0,1}, Compiled->False,
Ticks->False, ViewPoint->{-2.640, -0.129, 0.007}]
```

Figure 2.12: The Double Helix as a Lofted Surface.

◇ **Exercise 2.16:** Calculate the expression of a cone as a lofted surface. Assume that the vertex of the cone is located at the origin, and the base is a circle of radius R, centered on the z axis and located on the plane $z = H$.

◇ **Exercise 2.17:** Derive the expression for a square pyramid where each face is a lofted surface. Assume that the base is a square, $2a$ units on a side, centered about the origin on the xy plane. The top is point $(0, 0, H)$.

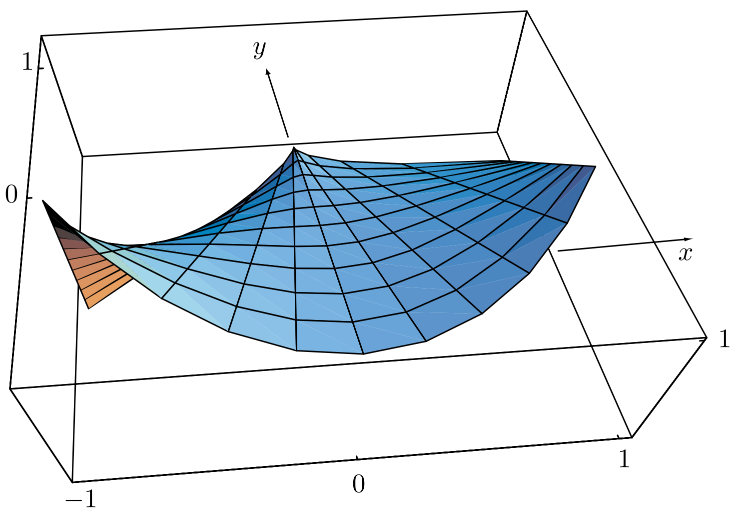
2.4.2 A Cusp

Given the two curves $\mathbf{P}_1(u) = (8,4,0)u^3-(12,9,0)u^2+(6,6,0)u+(-1,0,0)$ and $\mathbf{P}_2(u) = (2u-1, 4u(u-1), 1)$, the lofted surface defined by them is easy to calculate. Notice that the curves pass through the points $\mathbf{P}_1(0) = (-1,0,0)$, $\mathbf{P}_1(0.5) = (0,5/4,0)$, $\mathbf{P}_1(1) = (1,1,0)$, $\mathbf{P}_2(0) = (-1,0,1)$, $\mathbf{P}_2(0.5) = (0,-1,1)$, and $\mathbf{P}_2(1) = (1,0,1)$, which makes it easy to visualize the surface (Figure 2.13). The tangent vectors of the two curves are

$$\mathbf{P}_1^u(u) = (24,12,0)u^2-(24,18,0)u+(6,6,0), \quad \mathbf{P}_2^u(u) = (2, 8u-4, 0).$$

Notice that $\mathbf{P}_1^u(0.5)$ equals $(0,0,0)$, which implies that $\mathbf{P}_1(u)$ has a cusp at $u = 0.5$. The lofted surface defined by the two curves is

$$\mathbf{P}(u,w) = \big(4u^2(2u-3)(1-w)-4uw+6u-1, u^2(4u-9)(1-w)+4u^2w-10uw+6u, w\big).$$




```
(* Another lofted surface example *)
<<:Graphics:ParametricPlot3D.m
Clear[ls];
ls=Simplify[{8u^3-12u^2+6u-1,4u^3-9u^2+6u,0}(1-w)+{2u-1,4u(u-1),1}w];
ParametricPlot3D[ls, {u,0,1,.1},{w,0,1,.1}, Compiled->False,
 ViewPoint->{-0.139, -1.179, 1.475}, DefaultFont->{"cmr10", 10},
 AspectRatio->Automatic, Ticks->{{0,1},{0,1},{0,1}}];
```

Figure 2.13: A Lofted Surface Patch.

> Now, look Gwen, y'know if we're gonna keep living together in this loft, we're gonna have to have some rules.
>
> —Leah Remini (as Terri Reynolds) in *Fired Up* (1997)

◇ **Exercise 2.18:** Calculate the tangent vector of this surface in the u direction, and compute its value at the cusp.

> LERP, a quasi-acronym for Linear Interpolation, used as a verb or noun for the operation. "Bresenham's algorithm lerps incrementally between the two endpoints of the line."
>
> *The New Hacker's Dictionary version 4.2.2, a.k.a., The Jargon File*