

# Normal Distribution

Let "X" be a continuous random variable with interval  $(-\infty, +\infty)$  is said to be normal distribution having its probability density function (p.d.f) is given as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq +\infty$$

It has two parameters  $(\mu, \sigma^2)$ .

Where  $\mu = \text{Mean}$        $\sigma^2 = \text{Variance}$        $\sigma = \text{Standard deviation}$

$\pi = \text{Constant Approximately equal to } = \frac{22}{7} = 3.14159$

$e = \text{Constant Approximately equal to } = 2.71828$

$X = \text{Abscissa i.e. value marked on X - axis}$

$Y = \text{Ordinate height i.e. value marked on Y - axis}$

## Properties of normal distribution

i) It is continuous distribution and its range  $(-\infty, +\infty)$

ii) Total area under the curve is unity

iii) It is bell shape distribution

iv) It is symmetrical distribution and its mean, median and mode are identical

v) It is unimodal distribution and maximum ordinate of the curve at  $X = \mu$  is  $\frac{1}{\sigma\sqrt{2\pi}}$

vi) It has two points of inflection which are equidistant from mean "  $\mu$  " are

$$\left(\mu - \sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right) \text{ and } \left(\mu + \sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right)$$

vii) The mean deviation of normal distribution is approximately  $\frac{4}{5}$  of its Standard

deviation i.e.  $M.D = \frac{4}{5}\sigma = 0.7979\sigma$

viii) The Quartile deviation of normal distribution is approximately  $\frac{2}{3}$  of its Standard

deviation i.e.  $Q.D = \frac{2}{3}\sigma = 0.6745\sigma$

ix) All odd order moments about mean equal to zero i.e.  $\mu_1 = \mu_3 = \mu_5 = \dots = 0$

x) The expression of even order moments about mean are  $\mu_{2n} = \left(\frac{\sigma^2}{2}\right)^n \frac{(2n)!}{n!}$

xi) If X is  $N(\mu, \sigma^2)$  and if  $Y = a + bX$  then Y is  $N(a + \mu b, b^2 \sigma^2)$

xii) The sum of independent normal variables is a normal variable

xiii) The property of normal distribution

i)  $P(\mu - 0.6745\sigma < X < \mu + 0.6745\sigma) = 0.50$  or 50%

ii)  $P(\mu - \sigma < X < \mu + \sigma) = 0.6827$  or 68.27%

iii)  $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$  or 95.44%

iv)  $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$  or 99.73%

xiv) The two quartiles are equidistant from the mean

$Q_1 = \mu - 0.6745\sigma$       Lower quartile

$Q_3 = \mu + 0.6745\sigma$       Upper quartile

xv) The normal curve approaches but never touches the base line. So the curve is asymptotic to the horizontal line.

xvi) It has two parameters  $(\mu, \sigma^2)$

### Standard normal variate

Ans: Any variable having zero mean and unit variance is called standard normal variate or variable. i.e.  $Z = \frac{X - \mu}{\sigma}$

### Standard normal distribution

The normal probability distribution of "Z" which has zero mean and unit variance is called the standardized normal distribution. It is denoted by  $Z \rightarrow N(0,1)$  and its p.d.f

given as 
$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \quad -\infty \leq Z \leq +\infty$$

### Properties of standard normal distribution

- i) It is continuous distribution and its range  $(-\infty, +\infty)$
- ii) Total area under the curve is unity
- iii) It is bell shape distribution
- iv) It is symmetrical distribution and its mean, median and mode are equal to zero
- v) The mean deviation of standard normal distribution is approximately  $\frac{4}{5}$  i.e.

$$M.D = \frac{4}{5} = 0.7979$$

- vi) The Quartile deviation of standard normal distribution is approximately  $\frac{2}{3}$

i.e.  $Q.D = \frac{2}{3} = 0.6745$

- vii) All odd order moments about mean equal to zero i.e.  $\mu_1 = \mu_3 = \mu_5 = \dots = 0$

- viii) The two quartiles of standard normal distribution

$Q_1 = -0.6745$  Lower quartile

$Q_3 = 0.6745$  Upper quartile

- ix) It has two parameters  $Z \rightarrow N(0,1)$

### Theorem.No.1: Show that total area under the curve is unity

Proof: Let by definition

$$\text{Total Area} = \int_{-\infty}^{\infty} f(x) d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$\text{Total Area} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$\text{Total Area} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) \quad (i)$$

Put  $Z = \frac{X - \mu}{\sigma}$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$d(x) = Z d(z)$  And Limits remains same  $-\infty \leq Z \leq \infty$

$$\text{Total Area} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2} \sigma d(z)$$

$$\text{Total Area} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2} d(z) \quad (ii)$$

As  $f(z) = e^{-\frac{1}{2}Z^2}$

Put  $Z = -Z$

$f(-Z) = e^{-\frac{1}{2}(-Z)^2} = e^{-\frac{1}{2}Z^2} = f(Z)$  Its even function then eq.(ii) becomes

$$\text{Total Area} = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}Z^2} d(z) \quad (\text{iii})$$

$$\text{Put } t = \frac{1}{2}Z^2$$

$$\sqrt{2t} = Z$$

$$\frac{1}{2}(2t)^{-\frac{1}{2}} 2d(t) = d(z)$$

$$(2t)^{-\frac{1}{2}} d(t) = d(z) \quad \text{And Limits remains same } 0 \leq t \leq \infty$$

$$\text{Total Area} = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{-\frac{1}{2}} e^{-t} d(t)$$

$$\text{Total Area} = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-t} d(t)$$

$$\text{Total Area} = \frac{2}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t} d(t)$$

$$\text{Total Area} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} d(t) \quad (\text{iv})$$

