Exponential Distribution

Let 'X' be a positive continuous random variable with interval $(0,\infty)$ is said to be Exponential distribution, having its p.d.f

$$f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}} \qquad 0\le x\le\infty$$

It is only one parameter θ.

If $\frac{1}{\theta}=\theta$ then it is also exponential distribution

$$f(x)=\theta e^{-x\theta}$$

If θ=1 then it follow standard exponential distribution

$$f(x)=e^{-x}$$

This is also known as negative exponential distribution or single parameter exponential distribution.

Properties

i) Exponential distribution is a continuous distribution.
ii) The total area under the curve is unity.
iii) The range of the distribution is 0 to ∞.
iv) It has one parameter θ.
v) The mean of the exponential distribution is $\mathrm{E(x)}=\theta$.
vi) The variance of the exponential distribution is $\mathrm{Var(x)}=\theta^2$.
vii) The m.g.f of the exponential distribution is $\mathrm{m.g.f}=(1-\theta)^{-1}$.

Prove that total area under the curve is unity

Proof: Let by definition:

$$Area=\int f(x)dx$$

As x ≈ exp(θ)

$$f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}} \qquad 0\le x\le\infty$$

$$Area=\int_0^\infty \frac{1}{\theta}e^{\frac{-x}{\theta}}dx$$

$$Area=\frac{1}{\theta}\int_0^\infty e^{\frac{-x}{\theta}}dx \qquad \text{(A)}$$

As we know that gamma function is

$$\overline{)a}\,b^a=\int_0^\infty x^{a-1}e^{-x/b}dx \qquad \text{(B)}$$

Comparing (A) & (B) and we get

a = 1 & b = θ

$$\overline{)a}\,b^a=\overline{)1}\,\theta^1 \qquad \text{Put in (A)}$$

$$Area=\frac{1}{\theta}\theta=1 \qquad \text{Hence Prove}$$

Find mean & variance

Solution: Let by definition:

$$E(x)=\int xf(x)dx=\frac{1}{\theta}\int_0^\infty xe^{\frac{-x}{\theta}}dx$$

$$E(x)=\frac{1}{\theta}\int_0^\infty x^{2-1}e^{\frac{-x}{\theta}}dx \qquad \text{(A)}$$

As we know that gamma function is

$$\overline{)a}b^a=\int_0^\infty x^{a-1}e^{-x/b}dx \qquad \text{(B)}$$

Comparing (A) & (B) and we get

a = 2 & b = θ

$$\overline{)a}b^a=\overline{)2}\theta^2$$

Put in (A)

$$E(x)=\frac{1}{\theta}\overline{)2}\theta^2$$

$$E(x)=\theta$$

$$Var(x)=E(x^2)-[E(x)]^2$$

$$E(x^2)=\int x^2f(x)dx=\frac{1}{\theta}\int_0^\infty x^2e^{\frac{-x}{\theta}}dx$$

$$E(x^2)=\frac{1}{\theta}\int_0^\infty x^{3-1}e^{\frac{-x}{\theta}}dx \qquad \text{(A)}$$

As we know that gamma function is

$$\overline{)a}b^a=\int_0^\infty x^{a-1}e^{-x/b}dx \qquad \text{(B)}$$

Comparing (A) & (B) and we get

a = 3 & b = θ

$$\overline{)a}b^a=\overline{)3}\theta^3$$

Put in (A)

$$E(x^2)=\frac{1}{\theta}\overline{)3}\theta^3=2\theta^2$$

$$Var(x)=E(x^2)-[E(x)]^2==2\theta^2-(\theta)^2=\theta^2$$

Find r^th^ moments about origin. By use it finds mean & variance

Solution: Let by definition

$$\mu_r'=E(x^r)$$

$$\mu_r'=\int x^rf(x)dx$$

$$\mu_r'=\frac{1}{\theta}\int_0^\infty x^re^{-x/\theta}dx$$

$$\mu_r'=\frac{1}{\theta}\int_0^\infty x^{r+1-1}e^{-x/b}dx \qquad \text{(A)}$$

As we know that gamma function is

$$\Gamma a b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad (B)$$

Comparing (A) & (B) and we get

a = r+1 & b = θ

$$\Gamma a b^a = \Gamma r+1 \theta^{r+1} \qquad \text{Put in (A)}$$

$$\mu_r' = \frac{1}{\theta} \Gamma r+1 . \theta^{r+1}$$

$$\mu_r' = \frac{1}{\theta} \Gamma r+1 . \theta^{r+1}$$

$$\mu_r' = \Gamma r+1 . \theta^r \qquad (C)$$

Use rth moments to find mean & variance

$$mean = \mu_1' = E(x)$$

$$\mu_r' = \Gamma r+1 . \theta^r$$

Put r = 1 in eq (C)

$$\mu_1' = \Gamma 1+1 . \theta^1 = \theta$$

Now, put r = 2 in eq.(C)

$$\mu_2' = \Gamma 2+1 . \theta^2 = 2\Gamma 2 . \theta^2 = 2\theta^2$$

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = 2.\theta^2 - (\theta)^2 = \theta^2 = Var(X)$$

