# 3. Poisson probability distribution

When Probability of success “P” is very small and “n” is very large and then we use Poisson distribution and its p.d.f given as

$$P(X = x) = \frac{e^{-\mu}(\mu)^x}{x!}$$

Examples:

i) Number of typing errors per page in the books
ii) Number of persons born blind per year in large city
iii) Telephone calls in a small interval of time
iv) Number of students cheating others in MBBS entry test

**Properties of Poisson experiment**

i) The result of each trial can be classified as success or failure
ii) Events occur independently in fixed interval of time or space
iii) Probability of occurrence of the event remains equal in the fixed interval of time or space
iv) Probability that more than one event occurs in small interval of time or space is negligible

**Poisson random variable**

The random variable which denotes the number of successes of Poisson experiment is called Poisson random variable. It is a discrete variable and ranges from zero to infinity $(0 \; to \; \infty)$

**Properties of Poisson distribution**

i) Total probability of Poisson distribution is unity
ii) Mean ,variance and third moment about mean are equal
iii) It is discrete probability distribution
iv) If $\mu$ becomes very large then it approaches to normal distribution
v) If “X” and “Y” are independent Poisson random variables with parameters $\mu$ and “v” then “X+Y” is also Poisson distribution with parameter $\mu + v$

**Derivation of Poisson distribution**

Poisson distribution is a limiting form of binomial distribution when “P” is small and “n” is very large as $n \to \infty$ and $P \to 0$.

Proof:

Let “X” be a rasndom variable having a binomial distribution with probability distribution function.

$$P(X) = \overset{n}{\underset{x}{C}} P^x q^{n-x} \qquad \text{X:0,1,2,3,…,n}$$

$$P(X) = \frac{n!}{X!(n-X)!} P^x q^{n-x}$$

$$P(X) = \frac{n(n-1)(n-2)....(n-x+1)(n-x)!}{X!(n-X)!} P^x q^{n-x}$$

$$P(X) = \frac{n(n-1)(n-2)....(n-x+1)}{X!} P^x q^{n-x} \qquad (A)$$

As we know that mean of binomial distribution

$\mu = np$

$p = \frac{\mu}{n}$ $\qquad\qquad$ $q = 1 - p = 1 - \frac{\mu}{n}$

Then eq(A) becomes

$$P(X) = \frac{n(n-1)(n-2)....(n-x+1)}{X!}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^{n-x}$$

$$P(X) = \frac{\mu^X}{X!}\left[n(n-1)(n-2)....(n-x+1)\right]\left(\frac{1}{n^X}\right)\left(1-\frac{\mu}{n}\right)^{-x}\left(1-\frac{\mu}{n}\right)^n$$

$$P(X) = \frac{\mu^X}{X!}\left[n.n(1-\frac{1}{n})n(1-\frac{2}{n})....n(1-\frac{x-1}{n})\right]\left(\frac{1}{n^X}\right)\left(1-\frac{\mu}{n}\right)^{-x}\left(1-\frac{\mu}{n}\right)^n$$

$$P(X) = \frac{\mu^X}{X!}\left[n.n.n...n(X-times)(1-\frac{1}{n})(1-\frac{2}{n})....(1-\frac{x-1}{n})\right]\left(\frac{1}{n^X}\right)\left(1-\frac{\mu}{n}\right)^{-x}\left(1-\frac{\mu}{n}\right)^n$$

$$P(X) = \frac{\mu^X}{X!}\left[n^X(1-\frac{1}{n})(1-\frac{2}{n})....(1-\frac{x-1}{n})\right]\left(\frac{1}{n^X}\right)\left(1-\frac{\mu}{n}\right)^{-x}\left(1-\frac{\mu}{n}\right)^n$$

$$P(X) = \frac{\mu^X}{X!}\left[(1-\frac{1}{n})(1-\frac{2}{n})....(1-\frac{x-1}{n})\right]\left(1-\frac{\mu}{n}\right)^{-x}\left(1-\frac{\mu}{n}\right)^n$$

When $n \to \infty$ then $(1-\frac{1}{n})(1-\frac{2}{n})....(1-\frac{x-1}{n})(1-\frac{\mu}{n})^{-x} \to 0$

$$\lim it_{n\to\infty} b(x,n,p) = \lim it_{n\to\infty} \frac{\mu^X}{X!}\left[(1-\frac{1}{n})(1-\frac{2}{n})....(1-\frac{x-1}{n})\right]\left(1-\frac{\mu}{n}\right)^{-x}\left(1-\frac{\mu}{n}\right)^n$$

$$\lim it_{n\to\infty} b(x,n,p) = \frac{\mu^X}{X!}\lim it_{n\to\infty}\left[(1-\frac{1}{n})(1-\frac{2}{n})....(1-\frac{x-1}{n})(1-\frac{\mu}{n})^{-x}\right]\lim it_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n$$

