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A novel approach for the solution of a class of singular boundary value problems arising in physiology

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ABSTRACT

A new approach implementing a modified decomposition method in combination with the cubic B-spline collocation technique is introduced for the numerical solution of a class of singular boundary value problems arising in physiology. The domain of the problem is split into two subintervals; a modified decomposition procedure based on a special integral operator is implemented in the vicinity of the singular point and outside this domain the resulting boundary value problem is tackled by applying the B-spline scheme. Performance of this method is examined numerically; the examples reveal that the current approach converges to the exact solution rapidly and with $\mathcal{O}(h^2)$ accuracy. Results show that the method yields a numerical solution in very good agreement with the existing exact and approximate solutions.

1. Introduction

The aim of this paper is to introduce a new approach for the numerical solution of the following class of singular boundary value problems arising in physiology:

$$y'' + \frac{m}{x} y' = f(x, y(x)), \tag{1.1}$$

defined on the interval $[0, b]$ and subject to the following boundary conditions:

$$y'(0) = 0 \quad (\text{or } y(0) = \eta), \tag{1.2}$$

and

$$\alpha y(b) + \beta y'(b) = \gamma. \tag{1.3}$$

We assume that $f(x, y)$ is continuous, $\frac{\partial f}{\partial x}$ exists and is continuous on the domain $[0, b]$. The singular boundary value problem (1.1)–(1.3) arises in a number of applications, particularly for the cases when $m = 0, 1, 2$ and for certain linear and nonlinear functions $f(x, y)$. Of special interest is the case when $m = 2$ and

$$f(x, y) = \frac{ny}{y + k}, \quad n > 0,\ k > 0$$

which arises in the modeling of steady state oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics (see [1] and [2]). Another case of physical significance is when $m = 2$ and

$$f(x, y) = -le^{-lky}, \quad l > 0,\ k > 0$$

which occurs in the formulation of the distribution of heat sources in the human head (see [3,4]).

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In recent years, finding numerical solutions of singular differential equations, particularly those arising in physiology, has been the focus of a number of authors. Kanth and Bhattacharya [5] used a quasi-linearization technique to reduce a class of nonlinear singular boundary value problems arising in physiology to a sequence of linear problems; the resulting set of differential equations are modified at the singular point, then spline collocation is utilized to obtain the numerical solution. Pandey and Singh [6] described a finite difference method based on a uniform mesh for the solution of a class of singular boundary value problems arising in physiology; it was shown that the method is of second-order accuracy under quite general conditions. Caglar et al. [7] used B-spline functions to develop a numerical method for computing approximations to the solution of nonlinear singular boundary value problems associated with physiological science. The original differential equation was modified at the singular point, then the boundary value problem was treated by using the B-spline approximation. Asaithambi and Garner [8] presented a numerical technique for obtaining pointwise bounds for the solution of a class of nonlinear boundary-value problems appearing in physiology. Gustaffsson [9] presented a numerical method for solving singular boundary value problems. Kanth and Reddy [10] presented a numerical method for solving a two point boundary value problem in the interval $[0, 1]$ with regular singularity at $x = 0$. Kanth and Reddy [11] presented a numerical method for singular two point boundary value problems via Chebyshev economizition. A number of papers discussed the existence of solutions for the given problem, for instance, existence and uniqueness of the solution of (1.1)–(1.3), for the special case $m = 2,\ \alpha = \gamma$ and $\beta = 1$ has been given in [12]. See papers [12,7] and the references therein for further applications of such a problem and where it arises.

The purpose of this paper is to introduce a novel approach based on a combination of a modified decomposition approach and cubic B-splines collocation for the numerical solution of the class of singular second-order boundary value problems given in (1.1)–(1.3) that arise in physiology. The main thrust of this approach is to decompose the domain of the problem into two subintervals. The singularity, which lies in the first subinterval is removed via the application of a modified decomposition procedure based on a special integral operator that is applied to surmount the singularity. Then, in the second subdomain, which is outside the vicinity of the singularity, the resulting problem is treated via employing the B-spline collocation technique. The performance of the numerical scheme is assessed and tested on specific test problems. The oxygen diffusion problem in spherical cells and a nonlinear heat-conduction model of the human head are discussed as illustrative examples. The numerical outcomes indicate that the method yields highly accurate results and is computationally more efficient than existing ones.

