Chaos, Solitons and Fractals 42 (2009) 2934–2948

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

B-spline solution of a singularly perturbed boundary value problem arising in biology ☆

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ARTICLE INFO

Article history:
Accepted 1 April 2009

ABSTRACT

We use B-spline functions to develop a numerical method for solving a singularly perturbed boundary value problem associated with biology science. We use B-spline collocation method, which leads to a tridiagonal linear system. The accuracy of the proposed method is demonstrated by test problems. The numerical result is found in good agreement with exact solution.

1. Introduction

Any function of $L^2(R)$ can be expressed by the dilation and translation of wavelet functions, so it has drawn a great deal of attention from scientists and engineers. The stiffness matrix is sparse when it is used as trial functions. Wavelet especially adapt to solve the equation with the singular solution and a local severe gradients. Wavelets have many excellent properties such as orthogonality, compact support, exact representation of polynomials to a certain degree, flexibility to represent functions at different levels of resolution.

B-spline functions are useful wavelet basis functions and based on piece polynomials that possess attractive properties: piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination, which leads to matrices that are easier to diagonalize. The resulting matrices are sparse, but always, banded. B-splines were introduced by Schoenberg in 1946 [9]. Up to now, B-spline approximation method for numerical solutions have been researched by various researchers [1–7].

We consider a B-spline collocation method for following singularly perturbed boundary value problems arising in biology:

$$Ly(x) = -\varepsilon y'' + p(x)y' + q(x)y = f(x), \quad a < x < b \tag{1}$$

With boundary conditions

$$y(a) = A, \quad y(b) = B \tag{2}$$

where $0 < \varepsilon < 1$, ε is a small positive parameter, $p(x)$ and $q(x)$ are sufficiently smooth real-valued functions. This problem arising in transport phenomena in chemistry and biology [10,11] has been studied by several authors [8,12–14]. It is so attractive to mathematicians due to the fact that the solution exhibits a multiscale character, that is, there is a thin layer where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. So the usual numerical treatment of singular perturbation problems gives major computational difficulties. Typically, these problems arise very frequently in fluid dynamics, elasticity, quantum mechanics, chemical reactor theory and many other allied areas. In recent years, a large number of special purpose methods have been developed to provide accurate numerical solutions.

☆ Supported by the Natural Science Foundation of China (Nos. 10771167 and 10571142).
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doi:10.1016/j.chaos.2009.04.036

In this paper, the 3rd order B-spline function is used as a single mother wavelet and the numerical method based on B-splines is studied to solve a class of singular boundary value problems arising in biology. We use B-spline collocation method, which leads to a tridiagonal linear system. The present method is tested for its efficiency by considering two examples from biology.

2. Spline scaling function

An arbitrary Nth order spline function with compact support of N, a useful wavelet basis function, has excellent mathematical properties. It is a concatenation of N sections of $(N-1)$th order polynomials, continuous at the junctions or 'knots', and gives continuous $(N-1)$th derivatives at the junctions. The expression and recursion formula of B-spline function are as follows:

$$N_1(x) = \chi_{[0,1]}(x) = \begin{cases} 1, & 0 \leqslant x < 1 \\ 0, & \text{others} \end{cases} \tag{3}$$

$$N_2(x) = N_1 * N_1(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ 2-x & 1 \leqslant x < 2 \\ 0, & \text{others} \end{cases} \tag{4}$$

$$N_3(x) = \int_0^1 N_2(x-\tau)d\tau = \begin{cases} \frac{1}{2}x^2 & [0,1) \\ \frac{3}{4} - \left(x - \frac{3}{2}\right)^2 & [1,2) \\ \frac{1}{2}(x-3) & [2,3) \\ 0 & \text{others} \end{cases} \tag{5}$$

$$N_4(x) = \int_0^1 N_3(x-\tau)d\tau = \begin{cases} \frac{1}{6}x^3 & [0,1) \\ -\frac{1}{2}x^3 + 2x^2 - 2x + \frac{2}{3} & [1,2) \\ \frac{1}{2}x^3 - 4x^2 + 10x - \frac{22}{3} & [2,3) \\ -\frac{1}{6}x^3 + 2x^2 - 8x + \frac{32}{3} & [3,4) \\ 0 & \text{others} \end{cases} \tag{6}$$

$$N_m(x) = N_{m-1} * N_1(x) = \int_0^1 N_{m-1}(x-\tau)d\tau = \frac{x}{m-1}N_{m-1}(x) + \frac{m-x}{m-1}N_{m-1}(x-1) \qquad m \geqslant 2 \tag{7}$$

The 3rd order B-spline function $N_4(x)$ is usually used to calculate in practice, which is easy and efficient, possesses the following characters: piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination.

3. Wavelet theory

Preliminaries: Definition (multi-resolution): A sequence $\{V_j\}_{j\in Z}$ of closed subspaces of $L^2(R)$ is a multi-resolution approximation if the following properties are satisfied:

$$\begin{aligned} &\forall (j,k) \in Z^2 f(t) \in V_j \Longleftrightarrow f(t - 2^j k) \in V_j \\ &V_j \subset V_{j+1} \\ &f(t) \in V_j \Longleftrightarrow f(2t) \in V_{j+1} \\ &\lim_{j\to-\infty} V_j = \cap_{j=-\infty}^{+\infty} V_j = \{0\}, \\ &\overline{\cup V_j} = L^2(R) \end{aligned}$$

There exists ϕ such that $\{\phi(t-n)\}_{n\in z}$ is a Riesz basis of V_0.

Supposing that one-dimensional scaling functions $\phi(x)$ generate multi-resolution analyses $\{V_j\}$, The span of V_j is $\{2^{j/2}\varphi(2^j x - k)\}$. One-dimensional functions in the form is given

$$f_{j,k}(x) = 2^{j/2} f(2^j x - k) \tag{8}$$

From the wavelet theory, the approximate formulas of solution can be expressed in terms of the scaling function basis ϕ_k^J at scale J as

$$S(x) = \sum_k c_k \phi_k^J(x) \tag{9}$$

where $c_k = < S(x), \phi_k^J(x) > = \int S(x)\phi_k^J(x)dx$, $\phi_k^J(x) = 2^{J/2}\phi(2^J x - k)$, J is the scale parameter or the resolution, k is the parameter of the time or space location ($J,k \in Z$,integers) and it can be determined from the following two scale relations:

$$\phi(x) = \sum_{k=0}^{L-1} p_k \phi(2x - k) \tag{10}$$

where L is an even integer.