As we know that Gamma function is

$$\Gamma(\alpha)\beta^\alpha = \int_0^{\infty} t^{\alpha-1} e^{-t/\beta} d(t) \quad (\text{v})$$

Comparing (iv) and (v) then we get

$$\alpha = \frac{1}{2} \quad \text{And } \beta = 1 \quad \text{Then eq.(iv) becomes}$$

$$\text{Total Area} = \frac{1}{\sqrt{\pi}} \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1 \quad \text{Therefore } \left(\frac{1}{2}\right)^{\frac{1}{2}} = \sqrt{\pi}$$

$$\text{Total Area} = 1 \quad \text{Hence proved}$$

**Theorem.No.2: Show that mean of normal distribution is " $\mu$ "**

Proof: Let by definition

$$E(X) = \int_{-\infty}^{\infty} Xf(x)d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$E(X) = \int_{-\infty}^{\infty} X \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} X e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) \quad (\text{i})$$

$$\text{Put } Z = \frac{X - \mu}{\sigma}$$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = \sigma d(z) \quad \text{And Limits remains same } -\infty \leq Z \leq \infty$$

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma Z + \mu) e^{-\frac{1}{2}Z^2} \sigma d(z)$$

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma Z e^{-\frac{1}{2}Z^2} d(z) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}Z^2} d(z)$$

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z e^{\frac{1}{2}Z^2} d(z) + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}Z^2} d(z) \quad (\text{ii})$$

$$\text{As } f(z) = e^{\frac{1}{2}Z^2} \quad f(z) = Z e^{\frac{1}{2}Z^2}$$

Put  $Z = -Z$

$$f(-Z) = (-Z)e^{\frac{1}{2}(-Z)^2} = -Ze^{\frac{1}{2}Z^2} = -f(Z)$$

$$f(-Z) = e^{\frac{1}{2}(-Z)^2} = e^{\frac{1}{2}Z^2} = f(Z)$$

1<sup>st</sup> function is odd and 2<sup>nd</sup> is even function then eq. (ii) becomes

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} (0) + \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{\frac{1}{2}Z^2} d(z)$$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{\frac{1}{2}Z^2} d(z) \quad (\text{iii})$$

$$\text{Put } t = \frac{1}{2} Z^2$$

$$\sqrt{2t} = Z$$

$$\frac{1}{2} (2t)^{\frac{1}{2}} 2d(t) = d(z)$$

$$(2t)^{\frac{1}{2}} d(t) = d(z)$$

And Limits remains same  $0 \leq t \leq \infty$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{\frac{1}{2}} e^{-t} d(t)$$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-t} d(t)$$

$$E(X) = \frac{2\mu}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t} d(t)$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} d(t) \quad (\text{iv})$$

As we know that Gamma function is

$$\Gamma(\alpha)\beta^\alpha = \int_0^{\infty} t^{\alpha-1} e^{-t/\beta} d(t) \quad (\text{v})$$

Comparing (iv) and (v) then we get

$$\alpha = \frac{1}{2} \quad \text{And} \quad \beta = 1 \quad \text{Then eq. (iv) becomes}$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \left( \frac{1}{2} \right)^{\frac{1}{2}} = \frac{\mu}{\sqrt{\pi}} \left( \frac{1}{2} \right) = \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} = \mu \quad \text{Therefore} \quad \left( \frac{1}{2} \right)^{\frac{1}{2}} = \sqrt{\pi}$$

$$E(X) = \mu \quad \text{Hence proved}$$

**Theorem.No.3: Show that Variance of normal distribution is " $\sigma^2$ "**

Proof: Let by definition

$$\text{Var}(X) = E[X - \mu]^2 = \int_{-\infty}^{\infty} [X - \mu]^2 f(x) d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} [X - \mu]^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$\text{Var}(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} [X - \mu]^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) \quad (\text{i})$$

Put  $Z = \frac{X - \mu}{\sigma}$

$Z\sigma = x - \mu$

$X = Z\sigma + \mu$

$d(x) = \sigma d(z)$

And Limits remains same  $-\infty \leq Z \leq \infty$

$$Var(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma Z)^2 e^{-\frac{1}{2}Z^2} \sigma d(z)$$

$$Var(X) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 e^{-\frac{1}{2}Z^2} d(z) \tag{ii}$$

As  $f(z) = Z^2 e^{-\frac{1}{2}Z^2}$

Put  $Z = -Z$

$$f(-Z) = (-Z)^2 e^{-\frac{1}{2}(-Z)^2} = Z^2 e^{-\frac{1}{2}Z^2} = f(Z)$$

$$f(-Z) = e^{-\frac{1}{2}(-Z)^2} = e^{-\frac{1}{2}Z^2} = f(Z)$$

It is even function then eq. (ii) becomes

$$Var(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} Z^2 e^{-\frac{1}{2}Z^2} d(z) \tag{iii}$$

Put  $t = \frac{1}{2}Z^2$

$2t = Z^2$

$\sqrt{2t} = Z$

$$\frac{1}{2}(2t)^{-\frac{1}{2}} 2d(t) = d(z)$$

$(2t)^{-\frac{1}{2}} d(t) = d(z)$

And Limits remains same  $0 \leq t \leq \infty$

$$Var(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2t(2t)^{-\frac{1}{2}} e^{-t} d(t)$$

$$Var(X) = \frac{4\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}}}{2^{\frac{1}{2}}} e^{-t} d(t)$$

$$Var(X) = \frac{4\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t} d(t)$$

$$Var(X) = \frac{4\sigma^2}{2\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} d(t)$$

$$Var(X) = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}-1} e^{-t} d(t) \tag{iv}$$

