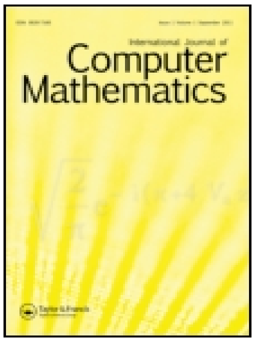

International Journal of Computer Mathematics

ISSN: 0020-7160 (Print) 1029-0265 (Online) Journal homepage: https://www.tandfonline.com/loi/gcom20

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To cite this article: Hikmet Caglar , Nazan Caglar , Mehmet Özer , Antonios Valarıstos & Antonios N. Anagnostopoulos (2010) B-spline method for solving Bratu's problem, International Journal of Computer Mathematics, 87:8, 1885-1891, DOI: 10.1080/00207160802545882

To link to this article: https://doi.org/10.1080/00207160802545882

International Journal of Computer Mathematics
Vol. 87, No. 8, July 2010, 1885–1891

B-spline method for solving Bratu's problem

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(*Received 27 May 2008; revised version received 21 August 2008; accepted 20 September 2008*)

In this paper, we propose a B-spline method for solving the one-dimensional Bratu's problem. The numerical approximations to the exact solution are computed and then compared with other existing methods. The effectiveness and accuracy of the B-spline method is verified for different values of the parameter, below its critical value, where two solutions occur.

Keywords: Bratu's problem; B-spline method; nonlinear boundary value problem; Laplace method; decomposition method

2000 AMS Subject Classifications: 34B15; 33F05; 65D20

1. Introduction

We consider the Liouville–Bratu–Gelfand equation [5,13,17,18]

$$\begin{aligned} &\Delta u + \lambda e^u = 0, \quad x \in \Omega \\ &u = 0, \quad x \in \partial\Omega \end{aligned} \tag{1}$$

where $\lambda > 0$ is a physical parameter and Ω is a bounded domain in R^N. We restrict our attention to the one-dimensional case, where $\Omega = (0, 1)$. This is known in the literature as the classical Bratu's problem [13,17]:

$$\begin{aligned} &u''(x) + \lambda e^u = 0, \quad 0 < x < 1 \\ &u(0) = u(1) = 0. \end{aligned} \tag{2}$$

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ISSN 0020-7160 print/ISSN 1029-0265 online

DOI: 10.1080/00207160802545882
http://www.informaworld.com

An extended summary of history of the equation can be found in Refs [13,17]. The exact solution of this boundary value problem (BVP) is given by:

$$u(x) = -2\ln\left[\frac{\cos h((x-1/2)\times\theta/2)}{\cosh(\theta/4)}\right],$$

where θ is the solution of $\theta = \sqrt{2\lambda}\cos h(\theta/4)$.

The exponential term guarantees nonlinearity and the bifurcation phenomenon that follows up. In particular for different values of λ, one can verify the following:

- for $\lambda > \lambda_c$, the problem has no solutions
- for $\lambda = \lambda_c$, the problem has a unique solution
- for $0 < \lambda < \lambda_c$, the problem has two bifurcated solutions, where the critical value $\lambda_c = 3.513830719$ has the solution of $1 = (1/4)\sqrt{2\lambda_c}\sin h(\theta/4)$ [2–4,10].

Such two-point BVPs are widely used in science and engineering to describe complicated physical and chemical models. Bratu's problem is also used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process [5,15,16,18–20], the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model [5,16,17–20], chemical reaction theory, radiative heat transfer and nanotechnology [12].

Several analytical and numerical methods, such as finite difference method [5,18], decomposition method [8], Laplace transform decomposition method [14,19], Adomian decomposition method [20], homotopy analysis method [15], weighted residual method [1], differential transformation method [11] and He's variational method [12] have been used to solve the Bratu's problem.

Our intention is to use B-spline method for solving the aforesaid problem. This method has been applied with success in several BVPs with different types of boundary conditions [6]. In this work, third-degree B-spline functions have been successfully implemented to the Bratu's problem. The obtained results are numerically compared with the corresponding results from other methods given in literature [8,14]. The numerical results given by the present method show that a good accuracy can be obtained. As shown in text, application of this method is very simple and clear.