$$\lim it_{n\to\infty} b(x,n,p) = \frac{\mu^X}{X!}\left[(1-\frac{1}{\infty})(1-\frac{2}{\infty})....(1-\frac{x-1}{\infty})(1-\frac{\mu}{\infty})^{-x}\right]\lim it_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n$$

$$\lim it_{n\to\infty} b(x,n,p) = \frac{\mu^X}{X!}\left[(1-0)(1-0)....(1-0)(1-0)^{-x}\right]\lim it_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{n\frac{\mu}{\mu}}$$

$$\lim it_{n\to\infty} b(x,n,p) = \frac{\mu^X}{X!}\left[(1)(1)....(1)(1)^{-x}\right]\left[\lim it_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{\frac{n}{\mu}}\right]^{\mu}$$

$$\lim it_{n\to\infty} b(x,n,p) = \frac{\mu^X}{X!}\left[e^{-1}\right]^{\mu}$$

$$\lim it_{n\to\infty} b(x,n,p) = \frac{\mu^X}{X!}e^{-\mu}$$

Required p.d.f of Poisson distribution

**Show that total probability of Poisson distribution is one**

Proof: Let by definition

$$Total\ \ probabilty = \sum_{X=0}^{\infty} P(X)$$

Let " $X \to P(\mu)$ $\qquad\qquad$ $X : 0,1,2,3,...,\infty$

$$P(X=x)=\frac{e^{-\mu}(\mu)^x}{x!}$$

$$Total\ \ probability=\sum_{X=0}\frac{e^{-\mu}(\mu)^x}{x!}$$

$$Total\ \ probability=e^{-\mu}\sum_{X=0}\frac{(\mu)^x}{x!}$$

$$Total\ probability=e^{-\mu}\left[\frac{(\mu)^0}{0!}+\frac{(\mu)^1}{1!}+\frac{(\mu)^2}{2!}+...\right]=e^{-\mu}\left(e^{\mu}\right)=e^{-\mu+\mu}=e^0=1$$

Hence Proved

**Find first four moments about mean of Poisson distribution**

Solution: Let by definition

$$\mu_1'=Mean=\sum_{X=0}^{\infty}XP(X)$$

Let " $X\to P(\mu)$ $\qquad X:0,1,2,3,...,\infty$

$$P(X=x)=\frac{e^{-\mu}(\mu)^x}{x!}$$

$$\mu_1'=Mean=\sum_{X=0}^{\infty}XP(X)=\sum_{X=0}X\,\frac{e^{-\mu}(\mu)^x}{x!}$$

$$\mu_1'=Mean=e^{-\mu}\sum_{X=1}X\,\frac{(\mu)^{x-1}(\mu)}{X(X-1)!}$$

$$\mu_1'=Mean=e^{-\mu}\mu\sum_{X=1}\frac{(\mu)^{x-1}}{(X-1)!}$$

$$\mu_1'=Mean=e^{-\mu}\mu\left[\frac{(\mu)^0}{0!}+\frac{(\mu)^1}{1!}+\frac{(\mu)^2}{2!}+...\right]=\mu e^{-\mu}\left(e^{\mu}\right)=\mu e^{-\mu+\mu}=\mu e^0=\mu$$

$$\mu_2'=E(X^2)=E\left[(X(X-1)+X\right]=\sum_{X=0}^{\infty}\left[(X(X-1)P(X)\right]+\sum_{X=0}^{\infty}XP(X)$$

$$\mu_2'=E(X^2)=e^{-\mu}\sum_{X=1}X(X-1)\frac{(\mu)^{x-2}(\mu)^2}{X(X-1)(X-2)!}+\sum_{X=0}^{\infty}XP(X)$$

$$\mu_2'=E(X^2)=e^{-\mu}\mu^2\sum_{X=2}\frac{(\mu)^{x-2}}{(X-2)!}+\mu$$

$$\mu_2'=E(X^2)=e^{-\mu}\mu^2\left[\frac{(\mu)^0}{0!}+\frac{(\mu)^1}{1!}+\frac{(\mu)^2}{2!}+...\right]+\mu=e^{-\mu}\mu^2\left(e^{\mu}\right)+\mu=\mu^2+\mu$$

$$\mu_3'=E(X^3)=E\left[(X(X-1)(X-2)+3X(X-1)+X\right]=E\left[X(X-1)(X-2)\right]+3E\left[X(X-1)\right]+E(X)$$

$$\mu_3'=e^{-\mu}\mu^3\sum_{X=3}X(X-1)(X-3)\frac{(\mu)^{x-3}}{X(X-1)(X-2)(X-3)!}+3E\left[X(X-1)\right]+E(X)$$

$$\mu_3'=e^{-\mu}\mu^3\sum_{X=3})\frac{(\mu)^{x-3}}{(X-3)!}+E\left[X(X-1)\right]+E(X)$$