As we know that Gamma function is

$$\int_0^{\infty} t^{\alpha-1} e^{-t/\beta} d(t) \tag{v}$$

Comparing (iv) and (v) then we get

$\alpha = \frac{3}{2}$  And  $\beta = 1$  Then eq. (iv) becomes

$$Var(X) = \frac{\sigma^2}{\sqrt{\pi}} \left( \frac{3}{2} \right)^{\frac{3}{2}} \frac{1}{2} = \frac{2\sigma^2}{\sqrt{\pi}} \left( \frac{1}{2} \right)^{\frac{3}{2}} + 1$$

$$Var(X) = \frac{2\sigma^2}{\sqrt{\pi}} \left( \frac{1}{2} \right)^{\frac{3}{2}} = \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \sigma^2 \quad \text{Therefore} \quad \left( \frac{1}{2} \right)^{\frac{3}{2}} = \sqrt{\pi}$$

$Var(X) = \sigma^2$  Hence proved

**Theorem.No.4: Show that Mean deviation of normal distribution is " $M.D = \frac{4}{5}\sigma$ "**

Proof: Let by definition

$$M.D = E|X - \mu| = \int_{-\infty}^{\infty} |X - \mu| f(x) d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$M.D = \int_{-\infty}^{\infty} |X - \mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$M.D = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |X - \mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) \quad (i)$$

$$\text{Put } Z = \frac{X - \mu}{\sigma}$$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = \sigma d(z)$$

And Limits remains same  $-\infty \leq Z \leq \infty$

$$M.D = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma Z| e^{-\frac{1}{2}Z^2} \sigma d(z)$$

$$\text{Var}(X) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |Z| e^{-\frac{1}{2}Z^2} d(z)$$

$$M.D = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{-\infty}^0 -Ze^{-\frac{1}{2}Z^2} d(z) + \int_0^{\infty} Ze^{-\frac{1}{2}Z^2} d(z) \right]$$

$$M.D = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_0^{\infty} Ze^{-\frac{1}{2}Z^2} d(z) + \int_0^{\infty} Ze^{-\frac{1}{2}Z^2} d(z) \right]$$

$$M.D = \frac{\sigma}{\sqrt{2\pi}} \left[ 2 \int_0^{\infty} Ze^{-\frac{1}{2}Z^2} d(z) \right]$$

$$M.D = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} Ze^{-\frac{1}{2}Z^2} d(z) \quad (ii)$$

$$\text{Put } t = \frac{1}{2}Z^2$$

$$2t = Z^2$$

$$\sqrt{2t} = Z$$

$$\frac{1}{2}(2t)^{-\frac{1}{2}} 2d(t) = d(z)$$

$$(2t)^{-\frac{1}{2}} d(t) = d(z)$$

And Limits remains same  $0 \leq t \leq \infty$

$$M.D = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2t} (2t)^{-\frac{1}{2}} e^{-t} d(t)$$

$$M.D = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sqrt{2} t^{\frac{1}{2}} t^{-\frac{1}{2}}}{2^{\frac{1}{2}}} e^{-t} d(t)$$

$$M.D = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sqrt{2} t^{\frac{1}{2} + \frac{1}{2}}}{\sqrt{2}} e^{-t} d(t)$$

$$M.D = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} t^0 e^{-t} d(t)$$

$$\text{Var}(X) = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} t^{1-1} e^{-t} d(t) \quad (\text{iii})$$

As we know that Gamma function is

$$\int_0^{\infty} t^{\alpha-1} e^{-t/\beta} d(t) \quad (\text{iv})$$

Comparing (iii) and (iv) then we get

$\alpha = 1$  And  $\beta = 1$  Then eq. (iv) becomes

$$M.D = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} 1 = \frac{\sqrt{2}\sqrt{2}\sigma}{\sqrt{2}\sqrt{\pi}} = \frac{\sqrt{2}\sigma}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}} \sigma = 0.7979\sigma = \frac{4}{5}\sigma \quad \text{Hence proved}$$

**Theorem.No.5: Show that mean, median and mode of ND is equal to  $\mu$ .**

Proof: Let by definition

$$E(X) = \int_{-\infty}^{\infty} Xf(x)d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$E(X) = \int_{-\infty}^{\infty} X \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} X e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) \quad (\text{i})$$

$$\text{Put } Z = \frac{X - \mu}{\sigma}$$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = \sigma d(z)$$

And Limits remains same  $-\infty \leq Z \leq \infty$

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma Z + \mu) e^{-\frac{1}{2}Z^2} \sigma d(z)$$

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma Z e^{-\frac{1}{2}Z^2} d(z) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}Z^2} d(z)$$

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z e^{-\frac{1}{2}Z^2} d(z) + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2} d(z) \quad (\text{ii})$$

$$\text{As } f(z) = e^{-\frac{1}{2}Z^2} \quad f(z) = Z e^{-\frac{1}{2}Z^2}$$

$$\text{Put } Z = -Z$$

$$f(-Z) = (-Z) e^{-\frac{1}{2}(-Z)^2} = -Z e^{-\frac{1}{2}Z^2} = -f(Z)$$

$$f(-Z) = e^{-\frac{1}{2}(-Z)^2} = e^{-\frac{1}{2}Z^2} = f(Z)$$

1<sup>st</sup> function is odd and 2<sup>nd</sup> is even function then eq. (ii) becomes