In the proposed algorithm, The 3rd order B-spline function $N_4(x)$ is used as a single mother wavelet, i.e., $\phi(x) = N_4(x)$ and dilation and translation of mother wavelet functions can construct any function of $L^2(R)$.

$$S(x) = \sum_k c_k \phi_k^J(x) = \sum_k c_k 2^{J/2} \phi(2^J x - k) = \sum_k a_k \phi\left(\frac{x - 2^{-J}k}{2^{-J}}\right) = \sum_k a_k N_4\left(\frac{x - x_k}{h}\right) \tag{11}$$

where $h = \frac{1}{2^J} x_k = \frac{k}{2^J} a_k = 2^{J/2} c_k$.

The 3rd order B-splines function is used to construct numerical solutions to singular boundary-value problems discussed in Section 4.

4. B-spline solutions for singular boundary value problems arising in biology

The region $[a,b]$ is partitioned into uniformly sized finite elements of length h by the knots x_j such that $a = x_0 < x_1 < x_2 < \cdots < x_N = b$. Let $\phi_m(x)$ be 3rd order B-spline function with knots at the points x_m, $m = 0, 1, \ldots, N$. The set of splines $\{\phi_{-1}, \phi_0, \phi_1, \cdots \phi_N, \phi_{N+1}\}$ forms a basis for functions defined over $[a,b]$.

So the global approximation $S(x)$ to the function $y(x)$ can be written in terms of the B-splines as

$$S(x) = \sum_{i=-1}^{N+1} a_i N_4\left(\frac{x - x_i}{h}\right), \tag{12}$$

where $h = \frac{b-a}{n}$, a_i are unknown real coefficients.

Using the 3rd order B-spline function Eq. (6) and the approximate solution Eq. (11), the nodal values $S(x_j)$,$S'(x_j)$ and $S''(x_j)$ at the node x_j are given in terms of element parameters by

$$S(x_j) = \frac{1}{6}(a_{j-1} + 4a_j + a_{j+1}) \tag{13}$$

$$S'(x_j) = \frac{1}{2h}(-a_{j-1} + a_{j+1}) \tag{14}$$

$$S''(x_j) = \frac{1}{h^2}(a_{j-1} - 2a_j + a_{j+1}) \tag{15}$$

where the symbols′ and ″ denote first and second differentiation with respect to x, respectively.

Substituting Eqs. (12)–(15) into Eqs. (1) and (2), we can obtain following linear equations

$$Ba = r \tag{16}$$

where

$$a = (a_{-1}, a_0, a_1, \cdots, a_N, a_{N+1})^T$$
$$r = \left(6y(a), 6h^2 f_0/\varepsilon, 6h^2 f_1/\varepsilon, \cdots 6h^2 f_N/\varepsilon, 6h^2 y(b)\right)^T$$
$$f_i = f(a + ih)$$

Note $N_4\left(\frac{x_j - x_i}{h}\right) = B_{ij}$

$$B = \begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ LB_{-1,0} & LB_{0,0} & LB_{1,0} & 0 & \cdots & \\ 0 & LB_{0,1} & LB_{1,1} & LB_{2,1} & \cdots & \\ \cdots & \cdots & \cdots & & & \\ & & & LB_{N-1,N} & LB_{N,N} & LB_{N+1,N} \\ & & & 1 & 4 & 1 \end{bmatrix}_{(N+3)*(N+3)}$$

where

$$LB_{j-1,j} = 6 - 3hp_j/\varepsilon + h^2 q_j/\varepsilon \quad LB_{j,j} = 4h^2 q_j/\varepsilon - 12, \quad LB_{j,j+1} = 6 + 3hp_j/\varepsilon + h^2 q_j/\varepsilon \quad p_j = p(a + jh) \quad q_j = q(a + jh)$$

It is easily seen that the matrix B is strictly diagonally dominant and hence nonsingular. Since B is nonsingular, we can solve the system $Ba = r$ for $a_{-1}, a_0, a_1, \cdots, a_N, a_{N+1}$. Hence the method of collocation using the 3rd order B-spline function $N_4(x)$ as a basis function applied to the singularly perturbed boundary value problem has a unique solution $S(x)$ given by Eq. (12).

5. Computation of error and order of convergence

The relative error of numerical solution is defined as

$$E^r = \frac{\sqrt{\sum_{i=1}^{N}(S(x_i) - y(x_i))^2}}{\sqrt{\sum_{i=1}^{N}(y(x_i))^2}} \tag{17}$$

The pointwise errors are given by

$$E(x_i) = |S(x_i) - y(x_i)| \tag{18}$$

For every ε the computed maximum pointwise errors are given by

$$E^N = \max_{0 \leqslant i \leqslant N} |y(x_i) - S(x_i)| \tag{19}$$

The numerical order of convergence is given by

$$Ord^N = \frac{Log(E^N/E^{2N})}{Log2} \tag{20}$$

The uniform maximum pointwise errors is given by

$$\eta^N = \max_{\varepsilon=1,10^{-1},\cdots 10^{-12}, p=1,10^{-1},\cdots 10^{-12},0} \|S(x) - y(x)\|_\infty$$

6. Numerical results and conclusion

In the section, we illustrate the numerical techniques discussed in the previous section by the following problems.

Example 1. Consider the convection-dominated equation:

$$\begin{aligned} &-\varepsilon y'' + y' + y = 1 \qquad (0 < x < 1) \\ &y(0) = y(1) = 0 \end{aligned} \tag{21}$$

The exact solution is given by

$$y(x) = (e^{\lambda_2} - 1)e^{\lambda_1 x}/(e^{\lambda_1} - e^{\lambda_2}) + (1 - e^{\lambda_1})e^{\lambda_2 x}/(e^{\lambda_1} - e^{\lambda_2}) + 1 \tag{22}$$

where $\lambda_1 = \left(1 + \sqrt{1 + 4\varepsilon}\right)/(2\varepsilon), \lambda_2 = \left(1 - \sqrt{1 + 4\varepsilon}\right)/(2\varepsilon)$.

Comparison of the numerical results and pointwise errors is given in Table 3. Comparison of errors is given in Tables 1 and 2. Comparison of exact solution and approximation solution for different values of ε is given in Figs. 1–3. Relation of relative errors and values of ε and h is seen in Figs. 4 and 5. Relation of maximum pointwise errors and values of ε and h is given in Figs. 6 and 7. Relation of Order of convergence and values of ε and h is given in Figs. 8 and 9.

It observed that

(1) when h decreases (i.e. collocation number increases) for fixed ε the pointwise errors decrease;
(2) when ε from 0.8 to 0.01 decreases for fixed h the pointwise errors increase;
(3) when ε from 0.01 to 0.0015 decreases for fixed h the pointwise errors are almost steady;
(4) when $\varepsilon = 0.0015$, $x \to 1$ the errors are very large;
(5) when h increase for fixed ε, relative errors decreases, but maximum pointwise errors increase;
(6) when ε increase for fixed h, relative errors and maximum pointwise errors decreases;
(7) when ε increase for fixed h, order of convergence decreases rapidly afterward decreases slowly to 0;
(8) when h increase for fixed ε, order of convergence increases afterward decreases slowly.