$$\mu_3' = \mu^3 + 3\mu^2 + \mu$$

$$\mu_4' == E(X^4) = E[(X(X-1)(X-2)(X-3) + 6X(X-1)(X-2) + 7X(X-1) + X]$$

$$\mu_4' = E[(X(X-1)(X-2)(X-3) + 6X(X-1)(X-2) + 7X(X-1) + X]$$

$$\mu_4' = e^{-\mu}\mu^4 \sum_{X=4} X(X-1)(X-3)\frac{(\mu)^{x-4}}{X(X-1)(X-2)(X-3)(X-4)!} + 6\mu^3 + 7\mu^2 + \mu$$

$$\mu_4' = e^{-\mu}\mu^4 \sum_{X=4} \frac{(\mu)^{x-4}}{(X-4)!} + 6\mu^3 + 7\mu^2 + \mu$$

$$\mu_4' = \mu^4 + 6\mu^3 + 7\mu^2 + \mu$$

**First four moments about mean**

$$\mu_1 = \mu_1' - \mu_1' = \mu - \mu = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \mu^2 + \mu - \mu^2 = \mu = Var(X)$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \mu^2 + \mu - \mu^2 = \mu = Var(X)$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = \mu^3 + 3\mu^2 + \mu - 3(\mu^2 + \mu)(\mu) + 2\mu^3 = \mu^3 + 3\mu^2 + \mu - 3\mu^3 - 3\mu^2 + 2\mu^3 = \mu$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

$$\mu_4 = \mu^4 + 6\mu^3 + 7\mu^2 + \mu - 4(\mu)(\mu^3 + 3\mu^2 + \mu) + 6(\mu)^2(\mu^2 + \mu) - 3(\mu)^4$$

$$\mu_4 = \mu^4 + 6\mu^3 + 7\mu^2 + \mu - 4\mu^4 - 12\mu^3 - 4\mu^2 + 6\mu^4 + 6\mu^3 - 3(\mu)^4 = 3\mu^2 + \mu$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu^2}{\mu^3} = \frac{1}{\mu}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\mu^2 + \mu}{\mu^2} = 3 + \frac{1}{\mu}$$

**Derive moment generating function use it to find mean and variance**

Proof: Let by definition of m.g.f

$$M_0(t) = E(e^{tx}) = \sum_{X=0}^{\infty} e^{tx} P(X)$$

Let " $X \to P(\mu)$ $\qquad X : 0,1,2,3,...,\infty$

$$P(X = x) = \frac{e^{-\mu}(\mu)^x}{x!}$$

$$M_0(t) = \sum_{X=0}^{\infty} e^{tx}\frac{e^{-\mu}(\mu)^x}{x!} = e^{-\mu}\sum_{X=0}^{\infty}\frac{(\mu e^t)^x}{x!}$$

$$M_0(t) = e^{-\mu}\sum_{X=0}^{\infty}\frac{(\mu e^t)^x}{x!} = e^{-\mu}\left[\frac{(\mu e^t)^0}{0!} + \frac{(\mu e^t)^1}{1!} + \frac{(\mu e^t)^2}{2!} + ...\right] = e^{-\mu}\left(e^{\mu e^t}\right) = e^{-\mu + \mu e^t} = e^{\mu(e^t - 1)}$$

Use it to find mean and variance

$$\mu_1' = Mean = \frac{\partial}{\partial t}[M_0(t)]_{t=0} = \frac{\partial}{\partial t}\left[e^{\mu(e^t-1)}\right]_{t=0}$$

$$\mu_1' = Mean = \left[e^{\mu(e^t-1)}\frac{\partial}{\partial t}\mu(e^t - 1)\right]_{t=0}$$

$$\mu_1' = Mean = \left[\mu e^{\mu(e^t-1)}e^t\right]_{t=0} = \mu e^{\mu(e^0-1)}e^0 = \mu e^{\mu(1-1)}e^0 = \mu e^0 = \mu$$

$$\mu_1' = Mean = \frac{\partial}{\partial t}\left[M_0(t)\right]_{t=0} = \frac{\partial}{\partial t}\left[e^{\mu(e^t-1)}\right]_{t=0}$$

$$\mu_1' = Mean = \left[e^{\mu(e^t-1)}\frac{\partial}{\partial t}\mu(e^t-1)\right]_{t=0}$$

$$\mu_2' = E(X^2) = \frac{\partial^2}{\partial t^2}\left[M_0(t)\right]_{t=0} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}M_0(t)\right]_{t=0} = \frac{\partial}{\partial t}\left[\mu e^{\mu(e^t-1)}e^t\right]_{t=0}$$

$$\mu_2' = E(X^2) = \frac{\partial}{\partial t}\left[\mu e^{\mu(e^t-1)}e^t\right]_{t=0} = \left[\mu e^{\mu(e^t-1)}\frac{\partial}{\partial t}e^t + \mu e^t\frac{\partial}{\partial t}e^{\mu(e^t-1)}\right]_{t=0}$$