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} (0) + \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}Z^2} d(z)$$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}Z^2} d(z) \quad (\text{iii})$$

$$\text{Put } t = \frac{1}{2} Z^2$$

$$\sqrt{2t} = Z$$

$$\frac{1}{2} (2t)^{-\frac{1}{2}} 2d(t) = d(z)$$

$$(2t)^{-\frac{1}{2}} d(t) = d(z)$$

And Limits remains same  $0 \leq t \leq \infty$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{-\frac{1}{2}} e^{-t} d(t)$$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-t} d(t)$$

$$E(X) = \frac{2\mu}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t} d(t)$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} d(t) \quad (\text{iv})$$

As we know that Gamma function is

$$\Gamma(\alpha)\beta^\alpha = \int_0^{\infty} t^{\alpha-1} e^{-t/\beta} d(t) \quad (\text{v})$$

Comparing (iv) and (v) then we get

$$\alpha = \frac{1}{2} \quad \text{And} \quad \beta = 1 \quad \text{Then eq. (iv) becomes}$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{\mu}{\sqrt{\pi}} \left(\frac{1}{2}\right) = \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} = \mu \quad \text{Therefore} \quad \left(\frac{1}{2}\right)^{\frac{1}{2}} = \sqrt{\pi}$$

$$E(X) = \mu$$

#### Now for median

Let by definition median of normal distribution (median=M)

$$\int_{-\infty}^M f(x)d(x) = \frac{1}{2}$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$\int_{-\infty}^M \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) = \frac{1}{2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^M e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x) = \frac{1}{2}$$

$$\text{Put} \quad Z = \frac{X - \mu}{\sigma}$$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = \sigma d(z)$$

$$\text{Limits} \quad \text{When } X \rightarrow -\infty \text{ Then } Z \rightarrow -\infty \quad \text{And } X \rightarrow M \text{ Then } Z \rightarrow \frac{M - \mu}{\sigma}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}(z)^2} \sigma d(Z) = \frac{1}{2}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}(z)^2} d(Z) = \frac{1}{2} \quad (\text{i})$$

As we know that normal distribution is symmetrical then we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{1}{2}(z)^2} d(Z) = \frac{1}{2} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}(z)^2} d(Z) \quad (\text{ii})$$

Comparing (i) and (ii) then we get

$$\frac{M - \mu}{\sigma} = 0$$

$$M = \mu$$



Median =  $\mu$  Hence it is also equal to “ $\mu$ ”

### Now for mode

The following two conditions are satisfied then mode is exist

$$f'(x) = 0$$

$f''(x) < 0$  Mean second derivative must be negative

As we know that probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

Differentiate with respect to “X”

$$f'(x) = \frac{d}{d(x)} f(x)$$

$$\frac{d}{d(x)} f(x) = \frac{d}{d(x)} \left( \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{d}{d(x)} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \frac{d}{d(x)} -1 \left( \frac{X-\mu}{\sigma} \right)^2 \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} -2 \left( \frac{X-\mu}{\sigma} \right) \frac{d}{d(x)} \frac{X-\mu}{\sigma} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} \frac{d}{d(x)} (X-\mu) \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} (1-0) \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu) \right) \tag{A}$$

Now we equating the first derivative zero

$$0 = \frac{-1}{\sigma^3\sqrt{2\pi}} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu) \right)$$

$$0 = e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu)$$

$$0 = (X-\mu)$$

$$X = \mu$$

Now we again differentiate eq. (A) with respect to “X”

$$f''(x) = \frac{d}{d(x)} \left( \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu) \right)$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \frac{d}{d(x)} \left( e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu) \right)$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \left( (X-\mu) \frac{d}{d(x)} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} + e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \frac{d}{d(x)} (X-\mu) \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( (X - \mu) e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \frac{d}{d(x)} \frac{-1}{2} \left( \frac{X - \mu}{\sigma} \right)^2 + e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} (1 - 0) \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( (X - \mu) e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \frac{-2}{2} \left( \frac{X - \mu}{\sigma} \right) \frac{d}{d(x)} \left( \frac{X - \mu}{\sigma} \right) + e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -(X - \mu) e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma} \right) \frac{1}{\sigma} \frac{d}{d(x)} (X - \mu) + e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -(X - \mu) e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma} \right) \frac{1}{\sigma} (1 - 0) + e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -(X - \mu) e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma^2} \right) + e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma} \right)^2 + e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{1}{\sigma^3 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \left( \frac{X - \mu}{\sigma} \right)^2 - 1 \right)$$

Put  $X = \mu$  then we get  $f''(x) < 0$  thus the *Mode* =  $\mu$ .

Hence in normal distribution

*Mean* = *Median* = *Mode* =  $\mu$  Proved

**Theorem.No.6: Show that maximum value of  $f(X)$  occurs at “ $X = \mu$ ”**

Proof: As we know that

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2}$$

Differentiate with respect to “X”

$$f'(x) = \frac{d}{d(x)} f(x)$$

$$\frac{d}{d(x)} f(x) = \frac{d}{d(x)} \left( \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma \sqrt{2\pi}} \frac{d}{d(x)} \left( e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \frac{d}{d(x)} \frac{-1}{2} \left( \frac{X - \mu}{\sigma} \right)^2 \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \frac{-2}{2} \left( \frac{X - \mu}{\sigma} \right) \frac{d}{d(x)} \frac{X - \mu}{\sigma} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma} \right) \frac{1}{\sigma} \frac{d}{d(x)} (X - \mu) \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma} \right) \frac{1}{\sigma} (1 - 0) \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2} \left( \frac{X - \mu}{\sigma} \right) \frac{1}{\sigma} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (X-\mu) \right) \quad (A)$$