Example 2. Solve the following non-homogeneous equation:

$$-\varepsilon y'' + py' + y = \cos \pi x \ (0 < x < 1). \tag{23}$$

Table 1
Comparison of pointwise errors of Example 1.

X	Error			
	(ε = 0.1, h = 1/32)	(ε = 0.1, h = 1/128)	(ε = 0.01, h = 1/32)	(ε = 0.01, h = 1/128)
1/16	0.0274	0.0068	0.0295	0.0073
2/16	0.0259	0.0064	0.0278	0.0069
4/16	0.0230	0.0057	0.0245	0.0061
6/16	0.0204	0.0050	0.0217	0.0054
12/16	0.0025	0.0004	0.0150	0.0037
14/16	0.0330	0.0094	0.0129	0.0033

Table 2
Comparison of pointwise errors of Example 1.

X	Error							
	$h = 1/128$							$h = 1/1024$
	$\varepsilon = 0.8$	$\varepsilon = 0.5$	$\varepsilon = 0.2$	$\varepsilon = 0.01$	$\varepsilon = 0.008$	$\varepsilon = 0.002$	$\varepsilon = 0.0015$	$\varepsilon = 0.0015$
1/16	0.0033	0.0044	0.0062	0.0073	0.0073	0.0074	0.0074	0.0009
2/16	0.0029	0.0040	0.0059	0.0069	0.0069	0.0069	0.0069	0.0009
4/16	0.0022	0.0031	0.0051	0.0061	0.0061	0.0061	0.0061	0.0008
6/16	0.0014	0.0022	0.0042	0.0054	0.0054	0.0054	0.0054	0.0007
12/16	0.0019	0.0025	0.0025	0.0037	0.0037	0.0037	0.0037	0.0005
14/16	0.0034	0.0051	0.0092	0.0033	0.0033	0.0033	0.0033	0.0004

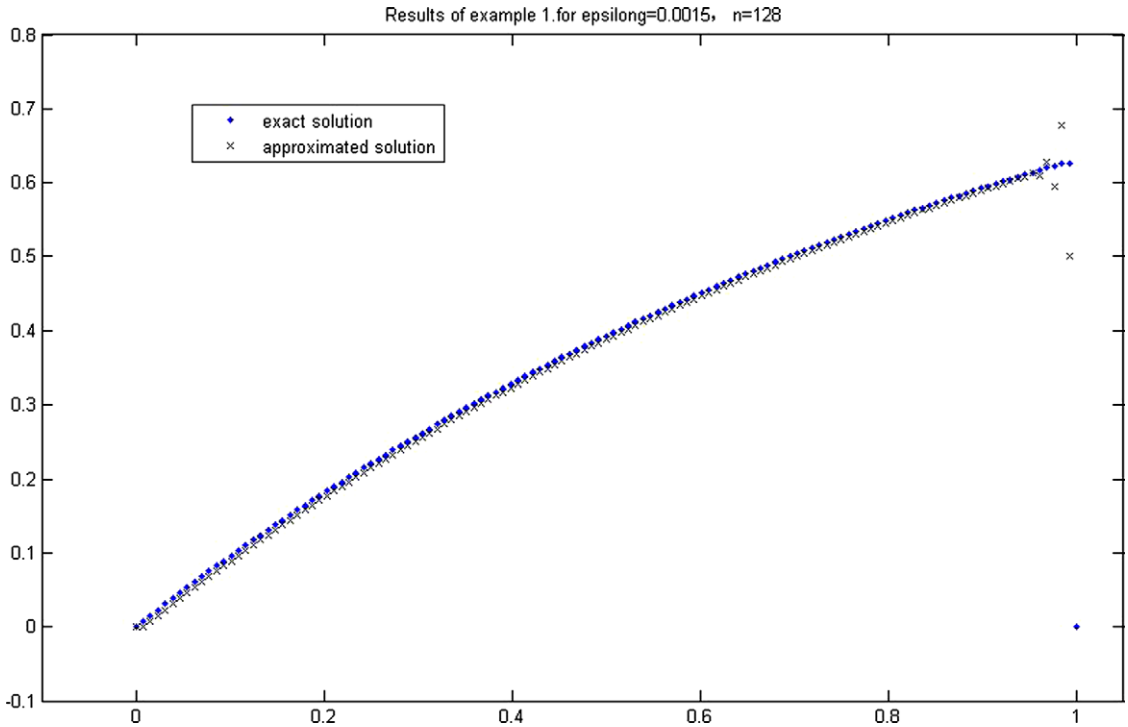

Fig. 1. Results of Problem 1 for $\varepsilon = 0.0015$, $n = 128$.

With boundary conditions

$$y(0) = y(1) = 0$$

The analytical solution is given by

$$y(x) = a\cos\pi x + b\sin\pi x + A\exp(\lambda_1 x) + B\exp[-\lambda_2(1-x)] \tag{24}$$

where

$$a = \frac{\varepsilon\pi^2 + 1}{p^2\pi^2 + (\varepsilon\pi^2+1)^2}, \quad b = \frac{p\pi}{p^2\pi^2 + (\varepsilon\pi^2+1)^2}$$
$$A = -a\frac{1+\exp(-\lambda_2)}{1-\exp(\lambda_1-\lambda_2)}, \quad B = a\frac{1+\exp(\lambda_1)}{1-\exp(\lambda_1-\lambda_2)}$$

And $\lambda_1 < 0$ and $\lambda_2 > 0$ are the real solutions of the characteristic equation

$$-\varepsilon\lambda^2 + p\lambda + 1 = 0$$

Comparison of exact solution and approximation solution for different values of ε and p is given in Figs. 10–12. Approximation solutions for different values of ε and for fix p is given in Fig. 13. Relation of relative errors and values of ε, p and h is seen in Figs. 14–16. Relation of maximum pointwise errors and values of ε, p and h is given in Figs. 17–19. The uniform maximum pointwise errors is given in Fig. 20.

