**2.18** Which of the following pairs of events are mutually exclusive?

(a) A golfer scoring the lowest 18-hole round in a 72-hole tournament and losing the tournament.

(b) A poker player getting a flush (all cards in the same suit) and 3 of a kind on the same 5-card hand.

(c) A mother giving birth to a baby girl and a set of twin daughters on the same day.

(d) A chess player losing the last game and winning the match.

**2.19** Suppose that a family is leaving on a summer vacation in their camper and that $M$ is the event that they will experience mechanical problems, $T$ is the event that they will receive a ticket for committing a traffic violation, and $V$ is the event that they will arrive at a campsite with no vacancies. Referring to the Venn diagram of Figure 2.5, state in words the events represented by the following regions:

(a) region 5;

(b) region 3;

(c) regions 1 and 2 together;

(d) regions 4 and 7 together;

(e) regions 3, 6, 7, and 8 together.

**2.20** Referring to Exercise 2.19 and the Venn diagram of Figure 2.5, list the numbers of the regions that represent the following events:

(a) The family will experience no mechanical problems and will not receive a ticket for a traffic violation but will arrive at a campsite with no vacancies.

(b) The family will experience both mechanical problems and trouble in locating a campsite with a vacancy but will not receive a ticket for a traffic violation.

(c) The family will either have mechanical trouble or arrive at a campsite with no vacancies but will not receive a ticket for a traffic violation.

(d) The family will not arrive at a campsite with no vacancies.

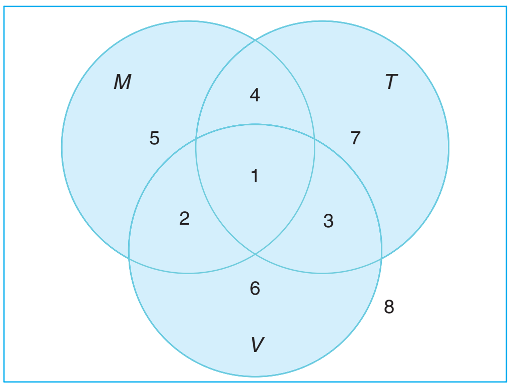


Figure 2.5: Venn diagram for Exercises 2.19 and 2.20.

## 2.3 Counting Sample Points

One of the problems that the statistician must consider and attempt to evaluate is the element of chance associated with the occurrence of certain events when an experiment is performed. These problems belong in the field of probability, a subject to be introduced in Section 2.4. In many cases, we shall be able to solve a probability problem by counting the number of points in the sample space without actually listing each element. The fundamental principle of counting, often referred to as the **multiplication rule**, is stated in Rule 2.1.

**Rule 2.1:** If an operation can be performed in $n_1$ ways, and if for each of these ways a second operation can be performed in $n_2$ ways, then the two operations can be performed together in $n_1 n_2$ ways.

**Example 2.13:** How many sample points are there in the sample space when a pair of dice is thrown once?

***Solution*:** The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

**Example 2.14:** A developer of a new subdivision offers prospective home buyers a choice of Tudor, rustic, colonial, and traditional exterior styling in ranch, two-story, and split-level floor plans. In how many different ways can a buyer order one of these homes?

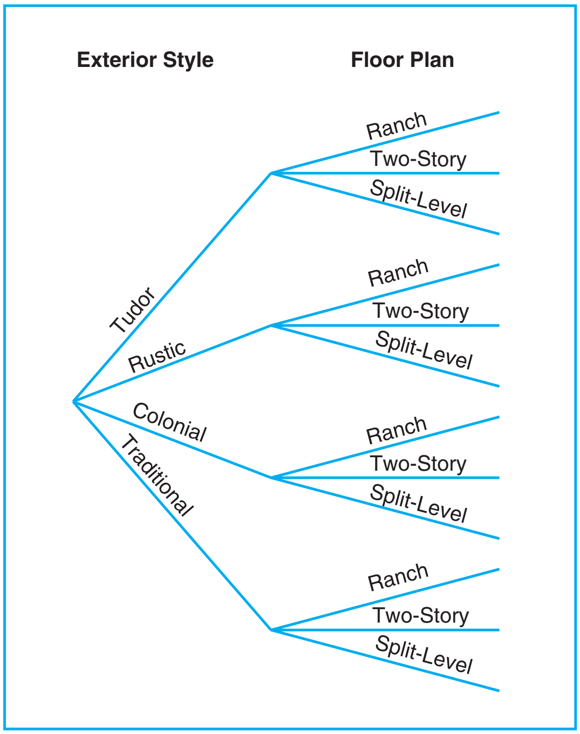


Figure 2.6: Tree diagram for Example 2.14.