Now we equating the first derivative zero

$$0 = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (X-\mu) \right)$$

$$0 = e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (X-\mu)$$

$$0 = (X-\mu)$$

$$X = \mu$$

Now we see that

For  $X < \mu$  Then  $f'(x) > 0$

For  $X > \mu$  Then  $f'(x) < 0$

For  $X = \mu$  Then  $f'(x) = 0$

Thus the maximum of the function  $f(x)$  at  $X = \mu$  which is  $\frac{1}{\sigma\sqrt{2\pi}}$

$$\left[ f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} = \frac{1}{\sigma\sqrt{2\pi}} \quad \text{at } X = \mu \right]$$

**Theorem.No.7: Show that the points of inflexion in ND are equidistant from mean**

Proof:

As we know that probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2}$$

Differentiate with respect to "X"

$$f'(x) = \frac{d}{d(x)} f(x)$$

$$\frac{d}{d(x)} f(x) = \frac{d}{d(x)} \left( \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{d}{d(x)} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \frac{d}{d(x)} \frac{-1}{2} \left( \frac{X-\mu}{\sigma} \right)^2 \right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \frac{-2}{2} \left( \frac{X-\mu}{\sigma} \right) \frac{d}{d(x)} \frac{X-\mu}{\sigma} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} \frac{d}{d(x)} (X-\mu) \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} (1-0) \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} \right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (X-\mu) \right) \quad (A)$$

Now we again differentiate eq. (A) with respect to "X"

$$f''(x) = \frac{d}{d(x)} \left( \frac{-1}{\sigma^3 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (X-\mu) \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \frac{d}{d(x)} \left( e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (X-\mu) \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( (X-\mu) \frac{d}{d(x)} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \frac{d}{d(x)} (X-\mu) \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( (X-\mu) e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \frac{d}{d(x)} - 1 \left( \frac{X-\mu}{\sigma} \right)^2 + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} (1-0) \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( (X-\mu) e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \frac{-2 \left( \frac{X-\mu}{\sigma} \right) \frac{d}{d(x)} \left( \frac{X-\mu}{\sigma} \right)}{2} + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -(X-\mu) e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} \frac{d}{d(x)} (X-\mu) + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -(X-\mu) e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma} \right) \frac{1}{\sigma} (1-0) + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -(X-\mu) e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma^2} \right) + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{-1}{\sigma^3 \sqrt{2\pi}} \left( -e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \frac{X-\mu}{\sigma} \right)^2 + e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \right)$$

$$f''(x) = \frac{1}{\sigma^3 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \left( \frac{X-\mu}{\sigma} \right)^2 - 1 \right)$$

Now we equating the 2nd derivative zero

$$0 = \frac{1}{\sigma^3 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} \left( \left( \frac{X-\mu}{\sigma} \right)^2 - 1 \right)$$

$$0 = \left( \left( \frac{X-\mu}{\sigma} \right)^2 - 1 \right)$$

$$1 = \left( \frac{X-\mu}{\sigma} \right)^2$$

$$\sigma^2 = (X-\mu)^2$$

$$\sqrt{\sigma^2} = \sqrt{(X-\mu)^2}$$

$$\pm \sigma = X - \mu$$

$$X = \mu \pm \sigma$$

$$\text{At } X = \mu - \sigma \text{ Then we get } \left[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma \sqrt{2\pi} e} \right]$$

$$\text{At } X = \mu + \sigma \text{ Then we get } \left[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{X-\mu}{\sigma} \right)^2} = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma \sqrt{2\pi} e} \right]$$

Hence the two points of inflection of normal curve are

$$\left[ (\mu - \sigma), \frac{1}{\sigma \sqrt{2\pi} e} \right] \quad \text{And} \quad \left[ (\mu + \sigma), \frac{1}{\sigma \sqrt{2\pi} e} \right]$$

**Theorem.No.8: Derive moment generating function of normal distribution**

Proof: Let by definition of m.g.f

$$M_0(t) = E(e^{tx})$$

$$M_0(t) = \int_{-\infty}^{\infty} e^{tx} f(x) d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$M_0(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$M_0(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

Put  $Z = \frac{X - \mu}{\sigma}$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = Z d(z) \quad \text{And Limits remains same } -\infty \leq Z \leq \infty$$

$$M_0(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma Z + \mu)} e^{-\frac{1}{2}(Z)^2} \sigma d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tZ\sigma} e^{t\mu} e^{-\frac{1}{2}(Z)^2} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z)^2 + tZ\sigma} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z^2 - 2tZ\sigma)} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z^2 - 2tZ\sigma + (\sigma)^2 - (\sigma)^2)} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z^2 - 2tZ\sigma + (\sigma)^2)} e^{-\frac{1}{2}(-(\sigma)^2)} d(z)$$

$$M_0(t) = \frac{e^{t\mu} e^{\frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z - \sigma)^2} d(z)$$

$$M_0(t) = \frac{e^{\mu t + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z - \sigma)^2} d(z)$$

Put  $w = z - t\sigma$

$$dw = dz$$

Limits remain same

$$M_0(t) = \frac{e^{\mu t + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} d(w)$$