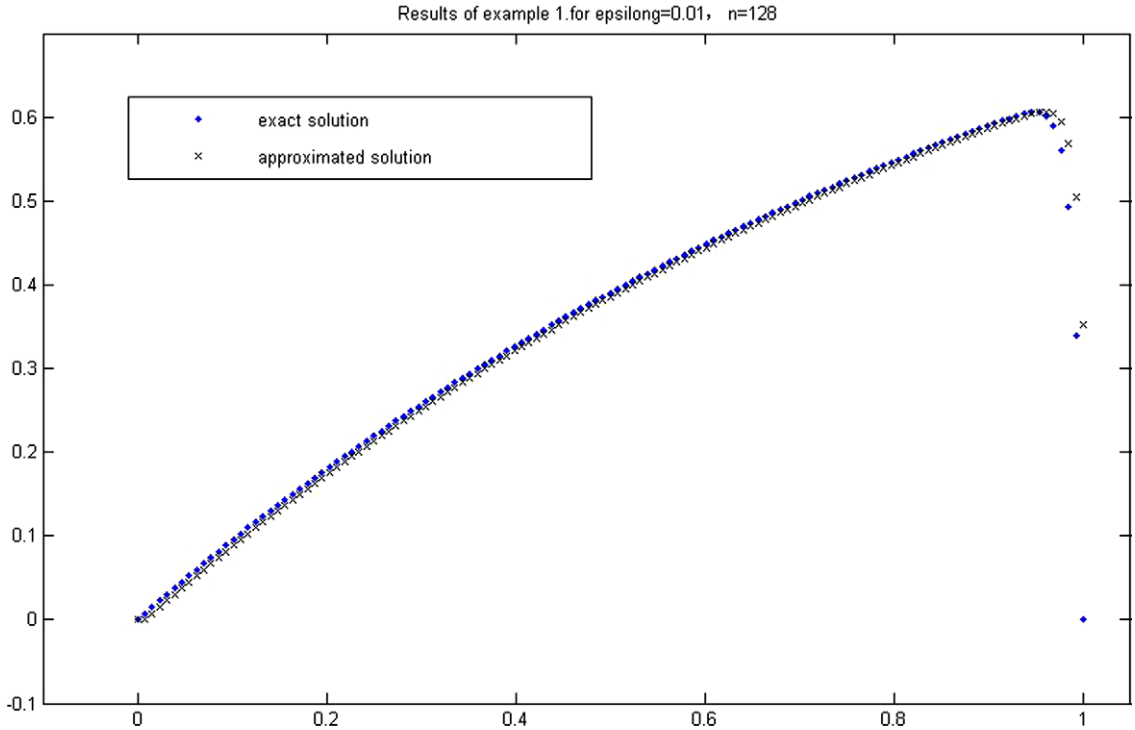

Fig. 2. Results of Problem 1 for $\varepsilon = 0.01$, $n = 128$.

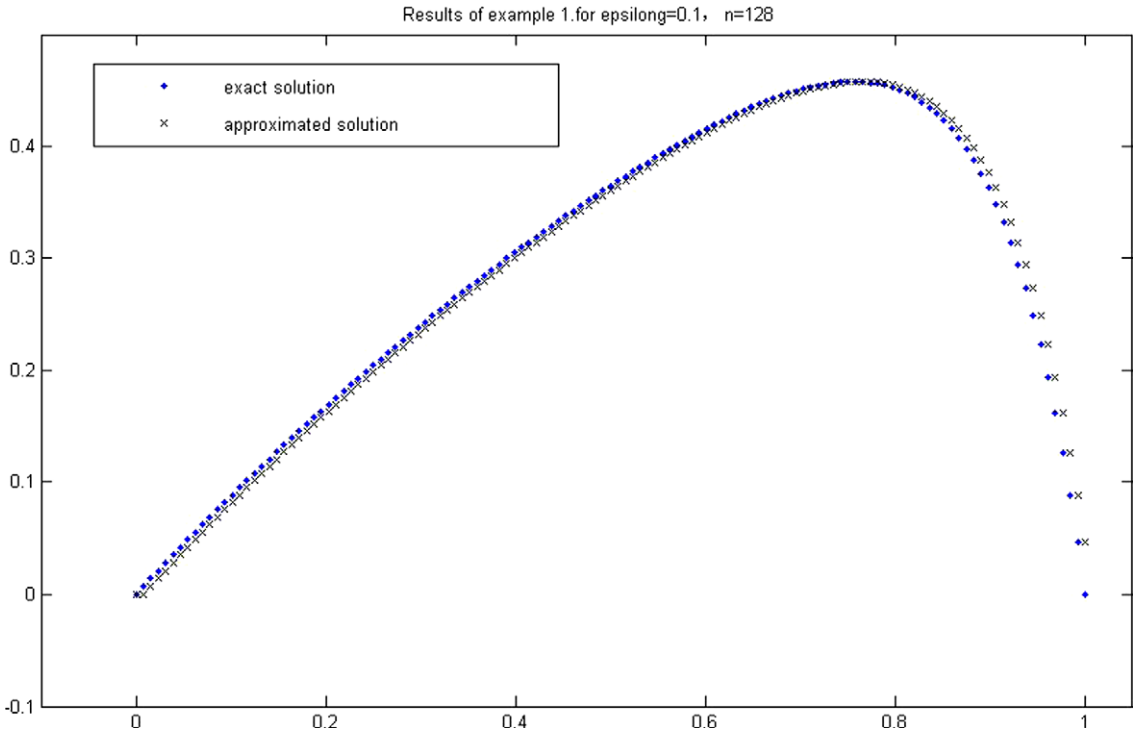

Fig. 3. Results of Problem 1 for $\varepsilon = 0.1$, $n = 128$.

It observed that

(1) the approximation solutions are in good agreement with exact solution(Figs. 12 and 13);
(2) when $\varepsilon = 10^{-6}$, $x \to 0$ and $x \to 1$ the errors are very large (Fig. 14);

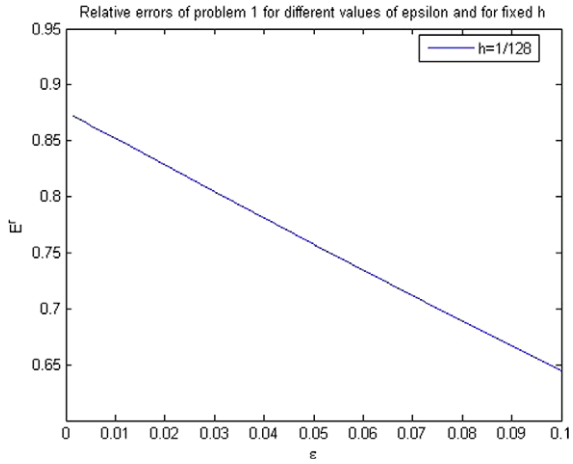

Fig. 4. Relative errors of Problem 1 for different values of ε and for fix h.

