SHORT REPORT

ScienceAsia 37 (2011): **79–82**

doi: 10.2306/scienceasia1513-1874.2011.37.079

Extended cubic uniform B-spline for a class of singular boundary value problems

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Received 23 Aug 2010
Accepted 13 Jan 2011

ABSTRACT: B-splines have been widely used to approximate solutions to differential equations. In this paper, a class of singular boundary value problems are treated by using extended cubic uniform B-spline approximations. The advantage of using an extended cubic B-spline rather the ordinary B-spline is that it introduces one additional free parameter. For a number of examples where exact solutions are known, the solutions obtained using the extended B-splines are found to be better approximations than those obtained using ordinary B-splines.

KEYWORDS: differential equation, free parameter

INTRODUCTION

Consider the homogeneous second order linear differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

If the functions $p(x)$ and $q(x)$ are both analytic at a point $x = x_0$, then the point x_0 is said to be an ordinary point. Otherwise, x_0 is a singular point. A singular boundary value problem occurs when a differential equation in a boundary value problem has a singular point at one boundary[1]. Such problems frequently arise in areas such as thermal explosions, electrohydrodynamics, chemical reactions, and atomic and nuclear physics[2–4].

Russell and Shampine[5] have discussed a classical three-point finite difference scheme for solving singular boundary value problems which gives good approximation solutions with a moderate step size. Ravi Kanth and Reddy have described a fourth-order finite difference method[6] and also the cubic approximation[7] for a class of singular boundary problems. Both methods can produce good results but at the cost of a high order of finite differencing. For solving the same problems, third-degree B-splines also approximate the exact solutions well[8]. Kumar[9] concluded that splines are a simpler and more practical way to solve singular boundary problems than finite difference methods. This provides the motivation for our work on using extended cubic uniform B-splines for solving singular boundary value problems. The advantage of using extended B-spline is that it possesses a free parameter, λ, to control the global shape parameter.

The series expansion procedure is a popular approach to remove the singularity at the singular point. Ravi Kanth and Reddy[10] used the Chebyshev economization near the singularity on $(0, \delta)$ and solved the regular boundary value problem in the interval $(\delta, 1)$ by employing the stable central difference method. Here, a simple and direct method is applied to evaluate the limits involving the singularity. By applying l'Hôpital's rule, the original differential equation is modified at the singular point. After the modification at the singular point, the boundary value problem is solved by using the extended cubic uniform B-spline.

In this paper, after defining the extended cubic uniform B-spline, we describe the numerical method for solving singular boundary value problems. The efficiency of the method is demonstrated using both homogeneous and non-homogeneous singular boundary problems.

EXTENDED CUBIC UNIFORM B-SPLINES

Definition 1 The blending function of the extended cubic uniform B-spline with degree 4, $E_i(x)$, is given by[11]

$$E_i = \frac{1}{24h^4}\begin{cases} 4h(1-\lambda)z_i^3 + 3\lambda z_i^4, & x \in I_i,\\ (4-\lambda)h^4 + 12h^3 z_{i+1} + 6h^2(2+\lambda)z_{i+1}^2 \\ \quad -12hz_{i+1}^3 - 3\lambda z_{i+1}^4, & x \in I_{i+1},\\ (4-\lambda)h^4 - 12h^3 z_{i+3} + 6h^2(2+\lambda)z_{i+3}^2 \\ \quad +12hz_{i+3}^3 - 3\lambda z_{i+3}^4, & x \in I_{i+2},\\ 4h(\lambda-1)z_{i+4}^3 + 3\lambda z_{i+4}^4, & x \in I_{i+3}, \end{cases} \tag{1}$$

Table 1 Values of E_i, E_i' and E_i''.

	x_{i+1}	x_{i+2}	x_{i+3}
E_i	$(4-\lambda)/24$	$(8+\lambda)/12$	$(4-\lambda)/24$
E_i'	$1/2h$	0	$-1/2h$
E_i''	$(2+\lambda)/2h^2$	$-(2+\lambda)/h^2$	$(2+\lambda)/2h^2$

where $z_i = x - x_i$, $I_j \equiv [x_j, x_{j+1}]$, and the parameter λ satisfies $-8 \leqslant \lambda \leqslant 1$.

To obtain the approximations of the solutions, the values of $E_i(x)$ and its derivatives at the knots are needed and these are given in Table 1. Values at other knots are zero.