As  $f(w) = e^{-\frac{1}{2}w^2}$

Put  $w = -w$

$$f(-w) = e^{-\frac{1}{2}(-w)^2} = e^{-\frac{1}{2}w^2} = f(w) \quad \text{Its even function then we gets}$$

$$M_0(t) = \frac{2e^{\mu t + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}w^2} d(w)$$

Put  $y = \frac{1}{2}w^2$

$$\sqrt{2y} = w$$

$$\frac{1}{2}(2y)^{-\frac{1}{2}} 2d(y) = d(w)$$

$$(2y)^{-\frac{1}{2}} d(y) = d(w)$$

And Limits remains same  $0 \leq y \leq \infty$

$$M_0(t) = \frac{2e^{\mu + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-y} (2y)^{-\frac{1}{2}} d(y)$$

$$M_0(t) = \frac{2e^{\mu + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_0^{\infty} \frac{y^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-y} d(y)$$

$$M_0(t) = \frac{2e^{\mu + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_0^{\infty} \frac{y^{\frac{1}{2}-1}}{\sqrt{2}} e^{-y} d(y)$$

$$M_0(t) = \frac{2e^{\mu + \frac{1}{2}(\sigma)^2}}{2\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y} d(y)$$

$$M_0(t) = \frac{e^{\mu + \frac{1}{2}(\sigma)^2}}{\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y} d(y) \quad (i)$$

As we know that Gamma function is

$$\Gamma(\alpha)\beta^\alpha = \int_0^{\infty} y^{\alpha-1} e^{-y/\beta} d(y) \quad (ii)$$

Comparing (i) and (v) then we get

$$\alpha = \frac{1}{2} \quad \text{And} \quad \beta = 1 \quad \text{Then eq. (ii) becomes}$$

$$M_0(t) = \frac{e^{\mu + \frac{1}{2}(\sigma)^2}}{\sqrt{\pi}} \left( \frac{1}{2} \right)^{\frac{1}{2}}$$

$$M_0(t) = \frac{e^{\mu + \frac{1}{2}(\sigma)^2}}{\sqrt{\pi}} \sqrt{\pi} \quad \text{Therefore} \quad \left( \frac{1}{2} \right)^{\frac{1}{2}} = \sqrt{\pi}$$

$$M_0(t) = e^{\mu + \frac{1}{2}(\sigma)^2} \quad \text{Required result}$$

### Theorem.No.9: Derive moment generating function of standardized ND

Proof: Let by definition of m.g.f

$$M_0(t) = E(e^{tZ})$$

$$M_0(t) = \int_{-\infty}^{\infty} e^{tz} f(z) d(z)$$

$$z \rightarrow N(0,1)$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \quad -\infty \leq Z \leq \infty$$

$$M_0(t) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} d(Z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-\frac{1}{2}(z)^2} d(Z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z)^2 + tz} d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2tz)} d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2tz + (t)^2 - (t)^2)} d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2tz + (t)^2)} e^{-\frac{1}{2}(-(t)^2)} d(z)$$

$$M_0(t) = \frac{e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} d(z)$$

$$M_0(t) = \frac{e^{\mu t + \frac{1}{2}(\sigma)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} d(z)$$

Put  $w = z - t$

$$dw = dz$$

Limits remain same

$$M_0(t) = \frac{e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} d(w)$$

$$\text{As } f(w) = e^{-\frac{1}{2}w^2}$$

Put  $w = -w$

$$f(-w) = e^{-\frac{1}{2}(-w)^2} = e^{-\frac{1}{2}w^2} = f(w) \quad \text{Its even function then we gets}$$

$$M_0(t) = \frac{2e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}w^2} d(w)$$

Put  $y = \frac{1}{2}w^2$

$$\sqrt{2y} = w$$

$$\frac{1}{2}(2y)^{-\frac{1}{2}} 2d(y) = d(w)$$

$$(2y)^{-\frac{1}{2}} d(y) = d(w)$$

And Limits remains same  $0 \leq y \leq \infty$

$$M_0(t) = \frac{2e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-y} (2y)^{-\frac{1}{2}} d(y)$$

$$M_0(t) = \frac{2e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_0^{\infty} \frac{y^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-y} d(y)$$

$$M_0(t) = \frac{2e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_0^{\infty} \frac{y^{\frac{1}{2}-1}}{\sqrt{2}} e^{-y} d(y)$$

$$M_0(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y} d(y)$$

$$M_0(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y} d(y) \quad \text{(i)}$$

As we know that Gamma function is

$$\Gamma(\alpha)\beta^\alpha = \int_0^{\infty} y^{\alpha-1} e^{-y/\beta} d(y) \quad \text{(ii)}$$

Comparing (i) and (v) then we get

$$\alpha = \frac{1}{2} \quad \text{And} \quad \beta = 1 \quad \text{Then eq. (ii) becomes}$$

$$M_0(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \left( \frac{1}{2} \right)^{\frac{1}{2}}$$

$$M_0(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \sqrt{\pi}$$

$$\text{Therefore} \quad \left( \frac{1}{2} \right)^{\frac{1}{2}} = \sqrt{\pi}$$

$$M_0(t) = e^{\frac{1}{2}t^2}$$

Required result

**Theorem.No.10: Show those odd order moments of a normal distribution are equal to zero and derive the expression of even order moments about mean.**

Proof: Let by definition of odd order moments

$$\mu_{2n+1} = E[X - \mu]^{2n+1} = \int_{-\infty}^{\infty} [X - \mu]^{2n+1} f(x) d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$\mu_{2n+1} = \int_{-\infty}^{\infty} [X - \mu]^{2n+1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$\mu_{2n+1} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} [X - \mu]^{2n+1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$\text{Put } Z = \frac{X - \mu}{\sigma}$$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = \sigma d(z)$$