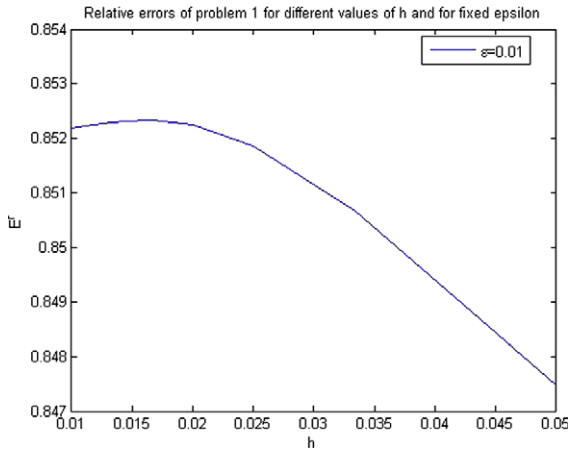

Fig. 5. Relative errors of Problem 1 for different values of h and for fix ε.

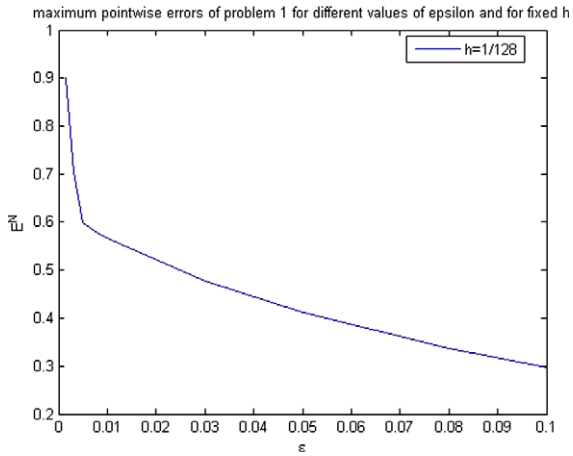

Fig. 6. Maximum pointwise errors of Problem 1 for different values of ε and for fix h.

(3) when ε decreases for fixed p the width of boundary layer becomes small and wave shape change more and more stiff at $x = 0$ and $x = 1$;
(4) when ε increase for fixed h and p, relative errors and maximum pointwise errors increase;

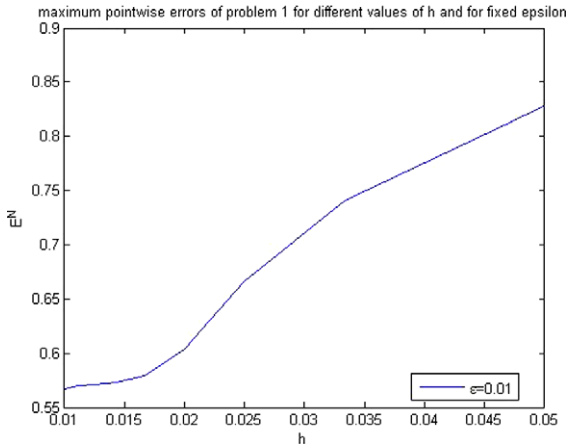

Fig. 7. Maximum pointwise errors of Problem 1 for different values of h and for fix ε.

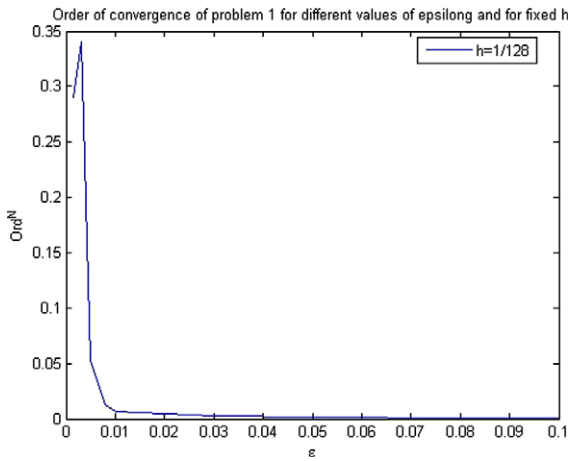

Fig. 8. Order of convergence of Problem 1 for different values of ε and for fix h.

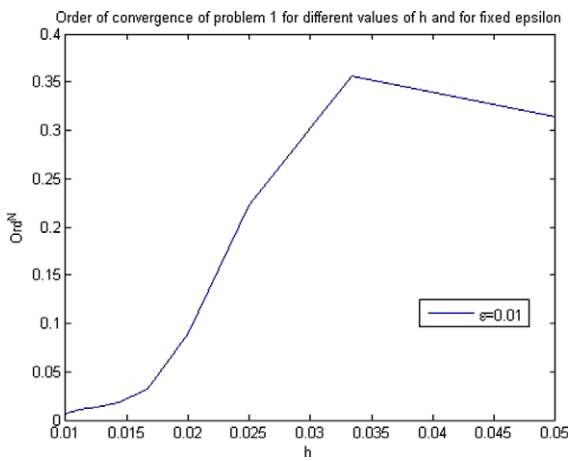

Fig. 9. Order of convergence of Problem 1 for different values of h and for fix ε.

(5) when h increase for fixed ε and p, relative errors increase and maximum pointwise errors increase rapidly afterward decreases slowly;
(6) when p increase for fixed ε and h, relative errors decreases slowly to $p = 10^{-15}$ afterward decreases rapidly and maximum pointwise errors decreases slowly.

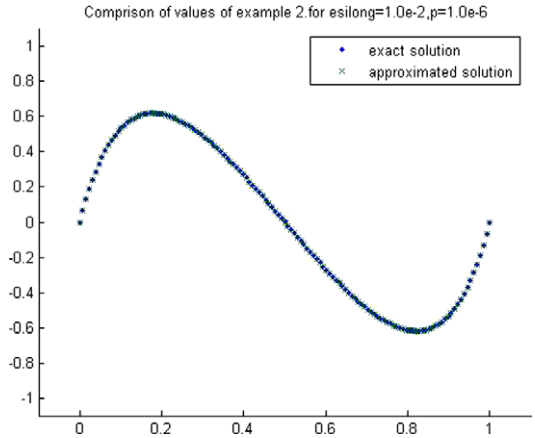

Fig. 10. Comparison of values of Problem 2 for $\varepsilon = 10^{-2}$, $p = 10^{-6}$.

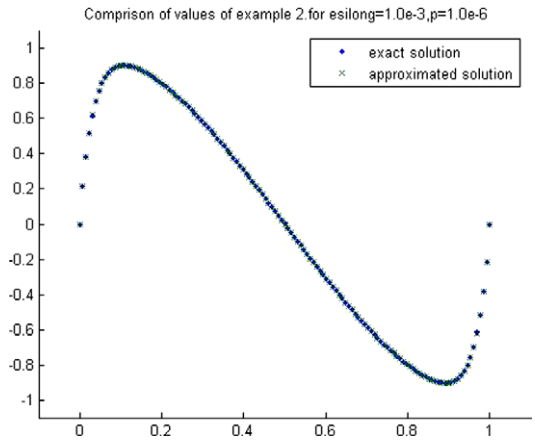

Fig. 11. Comparison of values of Problem 2 for $\varepsilon = 10^{-3}$, $p = 10^{-6}$.

