Uniform/Rectangular Distribution

Let 'X' be a continuous random variable with interval (a,b) is said to be rectangular or uniform distribution having its p.d.f

$$f(x)=\frac{1}{b-a} \qquad a\le x\le b$$

If $x\approx U(b)$ & If $x\approx U(\theta)$

$$f(x)=\frac{1}{\theta} \qquad 0\le x\le \theta$$

If $\theta=1$ then it follow standard form of uniform distribution

$$f(x)=1 \qquad 0\le x\le 1$$

Properties

i) Uniform distribution is a continuous distribution.
ii) The total area under the curve is unity.
iii) The mean of Uniform distribution is $E(x)=\frac{b+a}{2}$.
iv) The variance of Uniform distribution is $Var(x)=\frac{(a-b)^2}{12}$
v) The Harmonic Mean of Uniform distribution is $H.M=\frac{b-a}{\ln b-\ln a}$
vi) The Median of Uniform distribution is $m=\frac{a+b}{2}$

Prove that total area under the curve is unity

Solution: Let by definition

Total Area= $\int_{-\infty}^{\infty} f(x)dx$

As $x\approx U(a,b)$ with p.d.f

$$f(x)=\frac{1}{b-a} \qquad a\le x\le b$$

Area= $\int_a^b \frac{1}{b-a}dx=\frac{1}{b-a}\int_a^b dx=\frac{1}{b-a}x\Big|_a^b=\frac{1}{b-a}(b-a)=1$ Hence proved.

Find rth moments about origin of Uniform Distribution & find mean and Variance.

$$\mu_r'=E(x^r)=\int_{-\infty}^{\infty} x^r f(x)dx$$

As $x\approx U(a,b)$ with p.d.f

$$f(x)=\frac{1}{b-a} \qquad a\le x\le b$$

$$\mu_r'=\int_a^b x^r\frac{1}{b-a}dx$$

$$\mu_r'=\frac{1}{b-a}\int_a^b x^r dx$$

$$\mu_r'=\frac{1}{b-a}\frac{x^{r+1}}{r+1}\Big|_a^b$$

$$\mu_r'=\frac{1}{b-a}\left(\frac{b^{r+1}-a^{r+1}}{r+1}\right)$$

$$\mu_r'=\frac{\left(b^{r+1}-a^{r+1}\right)}{(b-a)(r+1)}$$

Put r=1

$$\mu_1' = \frac{(b^{1+1} - a^{1+1})}{(b-a)(1+1)} = \frac{(b^2 - a^2)}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{(b+a)}{2}$$

$$\text{var}(x) = \mu_2' - \left(\mu_1'\right)^2$$ Put r = 2

$$\mu_2' = \frac{(b^{2+1} - a^{2+1})}{(b-a)(2+1)} = \frac{(b^3 - a^3)}{3(b-a)} = \frac{(b-a)(b^2 + a^2 + ab)}{3(b-a)} = \frac{(b^2 + a^2 + ab)}{3}$$

$$\text{var}(x) = \mu_2' - \left(\mu_1'\right)^2 = \frac{(b^2 + a^2 + ab)}{3} - \left(\frac{(b+a)}{2}\right)^2$$

$$\text{var}(x) = \frac{(b^2 + a^2 + ab)}{3} - \frac{(b^2 + a^2 + 2ab)}{4}$$

$$\text{var}(x) = \frac{(4b^2 + 4ab + 4a^2 - 3a^2 - 3b^2 + 6ab)}{12} = \frac{a^2 + b^2 - 2ab}{12}$$

$$\text{var(x)} = \frac{(a-b)^2}{12}$$

Find Harmonic Mean of Uniform Distribution

Solution: Let by definition

$$\text{H.M} = \frac{1}{E\left(\frac{1}{x}\right)} \qquad \text{(i)}$$

As $x \approx U(a,b)$ with p.d.f

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

$$E\left(\frac{1}{x}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)dx$$

$$E\left(\frac{1}{x}\right) = \int_{a}^{b} \frac{1}{x}\frac{1}{b-a}dx$$

$$E\left(\frac{1}{x}\right) = \frac{1}{b-a}\int_{a}^{b} \frac{1}{x}dx = \frac{1}{b-a}\ \ln x\Big|_a^b$$

$$E\left(\frac{1}{x}\right) = \frac{1}{b-a}(\ln b - \ln a)$$

$$E\left(\frac{1}{x}\right) = \frac{(\ln b - \ln a)}{b-a}$$

Put in (i)

$$\text{H.M} = \frac{1}{\frac{(\ln b - \ln a)}{b-a}} = \frac{b-a}{(\ln b - \ln a)}$$