2. The third-degree B-splines

In this section, third-degree B-splines are used to construct numerical solutions to nonlinear Bratu's problem that is given in Equation (2). A detailed description of B-spline functions can be found in Ref. [7].

The third-degree B-splines are defined as:

$$B_0(x) = \frac{1}{6h^3}\begin{cases} x^3 & 0 \le x < h \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3 & h \le x < 2h \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3 & 2h \le x < 3h \\ -x^3 + 12hx^2 - 48h^2x + 64h^3 & 3h \le x < 4h \end{cases}, \tag{3}$$

$$B_{i-1}(x) = B_0(x-(i-1)h), \quad i = \dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

To solve nonlinear Bratu's problem, B_i, B_i' and B_i'' evaluated at the nodal points are needed. Their coefficients are tabulated in Table 1.

Table 1. Values of B_i, B_i' and B_i''.

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
B_i	0	1/6	4/6	1/6	0
B_i'	0	$3/6h$	$0/6h$	$-3/6h$	0
B_i''	0	$6/6h^2$	$-12/6h^2$	$6/6h^2$	0

3. B-spline solutions for Bratu's problem

Let

$$S(x) = \sum_{j=-3}^{n-1} C_j B_j(x) \tag{4}$$

be an approximate solution of Equation (2), where C_j are unknown real coefficients and $B_j(x)$ are third-degree B-spline functions. Let $x_0, x_1, \ldots, x_n$ be $n+1$ grid points in the interval $[a, b]$ so that $x_i = a + ih$, $i = 1, 2, \ldots, n$, $x_0 = a$, $x_n = b$, $h = (b-a)/n$. The approximate solution of Equation (4) is substituted in Equation (2) and evaluated at the grid points $x_0, x_1, \ldots, x_n$. This leads to a nonlinear system of equations of the form:

$$\sum_{j=-3}^{n-1} C_j B_j''(x_i) + \lambda \exp\left(\sum_{j=-3}^{n-1} C_j B_j(x_i)\right) = 0, \quad i = 0, 1, \ldots, n \tag{5}$$

$$\sum_{j=-3}^{n-1} C_j B_j\,(x_0) = 0, \quad \text{for } x = 0, \tag{6}$$

$$\sum_{j=-3}^{n-1} C_j B_j(x_n) = 0, \quad \text{for } x = 1. \tag{7}$$

The values of the spline functions at the knots $\{x_i\}_{i=0}^{n}$ are determined using Table 1 with substitution in Equations (5) to (7). Thus, a system of $n+3$ nonlinear equations in the $(n+3)$ unknowns $C_{-3}, C_{-2}, \ldots, C_{n-1}$ is obtained.

This system may be written in the matrix vector form as follows:

$$AC + \lambda B = 0, \tag{8}$$

where

$$A = \begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 & 0 & \cdots & 0 \\ \frac{6}{6h^2} & \frac{-12}{6h^2} & \frac{6}{6h^2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{6}{6h^2} & \frac{-12}{6h^2} & \frac{6}{6h^2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{6}{6h^2} & \frac{-12}{6h^2} & \frac{6}{6h^2} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ e^{C_{-3}+4C_{-2}+C_{-1}} \\ e^{C_{-2}+4C_{-1}+C_{0}} \\ e^{C_{-1}+4C_{0}+C_{1}} \\ \vdots \\ e^{C_{n-4}+4C_{n-3}+C_{n-2}} \\ e^{C_{n-3}+4C_{n-2}+C_{n-1}} \\ 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} C_{-3} & C_{-2} & C_{-1} & \cdots & C_{n-3} & C_{n-2} & C_{n-1} \end{bmatrix}^{\mathrm{T}}.$$

The approximate solution (Equation (4)) is obtained by solving the nonlinear system using Levenberg–Marquardt optimization method [9] and Matlab 6.5.

4. Numerical results

In this section, we illustrate the numerical techniques discussed previously, by applying our method to the planar one-dimensional Bratu's problem for three specific values of λ, which guarantee the existence of two locally unique solutions. In particular, having used $\lambda = 1$, 2 and 3.51, we have constructed comparison tables to indicate the accuracy of our method compared with the exact solution as well as the other methods' solutions. The numerical results are shown in Figure 1. All calculations have been performed using Matlab 6.5.