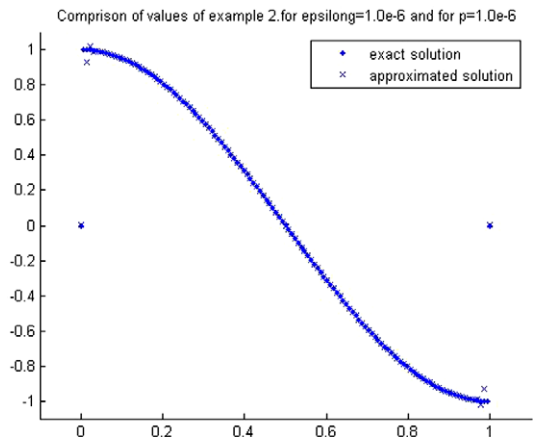

Fig. 12. Comparison of values of Problem 2 for $\varepsilon = 10^{-6}$, $p = 10^{-6}$.

Example 3. Solve the following nonlinear equation:

$$\varepsilon y'' + 2y' = -e^{y}, (0 \leqslant x \leqslant 1) \tag{25}$$

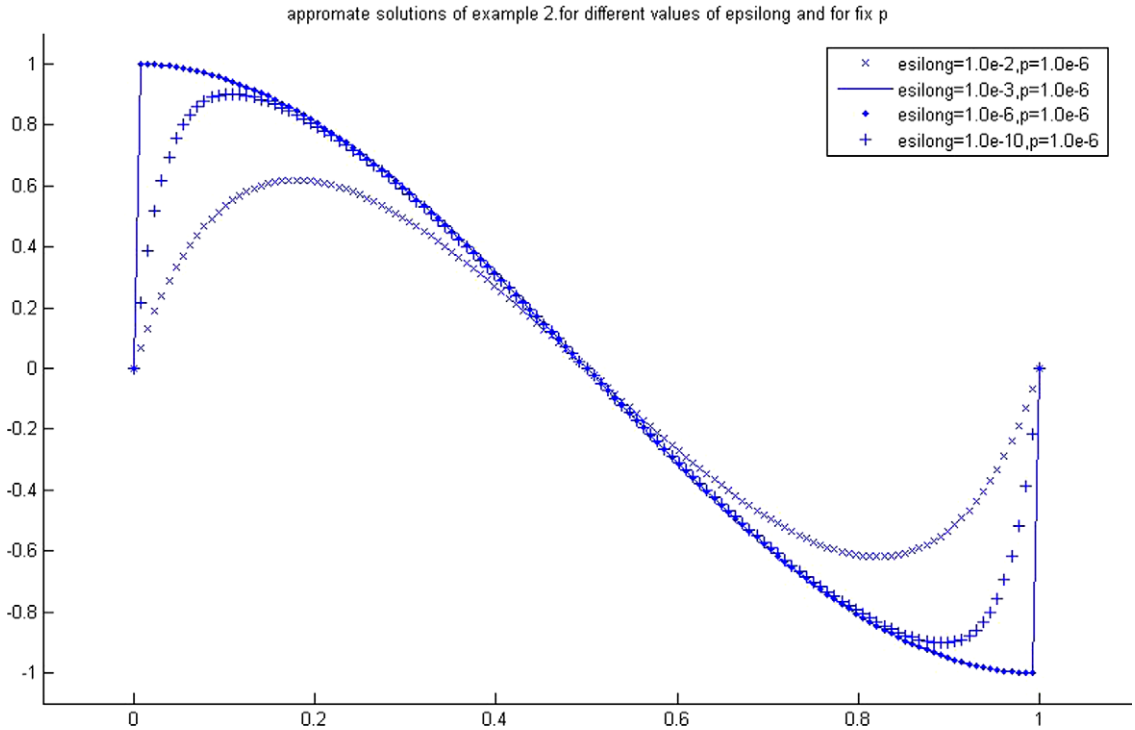

Fig. 13. Approximate solutions of Problem 2 for different values of ε and for fix p.

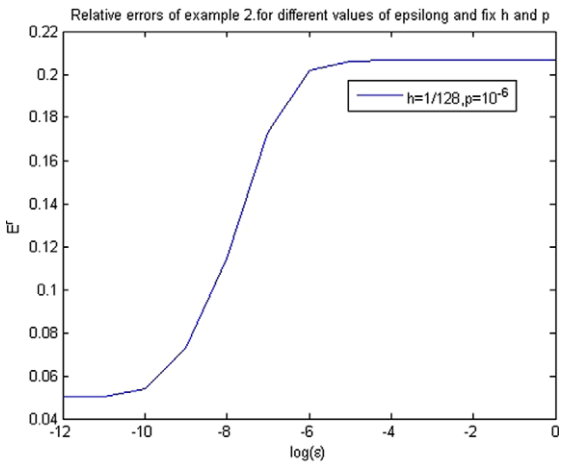

Fig. 14. Relative errors of Problem 2 for different values of ε and for fix h and p.

With boundary conditions

$$y(0) = y(1) = 0 \tag{26}$$

To compare, following valid approximation [15] is chosen as reference solution

$$y(x) = \ln\frac{2}{x+1} - e^{-2x/\varepsilon}\ln 2$$

According to method of reduction of order [16], the pair of initial value problems related to (25) and (26) are

$$(i)\ z'(x) = -\frac{2}{x+1} \quad \text{with} \quad z(1) = -\frac{\varepsilon}{2} \tag{27}$$

$$(ii)\ \varepsilon y'(x) + 2y(x) = z(x) \quad \text{with} \quad y(0) = 0 \tag{28}$$

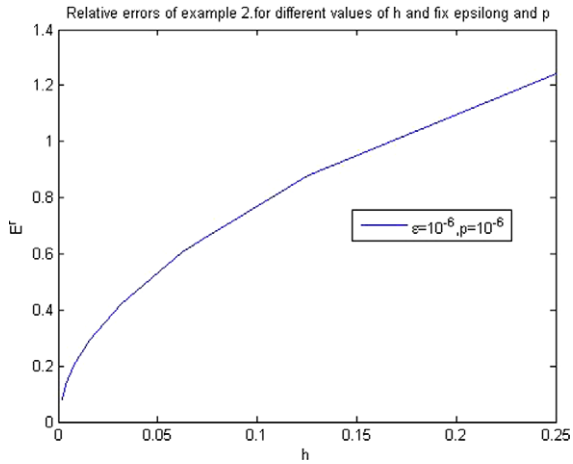

Fig. 15. Relative errors of Problem 2 for different values of h and for fix ε and p.