$$\mu_2' = E(X^2) = \left[\mu e^{\mu(e^t-1)}e^t + \mu e^t e^{\mu(e^t-1)}\frac{\partial}{\partial t}\mu(e^t-1)\right]_{t=0}$$

$$\mu_2' = E(X^2) = \left[\mu e^{\mu(e^t-1)}e^t + \mu e^t e^{\mu(e^t-1)}\mu e^t)\right]_{t=0}$$

$$\mu_2' = E(X^2) = \left[\mu e^{\mu(e^t-1)}e^t + \mu^2 e^{2t}e^{\mu(e^t-1)}\right]_{t=0}$$

$$\mu_2' = E(X^2) = \mu + \mu^2$$

$$Var(X) = \mu_2' - (\mu_1')^2 = \mu^2 + \mu - \mu^2 = \mu$$

**Derive characteristic function of Poisson distribution**

Proof: Let by definition of characteristic function

$$\theta(t) = E(e^{itx}) = \sum_{X=0}^{\infty} e^{itx}P(X)$$

0r

$\theta(t) = M_0(it)$ As $M_0(t) = e^{\mu(e^t-1)}$

$$\theta_x(t) = e^{\mu(e^{it}-1)}$$

**Derive probability generating function of Poisson distribution**

Proof: Let by definition of m.g.f

$$G(t) = E(\theta^x) = \sum_{X=0}^{\infty}\theta^x P(X)$$

Let " $X \to P(\mu)$ $X : 0,1,2,3,...,\infty$

$$P(X = x) = \frac{e^{-\mu}(\mu)^x}{x!}$$

$$G(t) = \sum_{X=0}^{\infty}\theta^x\frac{e^{-\mu}(\mu)^x}{x!} = e^{-\mu}\sum_{X=0}^{\infty}\frac{(\mu\theta)^x}{x!}$$

$$\theta(t) = e^{-\mu}\sum_{X=0}^{\infty}\frac{(\mu\theta)^x}{x!} = e^{-\mu}\left[\frac{(\mu\theta)^0}{0!} + \frac{(\mu\theta)^1}{1!} + \frac{(\mu\theta)^2}{2!} + ...\right] = e^{-\mu}\left(e^{\theta\mu}\right) = e^{-\mu+\theta\mu} = e^{\mu(\theta-1)}$$

$\theta(t) = e^{\mu(\theta-1)}$ Hence the required result

**Show that in Poisson distribution or recurrence formula of Poisson distribution**

$$P(X = x+1) = \frac{\mu}{x+1}P(x,\mu)$$

Proof: As we know that

$$P(X=x)=\frac{e^{-\mu}(\mu)^{x}}{x!} \qquad \text{(i)}$$

$$P(X=x+1)=\frac{e^{-\mu}(\mu)^{x+1}}{(x+1)!} \qquad \text{(ii)}$$

Dividing eq (ii) by (i)

$$\frac{P(X=x+1)}{P(X=x)}=\frac{\frac{e^{-\mu}(\mu)^{x+1}}{(x+1)!}}{\frac{e^{-\mu}(\mu)^{x}}{(x)!}}=\frac{\frac{(\mu)^{x}\mu}{(x+1)x!}}{\frac{(\mu)^{x}}{(x)!}}=\frac{\frac{(x)!(\mu)^{x}\mu}{(\mu)^{x}(x+1)x!}}{\frac{(\mu)^{x}}{(x)!}}=\frac{(x)!(\mu)^{x}\mu}{(\mu)^{x}(x+1)x!}$$

$$\frac{P(X=x+1)}{P(X=x)}=\frac{\mu}{(x+1)}$$

$$P(X=x+1)=\frac{\mu}{(x+1)}P(X=x)$$

Hence proved

**Show that recurrence relation in Poisson distribution**

$$\mu_{r+1}=\mu\left[r\mu_{r-1}+\frac{d}{d\mu}\mu_{r}\right]$$

Use it to find first four moments about mean and moments of skewness and kurtosis

Proof: Let we consider

$$\mu_r=E(X-Mean)^r$$

$$\mu_r=\sum(X-\mu)^r P(x)$$

As $X\to P(\mu)$ then its Probability function given as

$$P(X=x)=\frac{e^{-\mu}(\mu)^{x}}{x!} \quad \text{And } Mean=\mu$$

$$\mu_r=\sum(X-\mu)^r\frac{e^{-\mu}\mu^{X}}{X!}$$

Differentiate with respect to P

$$\frac{d}{d\mu}\mu_r=\frac{d}{d\mu}\left[\sum(X-\mu)^r\frac{e^{-\mu}(\mu)^{x}}{x!}\right]$$