In the past decade, there has been a great deal of interest (see [13–21]) in applying the decomposition method for solving a wide range of nonlinear equations, including algebraic, differential, partial-differential, differential-delay and integro-differential equations. In this paper, we employ a modified version of the decomposition approach in order to handle the singularity point at the origin, however, the setback of this method is that it diverges very rapidly as the applicable domain increases, that is, it yields only a local approximation. In contrast, the spline approach gives a global approximation regardless of the size of the interval, nevertheless it has a drawback in handling the singularity at the origin. To handle the deficiencies and balance the advantages of both methods we propose in this paper a combination of both methods as described above.

The balance of this paper is organized as follows. In Section 2, the mixed modified decomposition and cubic B-spline collocation approach is presented for the numerical solution of the class of singular second-order boundary value problems. In Section 3, a number of test problems are discussed to appraise the accuracy of the technique. Finally, Section 4 includes a conclusion that briefly summarizes the numerical results.

2. Numerical method

The essence of the new mixed decomposition-spline approach for the numerical solution of the nonlinear singular differential problem (1.1), is to split the domain $\overline{\Omega} = [0, b]$ into two subintervals as $\overline{\Omega} = \Omega_1 \cup \Omega_2 = [0, \delta] \cup [\delta, b]$. A modified decomposition scheme is implemented on the subdomain Ω_1, that is, in the vicinity of the singular point at $x = 0$. Then, outside this range a numerical solution of the problem is obtained by applying the B-spline collocation method on the subdomain Ω_2.

The decomposition method yields very accurate local approximations, even close to the singularity, but deteriorates as the applicable domain increases. In contrast, the spline collocation provides global estimation of the solution, however, the shortcoming of the method is that it gives an unsatisfactory approximation in the presence of a singularity. To handle the singularity and avoid the deficiencies of both methods, we propose the combination of both methods. In the next two subsections, we present both the cubic B-spline collocation and the modified decomposition techniques.

2.1. Spline approach

In this subsection, we present the cubic B-spline finite element collocation for the numerical solution of the class of nonlinear boundary value problems (1.1)–(1.3).

Consider the nodal points x_i on the interval $[a, b]$ where

$$0 \le a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

If the nodal points are equidistant from each other, we have $x_i = ih,\quad i = 0, 1, 2, \ldots, n$, where $h = \frac{b-a}{n}$ on the interval $[a, b]$. Let $\psi(t)$ be a shape function that satisfies the two boundary conditions (1.2)-(1.3) and is expressed as a linear combination

Table 1
ψ_i, ψ_i', and ψ_i'' evaluated at the nodal points.

Nodes	ψ_i	ψ_i'	ψ_i''
x_i	0	0	0
x_{i+1}	1	$\frac{3}{h}$	$\frac{6}{h^2}$
x_{i+2}	4	0	$-\frac{12}{h^2}$
x_{i+3}	1	$-\frac{3}{h}$	$\frac{6}{h^2}$
x_{i+4}	0	0	0

of $n+3$ shape functions given by

$$\psi(t) = \sum_{i=-3}^{n-1} A_i \psi_i(t). \tag{2.4}$$

The $A_i's$ are unknown real coefficients and the $\psi_i(x)$ are the cubic B-splines functions defined as follows:

$$\psi_i(x) = \frac{1}{h^3} \begin{cases} (x-x_i)^3, & [x_i, x_{i+1}] \\ h^3 + 3h^2(x-x_{i+1}) + 3h(x-x_{i+1})^2 - 3(x-x_{i+1})^3, & [x_{i+1}, x_{i+2}] \\ h^3 + 3h^2(x_{i+3}-x) + 3h(x_{i+3}-x)^2 - 3(x_{i+3}-x)^3, & [x_{i+2}, x_{i+3}] \\ (x_{i+4}-x)^3, & [x_{i+3}, x_{i+4}] \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

where $h = x_{i+1} - x_i$. From (2.5), the values of ψ_i, ψ_i' and ψ_i'' at the nodal points $x_i = ih$ are given according to Table 1.