***Solution*:** Since $n_1 = 4$ and $n_2 = 3$, a buyer must choose from

$$n_1 n_2 = (4)(3) = 12 \text{ possible homes.}$$

The answers to the two preceding examples can be verified by constructing tree diagrams and counting the various paths along the branches. For instance, in Example 2.14 there will be $n_1 = 4$ branches corresponding to the different exterior styles, and then there will be $n_2 = 3$ branches extending from each of these 4 branches to represent the different floor plans. This tree diagram yields the $n_1 n_2 = 12$ choices of homes given by the paths along the branches, as illustrated in Figure 2.6.

**Example 2.15:** If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

***Solution*:** For the chair position, there are 22 total possibilities. For each of those 22 possibilities, there are 21 possibilities to elect the treasurer. Using the multiplication rule, we obtain $n_1 \times n_2 = 22 \times 21 = 462$ different ways.

The multiplication rule, Rule 2.1 may be extended to cover any number of operations. Suppose, for instance, that a customer wishes to buy a new cell phone and can choose from $n_1 = 5$ brands, $n_2 = 5$ sets of capability, and $n_3 = 4$ colors. These three classifications result in $n_1 n_2 n_3 = (5)(5)(4) = 100$ different ways for a customer to order one of these phones. The **generalized multiplication rule** covering $k$ operations is stated in the following.

**Rule 2.2:** If an operation can be performed in $n_1$ ways, and if for each of these a second operation can be performed in $n_2$ ways, and for each of the first two a third operation can be performed in $n_3$ ways, and so forth, then the sequence of $k$ operations can be performed in $n_1 n_2 \cdots n_k$ ways.

**Example 2.16:** Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

***Solution*:** Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_l \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

**Example 2.17:** How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

***Solution*:** Since the number must be even, we have only $n_1 = 3$ choices for the units position. However, for a four-digit number the thousands position cannot be 0. Hence, we consider the units position in two parts, 0 or not 0. If the units position is 0 (i.e., $n_1 = 1$), we have $n_2 = 5$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position. Therefore, in this case we have a total of

$$n_1 n_2 n_3 n_4 = (1)(5)(4)(3) = 60$$

even four-digit numbers. On the other hand, if the units position is not 0 (i.e., $n_1 = 2$), we have $n_2 = 4$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position. In this situation, there are a total of

$$n_1 n_2 n_3 n_4 = (2)(4)(4)(3) = 96$$

even four-digit numbers.

Since the above two cases are mutually exclusive, the total number of even four-digit numbers can be calculated as $60 + 96 = 156$.

Frequently, we are interested in a sample space that contains as elements all possible orders or arrangements of a group of objects. For example, we may want to know how many different arrangements are possible for sitting 6 people around a table, or we may ask how many different orders are possible for drawing 2 lottery tickets from a total of 20. The different arrangements are called **permutations**.

**Definition 2.7:** A **permutation** is an arrangement of all or part of a set of objects.

Consider the three letters $a$, $b$, and $c$. The possible permutations are $abc$, $acb$, $bac$, $bca$, $cab$, and $cba$. Thus, we see that there are 6 distinct arrangements. Using Rule 2.2, we could arrive at the answer 6 without actually listing the different orders by the following arguments: There are $n_1 = 3$ choices for the first position. No matter which letter is chosen, there are always $n_2 = 2$ choices for the second position. No matter which two letters are chosen for the first two positions, there is only $n_3 = 1$ choice for the last position, giving a total of

$$n_1 n_2 n_3 = (3)(2)(1) = 6 \text{ permutations}$$

by Rule 2.2. In general, $n$ distinct objects can be arranged in

$$n(n-1)(n-2)\cdots(3)(2)(1) \text{ ways.}$$

There is a notation for such a number.