$$\frac{d}{d\mu}\mu_r=\left[\begin{array}{l}\sum\frac{e^{-\mu}(\mu)^{x}}{x!}\frac{d}{d\mu}(X-\mu)^r+\sum(X-\mu)^r\frac{(\mu)^{x}}{x!}\frac{d}{d\mu}e^{-\mu}\\+\sum(X-\mu)^r\frac{e^{-\mu}(\mu)^{x}}{x!}\frac{d}{d\mu}(\mu)^{x}\end{array}\right]$$

$$\frac{d}{d\mu}\mu_r = \left[\begin{array}{l}\sum \frac{e^{-\mu}(\mu)^x}{x!} r(X-\mu)^{r-1}(-1) + \sum (X-\mu)^r \frac{(\mu)^x}{x!} e^{-\mu}(-1) \\ + \sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!} X(\mu)^{x-1}(1)\end{array}\right]$$

$$\frac{d}{d\mu}\mu_r = \left[\begin{array}{l}-r\sum (X-\mu)^{r-1} \frac{e^{-\mu}(\mu)^x}{x!} - \sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!} \\ + \sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!} X(\mu)^{x-1}(1)\end{array}\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\sum (X-\mu)^{r-1} \frac{e^{-\mu}(\mu)^x}{x!} + \sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!}\left(-1 + X\mu^{-1}\right)\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\sum (X-\mu)^{r-1} \frac{e^{-\mu}(\mu)^x}{x!} + \sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!}\left(\frac{X}{\mu} - 1\right)\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\sum (X-\mu)^{r-1} \frac{e^{-\mu}(\mu)^x}{x!} + \sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!}\left(\frac{X-\mu}{\mu}\right)\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\sum (X-\mu)^{r-1} \frac{e^{-\mu}(\mu)^x}{x!} + \frac{1}{\mu}\sum (X-\mu)^r \frac{e^{-\mu}(\mu)^x}{x!}(X-\mu)\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\sum (X-\mu)^{r-1} \frac{e^{-\mu}(\mu)^x}{x!} + \frac{1}{\mu}\sum (X-\mu)^{r+1} \frac{e^{-\mu}(\mu)^x}{x!}\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\sum (X-\mu)^{r-1} P(X) + \frac{1}{\mu}\sum (X-\mu)^{r+1} P(X)\right]$$

$$\frac{d}{d\mu}\mu_r = \left[-r\mu_{r-1} + \frac{1}{\mu}\mu_{r+1}\right]$$

$$\frac{d}{d\mu}\mu_r = -r\mu_{r-1} + \frac{1}{\mu}\mu_{r+1}$$

$$\frac{d}{d\mu}\mu_r + r\mu_{r-1} = \frac{1}{\mu}\mu_{r+1}$$

$$\mu_{r+1} = \mu\left[r\mu_{r-1} + \frac{d}{d\mu}\mu_r\right]$$

Hence proved

Now first four moments about mean

Put $r = 0$

$\mu_1 = 0$ Where $\mu_0 = 1$

Put $r = 1$

$\mu_{1+1} = \mu\left[1\mu_0 + \frac{d}{d\mu}\mu_1\right]$ Where $\mu_0 = 1$ and $\mu_1 = 0$

$\mu_2 = \mu[1] = \mu$

Put $r = 2$

$\mu_{2+1} = \mu\left[2\mu_1 + \frac{d}{d\mu}\mu_1\right]$ Where $\mu_2 = \mu$ and $\mu_1 = 0$

$\mu_3 = \mu\left[2(0) + \frac{d}{d\mu}\mu\right] = \mu(0+1) = \mu$

Put $r = 3$

$\mu_{3+1} = \mu\left[3\mu_2 + \frac{d}{d\mu}\mu_3\right]$ Where $\mu_2 = \mu$ and $\mu_3 = \mu$

$\mu_4 = \mu\left[3\mu + \frac{d}{d\mu}\mu\right] = \mu(3\mu+1) = 3\mu^2 + \mu$

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu^2}{\mu^3} = \frac{1}{\mu}$

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\mu^2 + \mu}{\mu^2} = 3 + \frac{1}{\mu}$

**State and prove reproductive property of Poisson distribution**

Solution:

Statement: Let " $X_1$ *and* $X_2$ " be the two independent Poisson variate having with $\mu_1$ *and* $\mu_2$ parameters respectively. Then $X_1 + X_2$ also follow Poisson distribution with $\mu_1 + \mu_2$ parameters.