Find m.g.f of exponential distribution. Also find mean & variance by using m.g.f

Solution: Let by definition

$$M_x(t) = m_x(t) = E(e^{tx}) = \int e^{tx} f(x) dx$$

As x ≈ exp(θ)

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \qquad 0 \le x \le \infty$$

$$M_0(t) = \frac{1}{\theta} \int_0^\infty e^{tx} e^{-x/\theta} dx$$

$$M_0(t) = \frac{1}{\theta} \int_0^\infty e^{tx - x/\theta} dx$$

$$M_0(t) = \frac{1}{\theta} \int_0^\infty e^{-x/\theta (1-\theta t)} dx$$

$$M_0(t) = \frac{1}{\theta} \int_0^\infty x^{1-1} e^{-x/\theta(1-\theta t)^{-1}} dx$$

$$M(t) = \frac{1}{\theta} \int_0^\infty x^{1-1} e^{-x/\theta(1-\theta t)^{-1}} dx \qquad (A)$$

As we know that gamma function is

$$\Gamma a b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad (B)$$

Comparing (A) & (B) and we get

a = 1 & b = $\theta(1-\theta t)^{-1}$

$$\Gamma ab^{a} = \Gamma 1\{\theta(1-\theta t)^{-1}\}$$ Put in (A)

$$m_x(t) = \frac{1}{\theta}\Gamma 1\{\theta(1-\theta t)^{-1}\} = (1-\theta t)^{-1}$$

$$\mu_x(t) = (1-\theta t)^{-1}$$ Required m.g.f.

Use it to find mean & variance

$$E(x) = \mu_1' = \left[\frac{d}{dx} m_x(t)\right]_{t=0}$$

$$E(x) = \left[\frac{d}{dx}(1-\theta t)^{-1}\right]_{t=0}$$

$$E(x) = \left[\theta(1-\theta t)^{-2}\right]_{t=0} \quad \text{©}$$

$$E(x) = \theta$$ Required mean

Again differentiate eq(c) w.r.t to 't'

$$E(x^2) = \mu_2' = \left[\frac{d^2}{dx^2} M_x(t)\right]_{T=0}$$

$$E(x^2) = \frac{d}{dt}\left[\frac{d}{dx} M_x(t)\right]_{T=0}$$

$$E(x^2) = \frac{d}{dt}\left[\theta(1-\theta t)^{-2}\right]_{T=0}$$

$$E(x^2) = \left[-2\theta(1-\theta t)^{-3}(-\theta)\right]_{T=0}$$

$$E(x^2) = \mu_2' = 2\theta^2$$

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = 2.\theta^2 - (\theta)^2 = \theta^2 = Var(X)$$

Find cummulent generating function

Solution: Let by definition

$$k(t) = \log[M_x(t)]$$

$$k(t) = \log\left[(1-\theta t)^{-1}\right]$$

$$k(t) = -\log(1-\theta t)$$

Therefore

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\dots$$

$$k(t) = -\left[-(\theta t) - \frac{(\theta t)^2}{2} - \frac{(\theta t)^3}{3} - \frac{(\theta t)^4}{4}\dots\right]$$

$$k(t) = \left[\theta t + \theta^2\frac{t^2}{2} + \theta^3\frac{t^3}{3} + \theta^4\frac{t^4}{4}\dots\right]$$

$$k(t) = \left[\theta\frac{t}{1!} + \theta^2\frac{t^2}{2!} + 2\theta^3\frac{t^3}{3!} + 6\theta^4\frac{t^4}{4!}\dots\right] \quad \text{(A)}$$

As we know general expression of cummulent

$$k(t) = \left[k_1\frac{t}{1!} + k^2\frac{t^2}{2} + k_3\frac{t^3}{3} + k_4\frac{t^4}{4}\dots\right] \quad \text{(B)}$$

By comparing (A) & (B) and we get

$$k_1 = \mu_1' = \theta$$

$$k_2 = \mu_2 = \theta^2 = Var(X)$$
$$k_3 = \mu_3 = 2\theta^3$$
$$k_4 = \mu_4' = 6\theta^4$$
$$\mu_4 = k_4 + 3k_2^{\,2} = 6\theta^4 + 3(\theta^2)^2$$
$$\mu_4 = 6\theta^4 + 3\theta^4 = 9\theta^4$$

Moment Ratio

$$\beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}} = \frac{(2\theta^3)^2}{(\theta^2)^3} = \frac{4\theta^6}{\theta^6} = 4$$

$\mu_3 = 2\theta^3$ It is positive so distribution positively skewed

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{9\theta^4}{(\theta^2)^2} = \frac{9\theta^4}{\theta^4} = 9$$

As $\beta_2 > 3$ so the distribution is leptokurtic.