Note that when $\lambda = 0$, the basis function reduces to that of the cubic uniform B-spline. Also, as with the B-spline, the extended cubic uniform B-spline possesses the convex hull property, symmetry, and geometric invariability[11].

NUMERICAL METHOD

Assume that the singular two-point boundary value problem is in the form of

$$y''(x) + \frac{k}{x} y'(x) + r(x)y(x) = f(x), \quad 0 < x < 1, \tag{2a}$$

$$y'(0) = 0, \quad y(1) = \beta, \tag{2b}$$

where the parameter $k \geqslant 1$. Due to the singularity at $x = 0$, the boundary value problem is modified at the singular point, then transformed into the following form by using l'Hôpital's rule[6,8]:

$$\begin{aligned} (k+1)y''(x) + r(0)y(x) &= f(0), \quad \text{for } x = 0, \\ y''(x) + \frac{k}{x}y'(x) + r(x)y(x) &= f(x), \quad \text{for } x \neq 0. \end{aligned} \tag{3}$$

Suppose the domain $[a, b]$ of a curve is divided by the knots x_i into n segments $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$ where $x_i = a + ih$, and $h = (b-a)/n$. Then the approximate solution of (2a) is[12]

$$S(x) = \sum_{i=-3}^{n-1} C_i E_i(x) \tag{4}$$

where C_i are the unknown real coefficients and $E_i(x)$ are the basis function of the extended cubic uniform B-spline. In order to obtain the approximations of (2) at the point $x = x_i$, we substitute (4) into (2a). This gives

$$S''(x) + \frac{k}{x} S'(x) + r(x)S(x) = f(x) \tag{5}$$

which can be rewritten as

$$\sum_{i=-3}^{n-1} C_i E_i''(x) + \frac{k}{x} \sum_{i=-3}^{n-1} C_i E_i'(x) + r(x) \sum_{i=-3}^{n-1} C_i E_i(x) = f(x), \qquad x = 0, h, 2h, \dots, 1. \tag{6}$$

A linear system of order $(n+1)$ is obtained. However, two additional linear equations are needed to obtain the values of $n+3$ variables. Thus (4) is applied in the boundary conditions (2b) to obtain

$$\begin{aligned} \sum_{i=-3}^{n-1} C_i E_i'(x) &= 0 \quad \text{for } x = 0, \\ \sum_{i=-3}^{n-1} C_i E_i(x) &= \beta \quad \text{for } x = 1. \end{aligned} \tag{7}$$

Equations (6) and (7) lead to a tridiagonal matrix system which can be written as

$$AC = B \tag{8}$$

where

$$A = \begin{bmatrix} -12h & 0 & 12h & 0 & \dots & & 0 \\ \alpha_1 & \alpha_2 & \alpha_1 & 0 & \dots & & 0 \\ 0 & \gamma_1 & \gamma_2 & \gamma_3 & 0 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & \dots & & 0 & \alpha_3 & \alpha_4 & \alpha_3 \end{bmatrix},$$

$$C = \begin{bmatrix} C_{-3} \\ C_{-2} \\ \vdots \\ \vdots \\ C_{n-2} \\ C_{n-1} \end{bmatrix}, \quad B = 24h^2 \begin{bmatrix} 0 \\ f(x_0) \\ \vdots \\ \vdots \\ f(x_n) \\ \beta \end{bmatrix}$$

and

$$\begin{aligned} \alpha_1 &= 12(k+1)(2+\lambda) + r(0)(4-\lambda)h^2, \\ \alpha_2 &= -24(k+1)(2+\lambda) + 2r(0)(8+\lambda)h^2, \\ \alpha_3 &= h^2(4-\lambda), \\ \alpha_4 &= 2h^2(8+\lambda), \\ \gamma_1 &= 12(2+\lambda) + \frac{k}{x}(-12h) + r(x)(4-\lambda)h^2, \\ \gamma_2 &= -24(2+\lambda) + 2r(x)(8+\lambda)h^2, \\ \gamma_3 &= 12(2+\lambda) + \frac{k}{x}(12h) + r(x)(4-\lambda)h^2 \end{aligned}$$

Table 2 Comparison of error norms for cubic B-Spline (CuBS) and extended cubic B-Spline (ExCuBS).

h	CuBS		ExCuBS	
	L_∞ Norm	L_2 Norm	L_∞ Norm	L_2 Norm
0.1	1.1×10^{-4}	2.7×10^{-4}	1.3×10^{-5}	3.0×10^{-5}
0.05	2.8×10^{-5}	9.2×10^{-5}	1.2×10^{-7}	4.6×10^{-7}
0.02	4.5×10^{-6}	2.3×10^{-5}	9.8×10^{-9}	4.3×10^{-8}

Equation (8) can be solved using the Thomas algorithm[13] to obtain C_i in terms of λ. Finally, the approximate solution can be found easily after getting the appropriate λ value by optimization[14].