And Limits remains same  $-\infty \leq Z \leq \infty$

$$\mu_{2n+1} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} [\sigma Z]^{2n+1} e^{-\frac{1}{2}(Z)^2} \sigma d(Z)$$

$$\mu_{2n+1} = \frac{[\sigma]^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [Z]^{2n+1} e^{-\frac{1}{2}(Z)^2} d(Z) \quad (i)$$

$$\text{As } f(z) = Z^{2n+1} e^{-\frac{1}{2}Z^2}$$

$$\text{Put } Z = -Z$$

$$f(-Z) = -f(Z)$$

It is odd function then eq. (i) becomes

$$\mu_{2n+1} = \frac{[\sigma]^{2n+1}}{\sqrt{2\pi}} (0) = 0 \quad \text{Hence proved}$$

Let by definition of even order moments

$$\mu_{2n} = E[X - \mu]^{2n} = \int_{-\infty}^{\infty} [X - \mu]^{2n} f(x) d(x)$$

$$X \rightarrow N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$\mu_{2n} = \int_{-\infty}^{\infty} [X - \mu]^{2n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} [X - \mu]^{2n} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} d(x)$$

$$\text{Put } Z = \frac{X - \mu}{\sigma}$$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$$d(x) = \sigma d(z)$$

And Limits remains same  $-\infty \leq Z \leq \infty$

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} [\sigma Z]^{2n} e^{-\frac{1}{2}(Z)^2} \sigma d(Z)$$

$$\mu_{2n} = \frac{[\sigma]^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [Z]^{2n} e^{-\frac{1}{2}(Z)^2} d(Z) \quad (i)$$

$$\text{As } f(z) = Z^{2n} e^{-\frac{1}{2}Z^2}$$

$$\text{Put } Z = -Z$$



$$f(-Z) = f(Z)$$

It is even function then eq. (i) becomes

$$\mu_{2n} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^{\infty} [Z]^{2n} e^{-\frac{1}{2}(Z)^2} d(Z)$$

$$\mu_{2n} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^{\infty} [Z]^{2n} e^{-\frac{1}{2}(Z)^2} d(Z)$$

$$y = \frac{1}{2}Z^2 \quad \sqrt{2y} = Z$$

$$\frac{1}{2}(2y)^{\frac{1}{2}} 2d(y) = d(Z)$$

$$(2y)^{\frac{1}{2}} d(y) = d(Z) \quad \text{And Limits remains same } 0 \leq Z \leq \infty$$

$$\mu_{2n} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^{\infty} [\sqrt{2y}]^{2n} e^{-y} (2y)^{\frac{1}{2}} d(y)$$

$$\mu_{2n} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^{\infty} [2y]^n e^{-y} (2y)^{\frac{1}{2}} d(y)$$

$$\mu_{2n} = \frac{[2]^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} y^{n-\frac{1}{2}} e^{-y} d(y)$$

$$\mu_{2n} = \frac{[2]^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} y^{n+\frac{1}{2}-1} e^{-y} d(y)$$

Comparing by Gamma function then we get

$$\mu_{2n} = \frac{[2]^n \sigma^{2n}}{\sqrt{\pi}} \left( n + \frac{1}{2} \right)$$

$$\mu_{2n} = \frac{[2]^n \sigma^{2n}}{\sqrt{\pi}} \left( n - \frac{1}{2} \right) \left( n - \frac{3}{2} \right) \left( n - \frac{5}{2} \right) \dots \frac{5}{2}, \frac{3}{2}, \frac{1}{2} \left( \frac{1}{2} \right)$$

$$\mu_{2n} = \frac{[2]^n \sigma^{2n}}{\sqrt{\pi}} \left( n - \frac{1}{2} \right) \left( n - \frac{3}{2} \right) \left( n - \frac{5}{2} \right) \dots \frac{5}{2}, \frac{3}{2}, \frac{1}{2} \sqrt{\pi}$$

$$\mu_{2n} = [2]^n \sigma^{2n} \frac{(2n-1)(2n-3)(2n-5) \dots 5,3,1}{2 \times 2 \times 2 \times \dots \times 2}$$

$$\mu_{2n} = [2]^n \sigma^{2n} \frac{(2n-1)(2n-3)(2n-5) \dots 5,3,1}{2^n}$$

$$\mu_{2n} = \sigma^{2n} (2n-1)(2n-3)(2n-5) \dots 5,3,1$$

Multiplying and dividing by  $2n(2n-2)(2n-4) \dots 4,2$

$$\mu_{2n} = \sigma^{2n} \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5) \dots 5,4,3,2,1}{2n(2n-2)(2n-4) \dots 4,2}$$

$$\mu_{2n} = \sigma^{2n} \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5) \dots 5,4,3,2,1}{(2 \times 2 \times \dots \times 2)n(n-1)(n-2) \dots 2,1}$$

$$\mu_{2n} = \sigma^{2n} \frac{(2n)!}{2^n n!} = \left( \frac{\sigma^2}{2} \right)^n \frac{(2n)!}{n!}$$

**Theorem.No.12: Show that Mean deviation of normal distribution is**

$$"Q.D = \frac{2}{3} \sigma = 0.67\sigma"$$

$$\text{Proof: As we know that } Q.D = \frac{Q_3 - Q_1}{2} \quad (A)$$

First we find " $Q_3$  And  $Q_1$ "