Find mode of exponential distribution

Solution: Let by definition

If following two conditions are satisfied then mode exists.

$f(x') = 0$ or $\frac{d}{dx}\log f(x) = 0$

$f(x') < 0$ or $\frac{d^2}{dx^2}\log f(x) < 0$

As x ≈ exp(θ)

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \qquad 0 \le x \le \infty$$

Taking log on both sides

$$\log f(x) = \log\left[\frac{1}{\theta} e^{-x/\theta}\right]$$
$$\log f(x) = \left[\log\left(\frac{1}{\theta}\right) - \frac{x}{\theta}\log e\right] \qquad :- \log_e = 1$$
$$\log f(x) = \left[\log\left(\frac{1}{\theta}\right) - \frac{x}{\theta}\right]$$

Differentiate w.r.t to 'x'

$$\frac{d}{dx}\log f(x) = \frac{d}{dx}\left[\log\left(\frac{1}{\theta}\right) - \frac{x}{\theta}\right]$$
$$\frac{d}{dx}\log f(x) = \left[-\frac{1}{\theta}\right]$$

It means that mode of exponential distribution does not exist.

State & prove Memory less property of Exponential Distribution

Statement:

If 'x' follows the negative exponential distribution with p.d.f $f(x) = \theta e^{-x\theta}$ $0 \le x \le \infty$ then by definition

$$P\left[\frac{x > (a+b)}{x > a}\right] = P(x > b) \qquad \text{by law of compliment}$$
$$\frac{1 - P[x \le (a+b)]}{1 - P(x \le a)} = \frac{1 - F(a+b)}{1 - F(a)} \qquad \text{(i)}$$

Let 'x' be continuous random variable having the p.d.f $f(x) = \theta e^{-x\theta}$ its c.d.f is

$$F(x)=\int_0^x \theta e^{-x\theta}dx$$

$$F(x)=\theta\frac{e^{-x\theta}}{-\theta}\Bigg|_0^x$$

$$F(x)=1-e^{-x\theta}$$

$$F(a)=1-e^{-a\theta}$$

$$F(a+b)=1-e^{-(a+b)\theta}$$

$$F(b)=1-e^{-b\theta}$$

Put in (i)

$$\frac{1-F(a+b)}{1-F(a)}=\frac{1-\left(1-e^{-(a+b)\theta}\right)}{1-\left(1-e^{-a\theta}\right)}$$

$$\frac{1-F(a+b)}{1-F(a)}=\frac{1-1+e^{-(a+b)\theta}}{1-1+e^{-a\theta}}$$

$$\frac{1-F(a+b)}{1-F(a)}=\frac{e^{-(a+b)\theta}}{e^{-a\theta}}$$

$$\frac{1-F(a+b)}{1-F(a)}=e^{-a\theta-b\theta+a\theta}=e^{-b\theta}$$

$$\frac{1-F(a+b)}{1-F(a)}=1-\left(1-e^{-b\theta}\right)=1-F(b)=1-P(x\le b)=P(x>b)$$

2nd Method

Suppose A be the event such that (x>a) & B is another event such that x>(a+b). i.e B is a subset of A. Then we consider

$$P\left[\frac{B}{A}\right]=\frac{P(A\cap B)}{P(A)}=\frac{P(B)}{P(A)}$$

Because B is a subset of A, replacing A = x>a & B = x> a+b we get

$$P\left[\frac{x>(a+b)}{x>a}\right]=\frac{P(x>a+b)}{P(x>a)}$$

Then c.d.f of Negative Exp.Distribution

$$P(X\le x))=\int_{-\infty}^{\infty} f(x)dx=\int_0^x \theta e^{-x\theta}dx=\theta\frac{e^{-x\theta}}{-\theta}\Bigg|_0^x=1-e^{-x\theta}$$

Then by compliment we know that

$$P(X\le x))=1-e^{-x\theta}$$

$$P(x>a))=1-P(X\le x)=1-1-e^{-a\theta}=e^{-a\theta}$$

Similarly

$$P(x>a+b))==e^{-(a+b)\theta}$$

$$\frac{P(x>a+b)}{P(x>a)}=\frac{e^{-(a+b)\theta}}{e^{-a\theta}}=e^{-a\theta-b\theta+a\theta}=e^{-b\theta}=P(x>b)$$

Hence, Memory less property is proved.