Proof: As $X_1 \to P(\mu_1)$ then its Probability function given as

$P(X = x_1) = \frac{e^{-\mu_1}(\mu_1)^{x_1}}{x_1!}$ And $Mean = \mu_1$

And its m.g.f $M_{x_1}(t) = e^{\mu_1(e^t - 1)}$

As $X_2 \to P(\mu_2)$ then its Probability function given as

$P(X = x_2) = \frac{e^{-\mu_2}(\mu_2)^{x_2}}{x_2!}$ And $Mean = \mu_2$

And its m.g.f $M_0(t) = e^{\mu_2(e^t - 1)}$

$Z = X_1 + X_2$ Then m.g.f of "Z"

$M_Z(t) = E(e^{tZ})$

$M_Z(t) = E(e^{t(X_1 + X_2}))$

$M_Z(t) = E(e^{tX_1 + tX_2})$ As $X_1$ *and* $X_2$ are independent then

$M_Z(t) = E(e^{tX_1})E(e^{ttX_2})$

$M_Z(t) = M_{x_1}(t) M_{x_2}(t)$

$M_Z(t) = e^{\mu_1(e^t-1)} e^{\mu_2(e^t-1)} = e^{(\mu_1+\mu_2)(e^t-1)}$

Hence proved it is also Poisson distribution with parameters $\mu_1 + \mu_2$

**If $n \to \infty$, then $Poisson \to normality$ or state and prove asymptotic property of Poisson distribution.**

Proof:

As we know that $E(X) = \mu$ and $Var(X) = \mu$

Now we consider standard normal variate

$$Z = \frac{X-\mu}{\sigma}$$

$$Z = \frac{X-\mu}{\sqrt{\mu}}$$

Let by definition of m.g.f

$$m_Z(t) = E(e^{tZ}) = E(e^{t\frac{X-\mu}{\sqrt{\mu}}}) = E(e^{\frac{tX-t\mu}{\sqrt{\mu}}}) = E(e^{\frac{tX}{\sqrt{\mu}}} e^{\frac{-t\mu}{\sqrt{\mu}}}) = e^{\frac{-t\mu}{\sqrt{\mu}}} E(e^{\frac{tX}{\sqrt{\mu}}})$$

$$m_Z(t) = e^{\frac{-t\mu}{\sqrt{\mu}}} E(e^{\frac{tX}{\sqrt{\mu}}}) \qquad \text{(A)}$$

As we know that m.g.f of Poisson distribution

$$m_x(t) = e^{\mu(e^t-1)}$$

Replacing $t = \frac{t}{\sqrt{\mu}}$

$$m_x(t) = e^{\mu(e^{\frac{t}{\sqrt{\mu}}}-1)}$$

Then (A) becomes

$$m_Z(t) = e^{\frac{-t\mu}{\sqrt{\mu}}} e^{\mu(e^{\frac{t}{\sqrt{\mu}}}-1)}$$

Taking log on both sides

$$\log m_Z(t) = \log\left[e^{\frac{-t\mu}{\sqrt{\mu}}} e^{\mu(e^{\frac{t}{\sqrt{\mu}}}-1)}\right] = \log\left[e^{\frac{-t\mu}{\sqrt{\mu}}}\right] + \log\left[e^{\mu(e^{\frac{t}{\sqrt{\mu}}}-1)}\right] = \frac{-t\mu}{\sqrt{\mu}} \log e + {}^{\mu(e^{\frac{t}{\sqrt{\mu}}}-1)} \log e$$

$$\log m_Z(t) = \frac{-t\mu}{\sqrt{\mu}} + \mu(e^{\frac{t}{\sqrt{\mu}}} - 1)$$

$$\log m_Z(t) = \frac{-t\mu}{\sqrt{\mu}} + \mu\left(\frac{\left(\frac{t}{\sqrt{\mu}}\right)^0}{0!} + \frac{\left(\frac{t}{\sqrt{\mu}}\right)^1}{1!} + \frac{\left(\frac{t}{\sqrt{\mu}}\right)^2}{2!} + \frac{\left(\frac{t}{\sqrt{\mu}}\right)^3}{3!} + \ldots - 1\right)$$

$$\log m_Z(t) = \frac{-t\mu}{\sqrt{\mu}} + \mu\left(1 + \frac{t}{\sqrt{\mu}} + \frac{t^2}{2!\mu} + \frac{t^3}{\mu\sqrt{\mu}3!} + \ldots - 1\right)$$

$$\log m_Z(t) = \frac{-t\mu}{\sqrt{\mu}} + \mu\left(\frac{t}{\sqrt{\mu}} + \frac{t^2}{2!\mu} + \frac{t^3}{\mu\sqrt{\mu}3!} + \ldots\right)$$

$$\log m_Z(t) = \frac{-t\mu}{\sqrt{\mu}} + \frac{\mu t}{\sqrt{\mu}} + \frac{\mu t^2}{2!\mu} + \frac{\mu t^3}{\mu\sqrt{\mu}3!} + \ldots$$

$$\log m_Z(t) = \frac{t^2}{2} + \frac{t^3}{\sqrt{\mu}3!} + \ldots$$

As we know $\mu = nP$ Then $n \to \infty$ *then* $np \to \infty$

Applying limit

$$\log m_Z(t) = \lim t_{n\to\infty}(\frac{t^2}{2} + \frac{t^3}{\sqrt{\mu}3!} + \ldots)$$