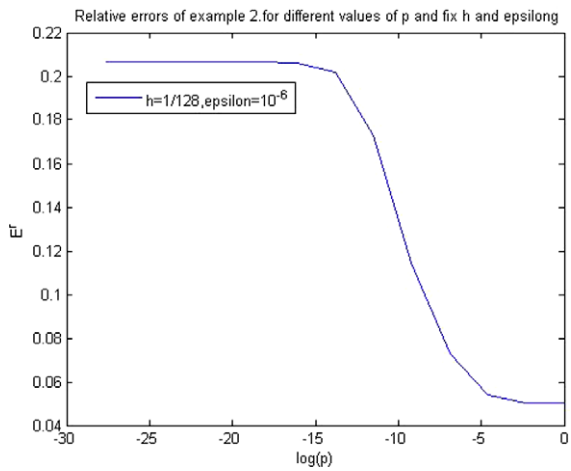

Fig. 16. Relative errors of Problem 2 for different values of p and for fix ε and h.

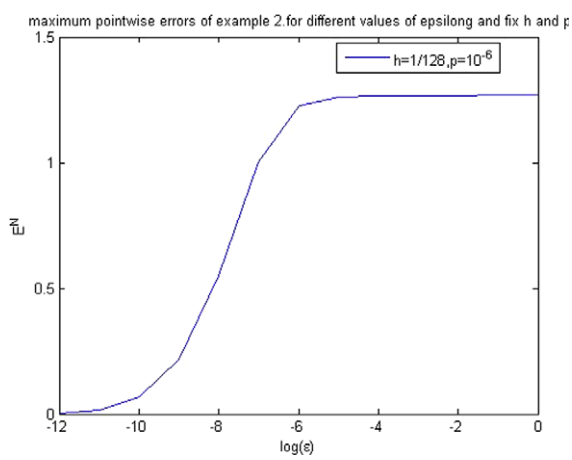

Fig. 17. Maximum pointwise of Problem 2 for different values of ε and for fix h and p.

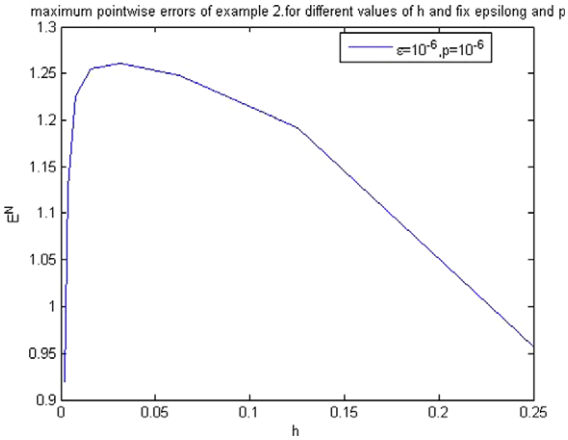

Fig. 18. Maximum pointwise errors of Problem 2 for different values of h and for fix ε and p.

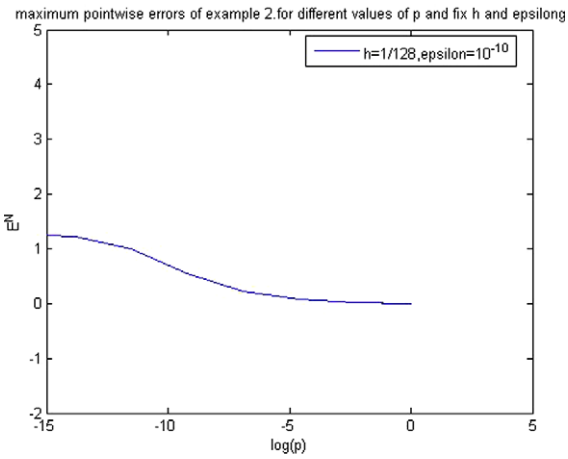

Fig. 19. Maximum pointwise errors of Problem 2 for different values of p and for fix ε and h.

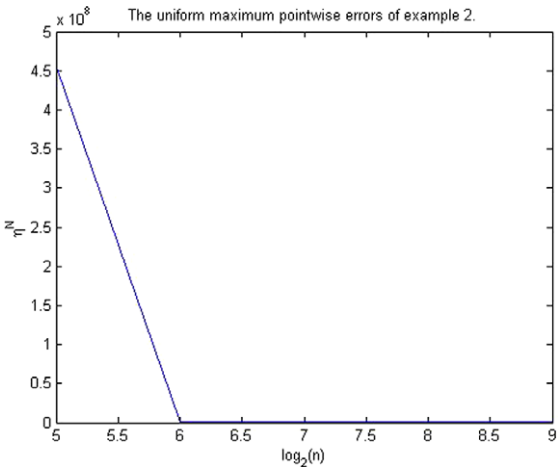

Fig. 20. The uniform maximum pointwise errors of Problem 2.

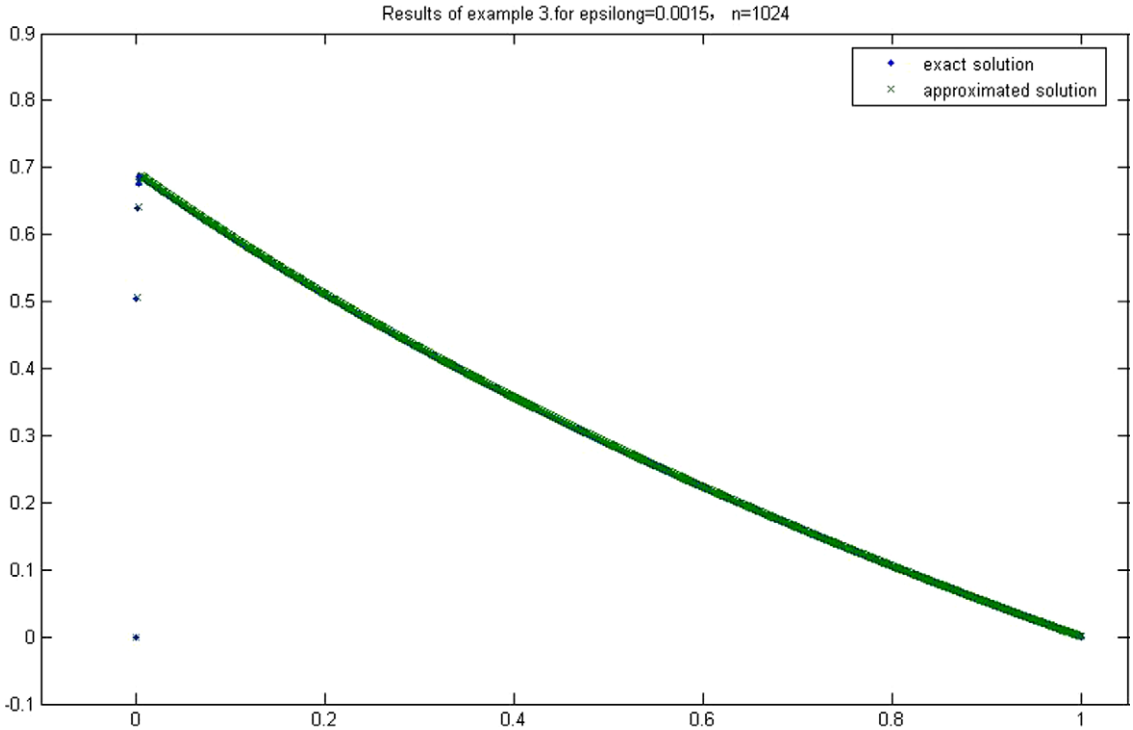

Fig. 21. Results of Problem 3 for $\varepsilon = 0.0015$, $n = 1024$.