**Definition 2.8:** For any non-negative integer $n$, $n!$, called "$n$ factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

Using the argument above, we arrive at the following theorem.

**Theorem 2.1:** The number of permutations of $n$ objects is $n!$.

The number of permutations of the four letters $a$, $b$, $c$, and $d$ will be $4! = 24$. Now consider the number of permutations that are possible by taking two letters at a time from four. These would be $ab$, $ac$, $ad$, $ba$, $bc$, $bd$, $ca$, $cb$, $cd$, $da$, $db$, and $dc$. Using Rule 2.1 again, we have two positions to fill, with $n_1 = 4$ choices for the first and then $n_2 = 3$ choices for the second, for a total of

$$n_1 n_2 = (4)(3) = 12$$

permutations. In general, $n$ distinct objects taken $r$ at a time can be arranged in

$$n(n-1)(n-2)\cdots(n-r+1)$$

ways. We represent this product by the symbol

$${}_nP_r = \frac{n!}{(n-r)!}.$$

As a result, we have the theorem that follows.

**Theorem 2.2:** The number of permutations of $n$ distinct objects taken $r$ at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

**Example 2.18:** In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

***Solution*:** Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$$_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800.$$

**Example 2.19:** A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions;

(b) $A$ will serve only if he is president;

(c) $B$ and $C$ will serve together or not at all;

(d) $D$ and $E$ will not serve together?

***Solution*:** (a) The total number of choices of officers, without any restrictions, is

$$_{50}P_2 = \frac{50!}{48!} = (50)(49) = 2450.$$

(b) Since $A$ will serve only if he is president, we have two situations here: (i) $A$ is selected as the president, which yields 49 possible outcomes for the treasurer's position, or (ii) officers are selected from the remaining 49 people without $A$, which has the number of choices $_{49}P_2 = (49)(48) = 2352$. Therefore, the total number of choices is $49 + 2352 = 2401$.

(c) The number of selections when $B$ and $C$ serve together is 2. The number of selections when both $B$ and $C$ are not chosen is $_{48}P_2 = 2256$. Therefore, the total number of choices in this situation is $2 + 2256 = 2258$.

(d) The number of selections when $D$ serves as an officer but not $E$ is $(2)(48) = 96$, where 2 is the number of positions $D$ can take and 48 is the number of selections of the other officer from the remaining people in the club except $E$. The number of selections when $E$ serves as an officer but not $D$ is also $(2)(48) = 96$. The number of selections when both $D$ and $E$ are not chosen is $_{48}P_2 = 2256$. Therefore, the total number of choices is $(2)(96) + 2256 = 2448$. This problem also has another short solution: Since $D$ and $E$ can only serve together in 2 ways, the answer is $2450 - 2 = 2448$.

Permutations that occur by arranging objects in a circle are called **circular permutations**. Two circular permutations are not considered different unless corresponding objects in the two arrangements are preceded or followed by a different object as we proceed in a clockwise direction. For example, if 4 people are playing bridge, we do not have a new permutation if they all move one position in a clockwise direction. By considering one person in a fixed position and arranging the other three in 3! ways, we find that there are 6 distinct arrangements for the bridge game.

**Theorem 2.3:** The number of permutations of $n$ objects arranged in a circle is $(n-1)!$.

So far we have considered permutations of distinct objects. That is, all the objects were completely different or distinguishable. Obviously, if the letters $b$ and $c$ are both equal to $x$, then the 6 permutations of the letters $a$, $b$, and $c$ become $axx$, $axx$, $xax$, $xax$, $xxa$, and $xxa$, of which only 3 are distinct. Therefore, with 3 letters, 2 being the same, we have $3!/2! = 3$ distinct permutations. With 4 different letters $a$, $b$, $c$, and $d$, we have 24 distinct permutations. If we let $a = b = x$ and $c = d = y$, we can list only the following distinct permutations: $xxyy$, $xyxy$, $yxxy$, $yyxx$, $xyyx$, and $yxyx$. Thus, we have $4!/(2!\,2!) = 6$ distinct permutations.