$$\log m_Z(t) = \frac{t^2}{2} + \lim t_{n\to\infty}\frac{t^3}{\sqrt{\mu}3!} + \lim t_{n\to\infty}\ldots$$

$$\log m_Z(t) = \frac{t^2}{2} + 0 + 0 + \ldots$$

$$\log m_Z(t) = \frac{t^2}{2}$$

Taking antilog on both sides

$m_Z(t) = e^{\frac{t^2}{2}}$ It is m.g.f of standardized normal distribution

**Hence proved** $n \to \infty$**, then** *Poisson* $\to$ *normality*

**Q.8.33 (b):** Suppose that "X" has a Poisson distribution. If $P(X = 1) = 0.3$ and $P(X = 2) = 0.2$ then calculate $P(X = 0)$ *and* $P(X = 3)$

Solution: As we know that $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}$

$$P(X = 1) = \frac{e^{-\mu}\mu^1}{1!} = e^{-\mu}\mu$$

$0.3 = e^{-\mu}\mu$ (i)

$$P(X = 2) = \frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-\mu}\mu^2}{2}$$

$0.2 \times 2 = e^{-\mu}\mu^2$

$0.4 = e^{-\mu}\mu\mu$ (ii)

Putting the value of $e^{-\mu}\mu = 0.3$ then we get $\mu$

$0.4 = 0.3\mu$

$$\mu = \frac{0.4}{0.3} = 1.33$$

Now we find

$$P(X = 0) = \frac{e^{-1.33}(1.33)^0}{0!} = ?$$

**Q.8.33 ©:** A random variable “X” has a Poisson distribution such that $P(X=2)=3P(X=4)$. Find $P(X=1)$

Solution: $P(X=2)=3P(X=4)$

$$\frac{e^{-\mu}\mu^2}{2!}=3\frac{e^{-\mu}\mu^4}{4!}$$

$$\frac{\mu^2}{2}=3\frac{\mu^4}{24}$$

$$\frac{\mu^2}{2}=\frac{\mu^4}{8}$$

$$\mu^2=4$$

$$\mu=2$$

$$P(X=1)=\frac{e^{-2}2^1}{1!}=0.271$$

**Q.8.40 (b):** In a Poisson distribution the first two frequencies were 250 and 160. Find the frequencies of the next two values of the variable.

Solution: Given that

$$NP(X=0)=N\frac{e^{-\mu}\mu^0}{0!}=Ne^{-\mu}$$

$$Ne^{-\mu}=250$$

$$NP(X=1)=N\frac{e^{-\mu}\mu^1}{1!}=Ne^{-\mu}\mu$$

$Ne^{-\mu}\mu=160$ Then putting the value of $Ne^{-\mu}=250$ and we get

$$250\mu=160$$

$\mu=\frac{160}{250}=0.64$ $\quad N=\frac{250}{e^{-0.64}}=474$ $\quad$ Now next two frequencies

$$NP(X=2)=474\frac{e^{-0.64}(0.64)^2}{2!}=474(0.1079)=51$$

$$NP(X=3)=474\frac{e^{-0.64}(0.64)^3}{3!}=(474)(0.023)=11$$

**Poisson process**

The Poisson process is a random physical mechanism in which events occur randomly on a time or distance scale. The formula of Poisson process is

$P(X=x)=\frac{e^{-\lambda t}(\lambda t)^x}{x!}$ Where $X=0,1,2,...,\infty$

$t = Number\ of\ units\ of\ time$

$X = Number\ of\ occurrance\ in\ t\ units\ of\ time$

$\lambda = Average\ number\ of\ occurrance\ per\ unit\ of\ time$

**Properties**

i) If "$\lambda$" *is an average number of occurrance per unit of time* or space. Then expected number of occurrence in fixed interval of time or space "t" is $\lambda t$

ii) The probability of occurrence two or more events in a small interval or time or space is negligible

iii) Events occurring in independent intervals of time or space are independent

**Example.8.21:** Telephone calls are being placed through a certain exchange at random times on the average of four per minute. Assuming a Poisson process, determine the probability that in a 15-seconds interval, there are 3 or more calls.

Solution:

$\lambda = 4\ minute\ or\ 60\ second$

$t=\frac{15}{60}\ seconds$

$\lambda t = 4\times\frac{15}{60}=1$ $P(X=x)=\frac{e^{-\lambda t}(\lambda t)^x}{x!}$ $P(X\geq 3)=1-P(X<3)$

**Example.8.22:** Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume the Poisson distribution, find the probability of at most one flaw in 225 square feet.