Find Median of Uniform Distribution

Let by definition of median

$$P(x < m) = \frac{1}{2}$$

As $x \approx U(a,b)$ with p.d.f

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

$$\int_a^m f(x)dx = \frac{1}{2}$$
$$\int_a^m \frac{1}{b-a}dx = \frac{1}{b-a}\int_a^m dx = \frac{1}{2}$$
$$\frac{1}{b-a}x\Big|_a^m = \frac{1}{2}$$
$$\frac{1}{b-a}(m-a) = \frac{1}{2}$$
$$(m-a) = \frac{b-a}{2}$$
$$\frac{b-a}{2}+a = m$$
$$\frac{b-a+2a}{2} = m$$
$$m = \frac{a+b}{2}$$

Show that odd order moment about mean are zero and even order moment about mean is $\mu_{2r} = \frac{2(b-a)^{2r+1}}{2^{2r+1}(b-a)(2r+1)}$

Solution: Let by definition

$$\mu_r = E(x-\mu)^r = \int_{-\infty}^{\infty}(x-\frac{\alpha+\beta}{2})^r f(x)dx \qquad \mu = \frac{\alpha+\beta}{2}$$

As $x \approx U(a,b)$ with p.d.f

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$
$$\mu_r = \int_a^b (x-\frac{b+a}{2})^r f(x)dx$$
$$\mu_r = \int_a^b (x-\frac{b+a}{2})^r \frac{1}{b-a}dx$$
$$\mu_r = \frac{1}{b-a}\int_a^b (x-\frac{b+a}{2})^r dx$$
$$\mu_r = \frac{1}{b-a}\left.\frac{(x-\frac{b+a}{2})^{r+1}}{r+1}\right|_a^b$$
$$\mu_r = \frac{1}{(b-a)(r+1)}(b-\frac{b+a}{2})^{r+1} - (a-\frac{b+a}{2})^{r+1}$$
$$\mu_r = \frac{1}{(b-a)(r+1)}\left[\frac{(b-a)^{r+1}}{2^{r+1}} - \frac{(a-b)^{r+1}}{2^{r+1}}\right]$$
$$\mu_r = \frac{1}{2^{r+1}(b-a)(r+1)}\left[(b-a)^{r+1} - (a-b)^{r+1}\right] \qquad \text{(i)}$$

For order moment we put r = 2r+1 in (i)

$$\mu_{2r+1} = \frac{1}{2^{2r+1+1}(b-a)(2r+1+1)}\left[(b-a)^{2r+1+1} - (a-b)^{2r+1+1}\right]$$
$$\mu_{2r+1} = \frac{1}{2^{2r+2}(b-a)(2r+2)}\left[(b-a)^{2r+2} - (a-b)^{2r+2}\right]$$

Therefore $-(b-a)^{2r+2} = (a-b)^{2r+2}$

If r=0,1,2,…. Then power is even

$$\mu_{2r+1} = \frac{1}{2^{2r+2}(b-a)(2r+2)}\left[(b-a)^{2r+2} - (b-a)^{2r+2}\right]$$

$$\mu_{2r+1} = \frac{1}{2^{2r+2}(b-a)(2r+2)}[0]$$

$$\mu_{2r+1} = 0$$

Hence prove that odd order moment about mean is zero

For even order moment put r = 2r in (i)

$$\mu_{2r} = \frac{1}{2^{2r+1}(b-a)(2r+1)}\left[(b-a)^{2r+1} - (a-b)^{2r+1}\right]$$

$$\mu_{2r} = \frac{1}{2^{2r+1}(b-a)(2r+1)}\left[(b-a)^{2r+1} - (-1)^{2r+1}(a-b)^{2r+1}\right]$$

If r=0,1,2,…. Then power is odd $(-1)^{2r+1} = (-1)^{odd} = -1$

$$\mu_{2r} = \frac{1}{2^{2r}2(b-a)(2r+1)}\left[(b-a)^{2r+1} + (a-b)^{2r+1}\right]$$

$$\mu_{2r} = \frac{2(b-a)^{2r+1}}{2^{2r}2(b-a)(2r+1)}$$

$$\mu_{2r} = \frac{(b-a)^{2r}(b-a)}{2^{2r}(b-a)(2r+1)}$$

$$\mu_{2r} = \frac{(b-a)^{2r}}{2^{2r}(2r+1)}$$

For moment ratios (b_1,b_2)

$$\mu_{2r} = \frac{(b-a)^{2r}}{2^{2r}(2r+1)}$$ Put r = 1

$$\mu_2 = \frac{(b-a)^2}{2^2(2+1)} = \frac{(b-a)^2}{12}$$

Now put r = 2

$$\mu_4 = \frac{(b-a)^4}{2^4(2.2+1)}$$

$$\mu_4 = \frac{(b-a)^4}{16(5)} = \frac{(b-a)^4}{80}$$

$$b_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{\left(\frac{(b-a)^2}{12}\right)^3} = 0$$ So, b_1 = 0.So, it is symmetrical distribution.

$$b_2 = \frac{\mu_4}{\mu_2} = \frac{\frac{(b-a)^4}{80}}{\left(\frac{(b-a)^2}{12}\right)^2} = \frac{144(b-a)^4}{80(b-a)^4} = \frac{144}{80} = 1.8$$ Require result.

$b_2 < 3$ So it is platy kurtic