SOLUTION If we regard the order in which the balls are selected as being significant, then as the first drawn ball may be any of the 11 and the second any of the remaining 10, it follows that the sample space consists of $11 \cdot 10 = 110$ points. Furthermore, there are $6 \cdot 5 = 30$ ways in which the first ball selected is white and the second black, and similarly there are $5 \cdot 6 = 30$ ways in which the first ball is black and the second white. Hence, assuming that "randomly drawn" means that each of the 110 points in the sample space is equally likely to occur, then we see that the desired probability is

$$\frac{30+30}{110} = \frac{6}{11} \quad ■$$

When there are more than two experiments to be performed the basic principle can be generalized as follows:

Generalized Basic Principle of Counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if, . . . , then there are a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

As an illustration of this, let us determine the number of different ways n distinct objects can be arranged in a linear order. For instance, how many different ordered arrangements of the letters a, b, c are possible? By direct enumeration we see that there are 6; namely, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. Each one of these ordered arrangements is known as a *permutation*. Thus, there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then chosen from the remaining one. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Suppose now that we have n objects. Similar reasoning shows that there are

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

different permutations of the n objects. It is convenient to introduce the notation $n!$, which is read "n factorial," for the foregoing expression. That is,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

Thus, for instance, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and so on. It is convenient to define $0! = 1$.

EXAMPLE 3.5b Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

SOLUTION There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4! 3! 2! 1! possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is $4!\,4!\,3!\,2!\,1! = 6{,}912$. ■

EXAMPLE 3.5c A class in probability theory consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same score, **(a)** how many different rankings are possible? **(b)** If all rankings are considered equally likely, what is the probability that women receive the top 4 scores?

SOLUTION

(a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, we see the answer to this part is $10! = 3{,}628{,}800$.

(b) Because there are 4! possible rankings of the women among themselves and 6! possible rankings of the men among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17{,}280$ possible rankings in which the women receive the top 4 scores. Hence, the desired probability is

$$\frac{6!4!}{10!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7} = \frac{1}{210} \quad ■$$

Suppose now that we are interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of three could be selected from the five items A, B, C, D, E? To answer this, reason as follows. Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to then select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3, say the group consisting of items A, B, and C, will be counted 6 times (that is, all of the permutations ABC, ACB, BAC, BCA, CAB, CBA will be counted when the order of selection is relevant), it follows that the total number of different groups that can be formed is $(5 \cdot 4 \cdot 3)/(3 \cdot 2 \cdot 1) = 10$.

In general, as $n(n-1) \cdots (n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is considered relevant (since the first one selected can be any one of the n, and the second selected any one of the remaining $n-1$, etc.), and since each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

NOTATION AND TERMINOLOGY

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{r!}{(n-r)!r!}$$

and call $\binom{n}{r}$ the number of *combinations* of n objects taken r at a time.

Thus $\binom{n}{r}$ represents the number of different groups of size r that can be selected from a set of size n when the order of selection is not considered relevant. For example, there are

$$\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

different groups of size 2 that can be chosen from a set of 8 people, and

$$\binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$$

different groups of size 2 that can be chosen from a set of 10 people. Also, since $0! = 1$, note that

$$\binom{n}{0} = \binom{n}{n} = 1$$

EXAMPLE 3.5d A committee of size 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

SOLUTION Let us assume that "randomly selected" means that each of the $\binom{15}{5}$ possible combinations is equally likely to be selected. Hence, since there are $\binom{6}{3}$ possible choices of 3 men and $\binom{9}{2}$ possible choices of 2 women, it follows that the desired probability is given by

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001} \quad \blacksquare$$

EXAMPLE 3.5e From a set of n items a random sample of size k is to be selected. What is the probability a given item will be among the k selected?

SOLUTION The number of different selections that contain the given item is $\binom{1}{1}\binom{n-1}{k-1}$. Hence, the probability that a particular item is among the k selected is

$$\binom{n-1}{k-1}\Big/\binom{n}{k} = \frac{(n-1)!}{(n-k)!(k-1)!}\Big/\frac{n!}{(n-k)!k!} = \frac{k}{n} \quad ■$$

EXAMPLE 3.5f A basketball team consists of 6 black and 6 white players. The players are to be paired in groups of two for the purpose of determining roommates. If the pairings are done at random, what is the probability that none of the black players will have a white roommate?

SOLUTION Let us start by imagining that the 6 pairs are numbered — that is, there is a first pair, a second pair, and so on. Since there are $\binom{12}{2}$ different choices of a first pair; and for each choice of a first pair there are $\binom{10}{2}$ different choices of a second pair; and for each choice of the first 2 pairs there are $\binom{8}{2}$ choices for a third pair; and so on, it follows from the generalized basic principle of counting that there are

$$\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{12!}{(2!)^6}$$

ways of dividing the players into a *first* pair, a *second* pair, and so on. Hence there are $(12)!/2^6 6!$ ways of dividing the players into 6 (unordered) pairs of 2 each. Furthermore, since there are, by the same reasoning, $6!/2^3 3!$ ways of pairing the white players among themselves and $6!/2^3 3!$ ways of pairing the black players among themselves, it follows that there are $(6!/2^3 3!)^2$ pairings that do not result in any black–white roommate pairs. Hence, if the pairings are done at random (so that all outcomes are equally likely), then the desired probability is

$$\left(\frac{6!}{2^3 3!}\right)^2 \Big/ \frac{(12)!}{2^6 6!} = \frac{5}{231} = .0216$$

Hence, there are roughly only two chances in a hundred that a random pairing will not result in any of the white and black players rooming together. ■

EXAMPLE 3.5g If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

SOLUTION Because each person can celebrate his or her birthday on any one of 365 days, there are a total of $(365)^n$ possible outcomes. (We are ignoring the possibility of someone having been born on February 29.) Furthermore, there are $(365)(364)(363) \cdot (365 - n + 1)$ possible outcomes that result in no two of the people having the same birthday. This is so