**Theorem 2.4:** The number of distinct permutations of $n$ things of which $n_1$ are of one kind, $n_2$ of a second kind, $\dots$, $n_k$ of a $k$th kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

**Example 2.20:** In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

***Solution*:** Directly using Theorem 2.4, we find that the total number of arrangements is

$$\frac{10!}{1!\,2!\,4!\,3!} = 12{,}600.$$

Often we are concerned with the number of ways of partitioning a set of $n$ objects into $r$ subsets called **cells**. A partition has been achieved if the intersection of every possible pair of the $r$ subsets is the empty set $\phi$ and if the union of all subsets gives the original set. The order of the elements within a cell is of no importance. Consider the set $\{a,\ e,\ i,\ o,\ u\}$. The possible partitions into two cells in which the first cell contains 4 elements and the second cell 1 element are

$$\{(a,e,i,o),(u)\},\{(a,i,o,u),(e)\},\{(e,i,o,u),(a)\},\{(a,e,o,u),(i)\},\{(a,e,i,u),(o)\}.$$

We see that there are 5 ways to partition a set of 4 elements into two subsets, or cells, containing 4 elements in the first cell and 1 element in the second.

The number of partitions for this illustration is denoted by the symbol

$$\binom{5}{4,1} = \frac{5!}{4!\ 1!} = 5,$$

where the top number represents the total number of elements and the bottom numbers represent the number of elements going into each cell. We state this more generally in Theorem 2.5.

**Theorem 2.5:** The number of ways of partitioning a set of $n$ objects into $r$ cells with $n_1$ elements in the first cell, $n_2$ elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

**Example 2.21:** In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

***Solution*:** The total number of possible partitions would be

$$\binom{7}{3,2,2} = \frac{7!}{3!\ 2!\ 2!} = 210.$$

In many problems, we are interested in the number of ways of selecting $r$ objects from $n$ without regard to order. These selections are called **combinations**. A combination is actually a partition with two cells, the one cell containing the $r$ objects selected and the other cell containing the $(n-r)$ objects that are left. The number of such combinations, denoted by

$$\binom{n}{r, n-r}, \text{ is usually shortened to } \binom{n}{r},$$

since the number of elements in the second cell must be $n - r$.

**Theorem 2.6:** The number of combinations of $n$ distinct objects taken $r$ at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

**Example 2.22:** A young boy asks his mother to get 5 Game-Boy™ cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

***Solution*:** The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3!\ (10-3)!} = 120.$$

The number of ways of selecting 2 cartridges from 5 is

$$\binom{5}{2} = \frac{5!}{2!\ 3!} = 10.$$

Using the multiplication rule (Rule 2.1) with $n_1 = 120$ and $n_2 = 10$, we have $(120)(10) = 1200$ ways.

**Example 2.23:** How many different letter arrangements can be made from the letters in the word *STATISTICS*?

***Solution*:** Using the same argument as in the discussion for Theorem 2.6, in this example we can actually apply Theorem 2.5 to obtain

$$\binom{10}{3,3,2,1,1} = \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!} = 50,400.$$

Here we have 10 total letters, with 2 letters ($S$, $T$) appearing 3 times each, letter $I$ appearing twice, and letters $A$ and $C$ appearing once each. On the other hand, this result can be directly obtained by using Theorem 2.4.

## Exercises

**2.21** Registrants at a large convention are offered 6 sightseeing tours on each of 3 days. In how many ways can a person arrange to go on a sightseeing tour planned by this convention?

**2.22** In a medical study, patients are classified in 8 ways according to whether they have blood type $AB^+$, $AB^-$, $A^+$, $A^-$, $B^+$, $B^-$, $O^+$, or $O^-$, and also according to whether their blood pressure is low, normal, or high. Find the number of ways in which a patient can be classified.

**2.23** If an experiment consists of throwing a die and then drawing a letter at random from the English alphabet, how many points are there in the sample space?

**2.24** Students at a private liberal arts college are classified as being freshmen, sophomores, juniors, or seniors, and also according to whether they are male or female. Find the total number of possible classifications for the students of that college.

**2.25** A certain brand of shoes comes in 5 different styles, with each style available in 4 distinct colors. If the store wishes to display pairs of these shoes showing all of its various styles and colors, how many different pairs will the store have on display?