Solution:

$\lambda = one\ in\ 150\ per\ square\ feet$

$t=\frac{225}{150}\ square\ feet$

$\lambda t = 1\times\frac{225}{150}=1.5$ $P(X=x)=\frac{e^{-\lambda t}(\lambda t)^x}{x!}$ $P(X\geq 1)=1-P(X<1)$

**Example:**

a) If there are 3 suicides per 30,000 population in a city. Find the probability that in a given year there are i) no ii) at least 4 suicides in a city of 120000

Solution:

$\lambda = 3\ suicide\ per\ 30000$ $t=\frac{120000}{30000}=4$ $\lambda t = 3\times 4 = 12$

b) Suppose there is an average of 2 suicides per year per 50000 populations. In a city of 100000, find the probability that in a given year there are i) 0 ii) 1 suicide

Solution:

$\lambda = 2\ suicide\ per\ 50000$

$t=\dfrac{1000000}{50000}=2$ $\qquad\lambda t=2\times 2=4$

c) The reception office at a building receives an average of 4.5 phone calls per half hour. Find the probability of receiving exactly 6 phone calls at this office during i) half hour ii) an hour

Solution:

$\lambda=4.5\ \ per\,half\ \ hour$

i) $t=\dfrac{30}{30}=1$ $\qquad\lambda t=4.5\times 1=4.5$

ii $t=\dfrac{60}{30}=2$ $\qquad\lambda t=4.5\times 2=9$

d) A manufacturer of plywood sheets detects 2 flaws per sheet of 200 square feet. Find the probability of at most one flaw in 300 square feet sheet purchases by a customer.

Solution:

$\lambda=2\ \ flaw\ per\ 200\, square\ \ feet$

$t=\dfrac{300}{200}=1.5$ $\qquad\lambda t=2\times 1.5=3$

e) A certain area in a city is, on the average, hot by 6 hurricanes a year. Find the probability that in the next 6 months that area will be hit by one hurricane.

Solution:

$\lambda=6\ \ per\ \ year\ \ or\ \ per\ 12\, month$

$t=\dfrac{6}{12}=0.5$ $\qquad\lambda t=6\times 0.5=3.0$

**Q.8.48 (b):** Suppose that customers enter a certain shop at the rate of 30 persons an hour. Using the Poisson distribution, calculate the probability that in a 2-minute interval, no customer will enter the shop.

Solution:

$\lambda=30\ \ persons\ \ per\ \ hour\ \ or\ \ 60\min ute$

$t=\dfrac{2}{60}\min ute$

$\lambda t=30\times\dfrac{2}{60}=1$ $\qquad P(X=x)=\dfrac{e^{-\lambda t}(\lambda t)^{x}}{x!}$ $\qquad P(X=0)=\dfrac{e^{-1}(1)^{0}}{0!}=0.368$

**Q.8.49 (a):** Flaws in plywood occur at random with an average of one flaw per 50 square feet. What is the probability that a $4\ feet\times 8\ feet$ sheet will have no flaws? At most one flaw?

b) A doctor receives an average of 3 telephone calls from 9 p.m until 9 a.m. the next morning. Assuming arrivals of calls are a Poisson process, what is the probability that the doctor will not be disturbed by a call if she goes to bed at midnight and rises at 6 a.m?

Solution:

a) $\lambda=one\ flaw\ per\ \ 50\, square\ \ feet$

$t=\dfrac{32}{50}\,per\,square\ 50\ \ sequare\ feet$

$\lambda t = 1 \times \frac{32}{50} = 0.64$

$P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$

$P(X \leq 1) = P(X = 1) + P(X = 0)$

b) $\lambda = 3 \ calls \ in \ 12 \ hour$

$t = \frac{6}{12}$

$\lambda t = 3 \times \frac{6}{12} = 1.5$ $\quad P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ $\quad P(X = 0) = ?$

**Q.8.50:** A computer system in a company has a breakdown once in 25 days, on the Average. Assuming breakdowns are a Poisson process, what is the probability of?

i) Exactly one breakdown in the next 10 days ii) more than one in next 10 days?

Solution:

i) $\lambda = one\,in \ per \ 25 \ days$

$t = \frac{10}{25}$

$\lambda t = 1 \times \frac{10}{25} = 0.4$ $\quad P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ $\quad P(X = 1) = ?$

ii) $P(X > 1) = 1 - P(X \leq 1) = 1 - [P(X = 1) + P(X = 0)]$

**Q.8.51:** The number of cars passing over a toll bridge during the time interval 10 to 11 a.m. is 300. The cars pass individually and collectively at random. Find the probability

i) Not more than 4 cars will pass during 1-minute interval 10:45 to 10:46

ii) 5 or more cars will pass during the same interval.

Solution:

i) $\lambda = 300 \ cars \ in \ one \ hour \ or \ 60 \min ute$

$t = \frac{1}{60}$

$\lambda t = 300 \times \frac{1}{60} = 5$

$P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$

$P(X \leq 4) = P(X + 4) + P(X = 3) + P(X = 2) + P(X = 1) + P(X = 0) = ?$

ii)

$P(X \geq 5) = 1 - P(X < 5) = 1 - [P(X + 4) + P(X = 3) + P(X = 2) + P(X = 1) + P(X = 0)] = ?$

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