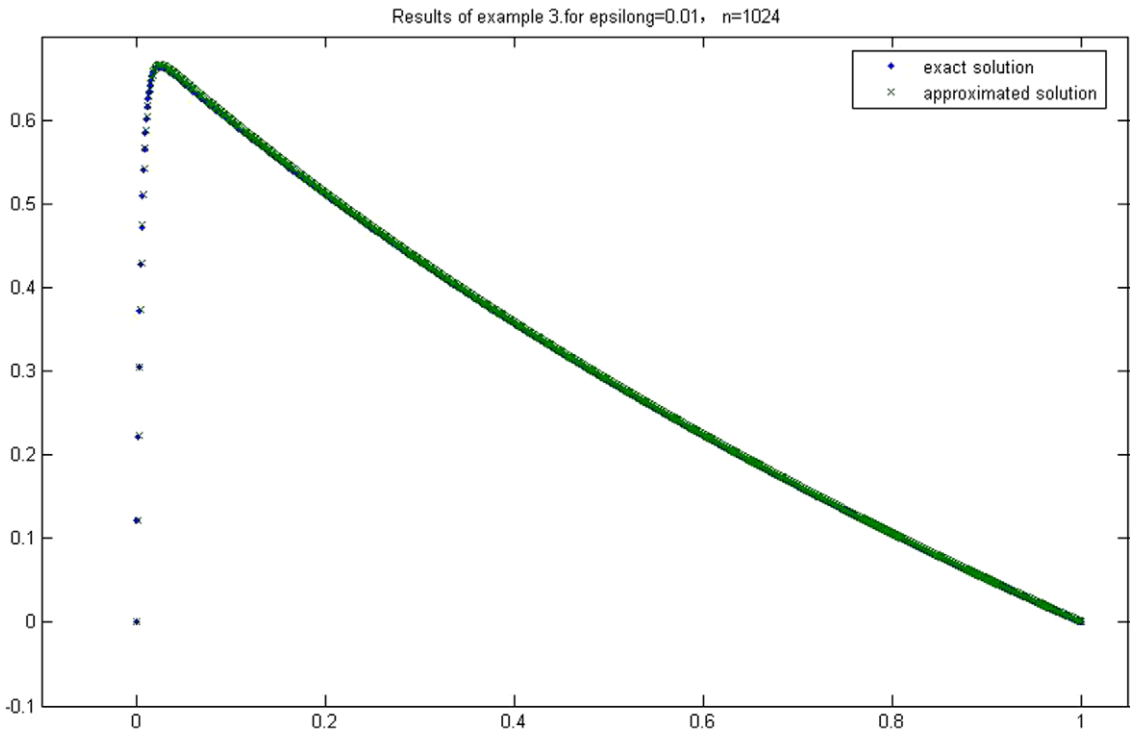

Fig. 22. Results of Problem 3 for $\varepsilon = 0.01$, $n = 1024$.

Table 3
Comparison of results and pointwise errors of Example 1.

X	$\varepsilon = 0.1$, $h = 1/128$			$\varepsilon = 0.01$, $h = 1/128$		
	Numerical	Exact	Error	Numerical	Exact	Error
1/16	0.0489	0.0556	0.0068	0.0379	0.0600	0.0073
2/16	0.1018	0.1082	0.0064	0.1096	0.1164	0.0069
4/16	0.1988	0.2045	0.0057	0.2132	0.2193	0.0061
6/16	0.2851	0.2901	0.0050	0.3048	0.3102	0.0054
12/16	0.3631	0.4578	0.0004	0.5205	0.5241	0.0037
14/16	0.4075	0.3981	0.0094	0.5763	0.5795	0.00333

Table 4
Numerical results of Example 3.

x	$\varepsilon = 0.0015$, $n = 1024$			$\varepsilon = 0.01$, $n = 1024$		
	$S(x)$	$y(x)$	Errors	$S(x)$	$y(x)$	Errors
100/1024	0.6021	0.6009	0.0012	0.6038	0.6009	0.0030
200/1024	0.5166	0.5156	0.0011	0.5181	0.5156	0.0025
300/1024	0.4379	0.4370	0.0010	0.4391	0.4370	0.0021
400/1024	0.3650	0.3641	0.0009	0.3659	0.3641	0.0018
500/1024	0.2970	0.2962	0.0008	0.2977	0.2962	0.0015
600/1024	0.2333	0.2326	0.0007	0.2339	0.2326	0.0013
700/1024	0.1734	0.1728	0.0007	0.1739	0.1728	0.0011
800/1024	0.1170	0.1164	0.00006	0.1172	0.1164	0.0009
900/1024	0.0635	0.0630	0.0005	0.0637	0.0630	0.0007
1000/1024	0.0128	0.0123	0.0005	0.0128	0.0123	0.0005

From (27) and (28) we have

$$\varepsilon y'(x) + 2y(x) = -2\ln(x+1) + 2\ln 2 - \frac{\varepsilon}{2} \tag{29}$$

Substitute (13) and (14) into (29), we obtain

$$\left(\frac{1}{3} - \frac{\varepsilon}{2h}\right)a_{j-1} + \frac{4}{3}a_j + \left(\frac{1}{3} + \frac{\varepsilon}{2h}\right)a_{j+1} = -2\ln(x+1) + 2\ln 2 - \frac{\varepsilon}{2}.$$

Comparison of reference solution and approximation solution for different values of ε is given in Figs. 21 and 22. The numerical results are given in Table 4.

The numerical results is found in good agreement with exact solution.

The numerical results show clearly the effect of ε on the boundary layer and the present method is relatively simple to collocate the solution at the mesh points, to set up the collocation system and to solve singular boundary value problems arising in biology, and it is applicable technique and approximates the exact solution very well.

Acknowledgements

The authors thank the editor and the reviewers for their valuable comments and suggestions to improve the results of this paper.

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