**2.26** A California study concluded that following 7 simple health rules can extend a man's life by 11 years on the average and a woman's life by 7 years. These 7 rules are as follows: no smoking, get regular exercise, use alcohol only in moderation, get 7 to 8 hours of sleep, maintain proper weight, eat breakfast, and do not eat between meals. In how many ways can a person adopt 5 of these rules to follow

(a) if the person presently violates all 7 rules?

(b) if the person never drinks and always eats breakfast?

**2.27** A developer of a new subdivision offers a prospective home buyer a choice of 4 designs, 3 different heating systems, a garage or carport, and a patio or screened porch. How many different plans are available to this buyer?

**2.28** A drug for the relief of asthma can be purchased from 5 different manufacturers in liquid, tablet, or capsule form, all of which come in regular and extra strength. How many different ways can a doctor prescribe the drug for a patient suffering from asthma?

**2.29** In a fuel economy study, each of 3 race cars is tested using 5 different brands of gasoline at 7 test sites located in different regions of the country. If 2 drivers are used in the study, and test runs are made once under each distinct set of conditions, how many test runs are needed?

**2.30** In how many different ways can a true-false test consisting of 9 questions be answered?

**2.31** A witness to a hit-and-run accident told the police that the license number contained the letters RLH followed by 3 digits, the first of which was a 5. If the witness cannot recall the last 2 digits, but is certain that all 3 digits are different, find the maximum number of automobile registrations that the police may have to check.

**2.32** (a) In how many ways can 6 people be lined up to get on a bus?
(b) If 3 specific persons, among 6, insist on following each other, how many ways are possible?
(c) If 2 specific persons, among 6, refuse to follow each other, how many ways are possible?

**2.33** If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct,
(a) in how many different ways can a student check off one answer to each question?
(b) in how many ways can a student check off one answer to each question and get all the answers wrong?

**2.34** (a) How many distinct permutations can be made from the letters of the word *COLUMNS*?
(b) How many of these permutations start with the letter $M$?

**2.35** A contractor wishes to build 9 houses, each different in design. In how many ways can he place these houses on a street if 6 lots are on one side of the street and 3 lots are on the opposite side?

**2.36** (a) How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, and 6 if each digit can be used only once?
(b) How many of these are odd numbers?
(c) How many are greater than 330?

**2.37** In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

**2.38** Four married couples have bought 8 seats in the same row for a concert. In how many different ways can they be seated
(a) with no restrictions?
(b) if each couple is to sit together?
(c) if all the men sit together to the right of all the women?

**2.39** In a regional spelling bee, the 8 finalists consist of 3 boys and 5 girls. Find the number of sample points in the sample space $S$ for the number of possible orders at the conclusion of the contest for
(a) all 8 finalists;
(b) the first 3 positions.

**2.40** In how many ways can 5 starting positions on a basketball team be filled with 8 men who can play any of the positions?

**2.41** Find the number of ways that 6 teachers can be assigned to 4 sections of an introductory psychology course if no teacher is assigned to more than one section.

**2.42** Three lottery tickets for first, second, and third prizes are drawn from a group of 40 tickets. Find the number of sample points in $S$ for awarding the 3 prizes if each contestant holds only 1 ticket.

**2.43** In how many ways can 5 different trees be planted in a circle?

**2.44** In how many ways can a caravan of 8 covered wagons from Arizona be arranged in a circle?

**2.45** How many distinct permutations can be made from the letters of the word $INFINITY$?

**2.46** In how many ways can 3 oaks, 4 pines, and 2 maples be arranged along a property line if one does not distinguish among trees of the same kind?

**2.47** How many ways are there to select 3 candidates from 8 equally qualified recent graduates for openings in an accounting firm?

**2.48** How many ways are there that no two students will have the same birth date in a class of size 60?

## 2.4 Probability of an Event

Perhaps it was humankind's unquenchable thirst for gambling that led to the early development of probability theory. In an effort to increase their winnings, gamblers called upon mathematicians to provide optimum strategies for various games of chance. Some of the mathematicians providing these strategies were Pascal, Leibniz, Fermat, and James Bernoulli. As a result of this development of probability theory, statistical inference, with all its predictions and generalizations, has branched out far beyond games of chance to encompass many other fields associated with chance occurrences, such as politics, business, weather forecasting,