Let " $Q_1$ " be a point that contain 25% area below it. Then we get the ordinate from  $\mu$  to

$$Q_1 \text{ is } 0.5 - 0.25 = 0.25$$

By S.N.V

$$Z = \frac{X - \mu}{\sigma}$$

$$P(Z / P = 0.250) = \frac{Q_1 - \mu}{\sigma} \quad \text{Using area table inversely}$$

$$0.67 = \frac{X - \mu}{\sigma} \quad \text{Sign is negative because left sided}$$

$$-0.67\sigma = Q_1 - \mu$$

$$Q_1 = \mu - 0.67\sigma$$

Let “ $Q_3$ ” be a point that contain 75% area below it. Then we get ordinate from  $Q_3$  to  $\mu$  is

$$0.75 - 0.5 = 0.25$$

By S.N.V

$$Z = \frac{X - \mu}{\sigma}$$

$$P(Z / P = 0.250) = \frac{Q_3 - \mu}{\sigma} \quad \text{Using area table inversely}$$

$$0.67 = \frac{Q_3 - \mu}{\sigma} \quad \text{Sign is positive because right sided}$$

$$0.67\sigma = Q_3 - \mu$$

$$Q_3 = \mu + 0.67\sigma$$

Substitute the values in (A) we get

$$Q.D = \frac{\mu + 0.67\sigma - (\mu - 0.67\sigma)}{2}$$

$$Q.D = \frac{\mu + 0.67\sigma - \mu + 0.67\sigma}{2} = \frac{\mu 0.67\sigma + 0.67\sigma}{2} = \frac{2(0.67\sigma)}{2} = 0.67\sigma = \frac{2}{3}\sigma \quad \text{Proved}$$

### Importance of normal distribution

There are some importance of normal distribution are given below

- i) Normal distribution is called probability distribution for errors of measurements.
- ii) It is used in solving problems both in probability and in statistical inference.
- iii) Many natural facts follow normal distribution.
- iv) This distribution helps for drawing conclusions about population on the basis of sample information.
- v) This distribution is used in many other subjects
- vi) Many other distribution derived from normal distribution
- vii) Shape of normal curve determined by its parameters ( $\mu$  and  $\sigma^2$ )
  - a)  $\mu$  Changes the position of normal curve along horizontal axis
  - b)  $\sigma^2$  Indicates horizontal spread
- viii) The graph of normal distribution is bell-shaped and symmetrical
- ix) Normal distribution has two parameters  $\mu$  and  $\sigma^2$ . It is denoted by  $X \rightarrow N(\mu, \sigma^2)$
- X) It is very important in applied statistics

### Binomial approximation to normal

When total number of trials “ $n$  is so much large ( $n \geq 30$ ) and  $p \equiv q = \frac{1}{2}$  binomial

probability tends to a continuous distribution known as normal distribution

$$\text{i.e. } \lim_{n \rightarrow \infty} b(x; n, p) \rightarrow N(\mu, \sigma^2)$$

### Poisson approximation to normal distribution

When  $\mu$  the mean of Poisson distribution is so much large (as it approaches to  $\infty$ ) then

Poisson distribution tends to a continuous distribution known as normal distribution

$$\lim_{t \rightarrow \infty} P(x; \mu) = N(\mu, \sigma^2)$$

### Asymptotic curve

The normal curve approaches but never touches the base line (horizontal axis). So the curve is asymptotic to the horizontal axis as  $X \rightarrow \pm\infty$



$$\frac{1}{12} M.D = \frac{1}{15} \sigma \quad (B)$$

From (A) and (B)

$\frac{1}{10} Q.D = \frac{1}{12} M.D = \frac{1}{15} \sigma$  Hence the Q.D, M.D and S.D are approximately in the ratio 10:12:15 Proved

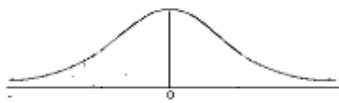
## Short Question

Q.1: Give the background of normal distribution?

Ans: Normal distribution is also called “Gaussian distribution in the honor of Karl F Gauss who derived its equation. Karl Pearson named it normal distribution, in 1893. It was discovered by “Abraham de moivre (1667-1754) as a limiting form of binomial distribution, when  $n \geq 30$  and  $p \equiv q$

Q.2: What is the shape of normal curve?

Ans: The shape of normal curve is uni-modal, symmetrical and bell shaped as shown in the figure.



Q.3: What is the role of mean”  $\mu$ ” and standard deviation “ $\sigma$ ” in the normal curve?

Ans: Mean”  $\mu$ ” and variance “ $\sigma^2$ ” are the parameters of normal distribution. Where  $\mu = Mean$  is known as shape parameters and  $\sigma$  is known as scale parameters.

Q.4: When normal distribution changes into standard normal distribution?

Ans: When the normal random variable “X” is expressed in terms of deviations from mean in units of standard deviation the normal distribution changes into standard normal distribution.

Q.5: Define points of inflection in a normal distribution.

Ans: Those points at which concavity of curve changes are called points of inflection.

There are two points of inflection which are equidistant from mean. The two points are

given as  $\left( \mu - \sigma, \frac{1}{\sigma\sqrt{2\pi e}} \right)$  and  $\left( \mu + \sigma, \frac{1}{\sigma\sqrt{2\pi e}} \right)$

Q.6: Explain why odd order moments about mean equal to zero for the normal distribution.

Ans: In normal curve, if nth moment is odd, the value of odd moment will always equal to zero. This is because the normal curve is symmetrical and for symmetrical distribution sum of the positive deviations from  $\mu$  will always equal to sum of the negative deviations from  $\mu$  and these deviations are cancel out to each other.