NUMERICAL RESULTS

In this section, a class of singular boundary value problem which are discussed widely in the literature[6–8] are solved by applying the extended cubic uniform B-spline. The accuracy of the method can be tested by calculating the error norms

$$L_\infty = \max_i |y_i - S_i|, \quad L_2 = \sqrt{\sum_{i=1}^{N} (y_i - S_i)^2}$$

where y and S denote the exact and approximate solutions, respectively.

Example 1 Consider Bessel's equation of order zero

$$y''(x) + \frac{1}{x} y'(x) + y(x) = 0,$$
$$y'(0) = 0, \quad y(1) = 1.$$

The solutions can be approximated by applying (6) and (7). The exact solution for the problem is $y(x) = J_0(x)/J_0(1)$. The computational errors, L_∞ norm, and L_2 norm for different values of step size, h, are given in Table 2. It can be seen that for the extended cubic B-spline, different values of λ are obtained for different values of h.

Example 2 The exact solution of

$$y''(x) + \frac{2}{x} y'(x) - 4y(x) = -2, \quad 0 < x \leqslant 1,$$
$$y'(0) = 0, \quad y(1) = 5.5,$$

is

$$y(x) = 0.5 + \frac{5 \sinh 2x}{x \sinh 2}.$$

The absolute errors are tabulated in Table 3. The error for the cubic B-spline is -2.97×10^{-4} at $x = 0$ and decreases monotonically to zero at $x = 1$. The errors for the extended cubic B-spline are much smaller than that obtained for the cubic B-spline.

Table 3 Absolute errors for extended B-Spline (ExCuBS) and B-Spline (CuBS) compared with the analytical solutions ($h = 0.05$).

x_i	Exact	CuBS ($\lambda = 0$)	ExCuBS ($\lambda = 0.00105$)
0	3.26	3.0×10^{-4}	9.1×10^{-7}
0.1	3.28	3.0×10^{-4}	1.1×10^{-6}
0.2	3.33	2.9×10^{-4}	1.6×10^{-6}
0.5	3.74	2.6×10^{-4}	4.1×10^{-6}
0.9	5.01	9.0×10^{-5}	3.4×10^{-6}

Table 4 Computational errors for extended B-Spline (ExCuBS) compared with the B-Spline approximations (CuBS) and the analytical solutions ($h = 0.05$).

x_i	Exact	CuBS ($\lambda = 0$)	ExCuBS ($\lambda = 0.00041$)
0.00	-0.267	-2.7×10^{-5}	1.2×10^{-6}
0.05	-0.266	-2.7×10^{-5}	1.2×10^{-6}
0.10	-0.265	-2.7×10^{-5}	1.2×10^{-6}
0.20	-0.257	-2.6×10^{-5}	1.0×10^{-6}
0.50	-0.204	-2.2×10^{-5}	-1.0×10^{-7}

Example 3 The exact solution of

$$y''(x) + \frac{1}{x} y'(x) = \left(\frac{8}{8 - x^2}\right)^2,$$
$$y'(0) = 0, \quad y(1) = 0,$$

is

$$y(x) = 2 \log\left(\frac{7}{8 - x^2}\right).$$

Table 4 presents the exact solutions and the corresponding errors. Again, the extended B-splines give the more accurate approximations.

CONCLUSIONS

In this paper, the extension of cubic uniform B-spline with blending function of degree 4 has been used to solve a family of two-point singular boundary value problems. With the flexibility of extensions, the approximations of the solution can be done by adjusting the free parameter, λ. The numerical results showed that extended cubic B-spline approximates the exact solution of the singular two-point boundary value problems considered very well.

Acknowledgements: The authors gratefully acknowledge the financial support from University Sciences Malaysia and thank the School of Mathematical Sciences for the use of its facilities.

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