In Table 2, we have tabulated the exact solution together with the solutions given by Laplace method, Decomposition and B-spline methods are as exhibited before, in the case of $\lambda = 1$. In Table 3, we have calculated the absolute error of each method compared with the exact solution. As one clearly observes, the magnitude of the errors remains uniform throughout the unit interval, whereas in the Laplace method the magnitude of the error increases when x exceeds 0.5.

In Table 4, we have tabulated the exact solution together with the solutions given by Laplace method, Decomposition and B-spline methods are as exhibited before, in the case of $\lambda = 2$. In

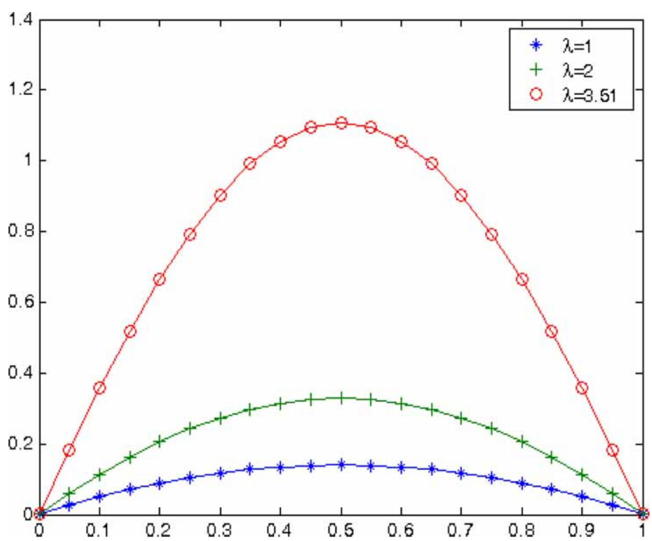

Figure 1. Results for $\lambda = 1, 2$ and $3.51 (n = 21)$.

Table 2. Laplace and decomposition method approximation and B-spline method of Bratu's problem for the case λ = 1.

x	Exact solution	Laplace [17]	Decomposition [16]	B-spline
0.1	0.0498467900	0.0498448112	0.0471616875	0.0498438103
0.2	0.0891899350	0.0891859956	0.0871680000	0.0891844690
0.3	0.1176090956	0.1176032408	0.1177614375	0.1176017599
0.4	0.1347902526	0.1347825488	0.1369920000	0.1347817559
0.5	0.1405392142	0.1405297477	0.1435546875	0.1405303221
0.6	0.1347902526	0.1347791409	0.1369920000	0.1347817559
0.7	0.1176090956	0.1175965240	0.1177614375	0.1176017599
0.8	0.0891899350	0.0891899350	0.0871680000	0.0891844690
0.9	0.0498467900	0.0498348222	0.0471616875	0.0498438103

Table 3. The absolute errors for Laplace and decomposition method approximation and B-spline method of Bratu's problem for the case λ = 1.

x	Laplace [17]	Decomposition [16]	B-spline
0.1	1.978800000003445e–6	2.685102500000001e–3	2.979700000002583e–6
0.2	3.939399999999815e–6	2.021935000000003e–3	5.465999999995641e–6
0.3	5.854800000010263e–6	1.523418999999915e–4	7.335699999999612e–6
0.4	7.703799999980721e–6	2.201747400000009e–3	8.496699999999136e–6
0.5	9.466499999999378e–6	3.015473299999988e–3	8.892100000018610e–6
0.6	1.111169999998274e–5	2.201747400000009e–3	8.496699999999136e–6
0.7	1.257160000001090e–5	1.523418999999915e–4	7.335699999999612e–6
0.8	1.347531e–5	2.021935000000003e–3	5.465999999995641e–6
0.9	1.196780000000536e–5	2.685102500000001e–3	2.979700000002583e–6

Table 4. Laplace and decomposition method approximation and B-spline method of Bratu's problem for the case λ = 2.

x	Exact solution	Laplace [17]	Decomposition [16]	B-spline
0.1	0.1144107440	0.1122817141	0.0991935000	0.1143935651
0.2	0.2064191156	0.2022094162	0.1917440000	0.2063865190
0.3	0.2738793116	0.2676925058	0.2679915000	0.2738344125
0.4	0.3150893646	0.3070874506	0.3183360000	0.3150365062
0.5	0.3289524214	0.3193532294	0.3359375000	0.3288968072
0.6	0.3150893646	0.3041598403	0.3183360000	0.3150365062
0.7	0.2738793116	0.2619458909	0.2679915000	0.2738344125
0.8	0.2064191156	0.1940413072	0.1917440000	0.2063865190
0.9	0.1144107440	0.1035373785	0.0991935000	0.1143935651

Table 5. The absolute errors for Laplace and decomposition method approximation and B-spline method of Bratu's problem for the case λ = 2.

x	Laplace [17]	Decomposition [16]	B-spline
0.1	2.129029899999996e–3	1.521724399999999e–2	1.717889999999778e–5
0.2	4.209699400000017e–3	1.467511560000001e–2	3.259660000001774e–5
0.3	6.186805800000028e–3	5.887811600000015e–3	4.489909999999542e–5
0.4	8.001913999999999e–3	3.246635400000031e–3	5.285839999996655e–5
0.5	9.599191999999979e–3	6.985078600000028e–3	5.561419999999817e–5
0.6	1.092952429999999e–2	3.246635400000031e–3	5.285839999996655e–5
0.7	1.193342070000003e–2	5.887811600000015e–3	4.489909999999542e–5
0.8	1.237780840000000e–2	1.467511560000001e–2	3.259660000001774e–5
0.9	1.087336550000000e–2	1.521724399999999e–2	1.717889999999778e–5

Table 5, we have calculated the absolute error of each method compared with the exact solution. As one clearly observes, the magnitude of the errors using the B-spline method becomes significantly smaller and remains uniform throughout the unit interval.

Finally in Tables 6 and 7, we have exhibited comparisons between the exact solution and B-spline method solutions in the case of $\lambda = 3.51$. Naturally, one can observe that the magnitude of the errors increase as λ increases, due to the involvement of λ with the nonlinear term in Equation (2). The effect of the number of mesh points, n, is also shown (for the case $\lambda = 2$). As clearly seen in Table 8, the numerical results become better as the value of n increases.

Table 6. B-spline method of Bratu's problem for the case $\lambda = 3.51$.

x	Exact solution	B-spline
0.1	0.395805699	0.357388461
0.2	0.739097410	0.664283874
0.3	1.008758260	0.902930838
0.4	1.182536660	1.055419782
0.5	1.242742690	1.107989815
0.6	1.182536660	1.055419782
0.7	1.008758260	0.902930838
0.8	0.739097410	0.664283874
0.9	0.395805699	0.357388461

Table 7. The absolute errors B-spline method of Bratu's problem for the case $\lambda = 3.51$.

x	B-spline
0.1	3.84172369550e–2
0.2	7.48135367780e–2
0.3	1.05827422823e–1
0.4	1.27116880861e–1
0.5	1.34752877607e–1
0.6	1.27116880864e–1
0.7	1.05827422823e–1
0.8	7.48135367760e–2
0.9	3.84172369530e–2

Table 8. Numerical solutions of different values of $n$for the case $\lambda = 2$.

x	Exact solution	$n = 21$	$n = 61$	$n = 91$
0.1	0.1144107440	0.1143420832	0.1143935652	0.1144073490
0.2	0.2064191156	0.2062888337	0.2063865191	0.2064126752
0.3	0.2738793116	0.2736998743	0.2738344125	0.2738704396
0.4	0.3150893646	0.3148781299	0.3150365063	0.3150789192
0.5	0.3289524214	0.3287301762	0.3288968073	0.3289414316
0.6	0.3150893646	0.3148781299	0.3150365063	0.3150789192
0.7	0.2738793116	0.2736998743	0.2738344125	0.2738704396
0.8	0.2064191156	0.2062888337	0.2063865191	0.2064126752
0.9	0.1144107440	0.1143420832	0.1143935652	0.1144073490

5. Conclusions

In this paper, B-spline method is developed for the approximate solution of nonlinear Bratu's problem. The numerical results obtained by using the method described in this study give acceptable results. We have concluded that the numerical results converge to the exact solution when h goes to zero. It has been observed that the B-spline method yields much better approximations to the exact solutions of Bratu's problem than those shown by Khuri [14] and Deeba *et al*. [8].

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