To construct such an approximate solution, we substitute the approximate solution (2.4) into Eq. (1.1). This yields

$$\sum_{i=-3}^{n-1} A_i \left[\psi_i''(x_j) + \frac{m}{x_j}\psi_i'(x_j)\right] = f\left(x_j, \sum_{i=-3}^{n-1} A_i \psi_i(x_j)\right), \quad j = 0, 1, 2, \ldots, n. \tag{2.6}$$

The above system consists of $n+1$ equations in $n+3$ unknowns. The boundary conditions in (1.2)-(1.3) give the following two equations:

For $y'(a) = 0$ (or $y(a) = \eta$) we have

$$\sum_{i=-3}^{n-1} A_i \psi_i'(x_0) = 0 \quad \left(\text{or} \sum_{i=-3}^{n-1} A_i \psi_i(x_0) = \eta\right). \tag{2.7}$$

For $\alpha y(b) + \beta y'(b) = \gamma$ we have

$$\sum_{i=-3}^{n-1} A_i \left(\alpha \psi_i(x_n) + \beta \psi_i'(x_n)\right) = \gamma. \tag{2.8}$$

The values of $\psi_i(x_j)$, $\psi_i'(x_j)$ and $\psi_i''(x_j)$ at the nodal points x_j, $j = 0, 1, \ldots, n$ are determined from Table 1.

The system of equations in (2.6)–(2.8) can be written in matrix form as follows:

$$\mathbf{Cb} = \mathbf{d} \tag{2.9}$$

where

$$\mathbf{C} = \begin{bmatrix} -\frac{3}{h} & 0 & \frac{3}{h} & 0 & 0 & \cdots & 0 \\ r_0 & w_0 & v_0 & 0 & 0 & \cdots & 0 \\ 0 & r_1 & w_1 & v_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & r_n & w_n & v_n \\ 0 & 0 & 0 & \cdots & \alpha - \frac{3\beta}{h} & 4\alpha & \alpha + \frac{3\beta}{h} \end{bmatrix}$$

$$r_i = \frac{6 - 3hG_i}{h^2}, \qquad w_i = \frac{-12}{h^2}, \qquad v_i = \frac{6 + 3hG_i}{h^2}$$

where

$$G_i = \frac{m}{x_i}$$

$$\mathbf{d} = \begin{bmatrix} 0 \\ F\left(x_0,\ a_{-3} + 4a_{-2} + a_{-1}\right) \\ F\left(x_1,\ a_{-2} + 4a_{-1} + a_0\right) \\ F\left(x_2,\ a_{-1} + 4a_0 + a_1\right) \\ \cdot \\ \cdot \\ \cdot \\ F\left(x_{n-1},\ a_{n-4} + 4a_{n-3} + a_{n-2}\right) \\ F\left(x_n,\ a_{n-3} + 4a_{n-2} + a_{n-1}\right) \\ \gamma \end{bmatrix}$$

and

$$\mathbf{b}^{\mathrm{T}} = \begin{bmatrix} A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots & A_{n-3} & A_{n-2} & A_{n-1} \end{bmatrix}.$$

The system of equations given in (2.9) is solved using the computer algebra system *Maple*.

In the case of the second condition in (2.7), the above system has to be slightly modified. The first three entries in the first row of matrix **C** have to be replaced by: 1, 4 and 1, respectively. Also, the first row in the vector **d** should be η instead of 0.

2.2. Modified decomposition method formulation

In this subsection, we describe briefly the decomposition algorithm as it applies to the following generalized nonlinear equation of the form

$$u - N(u) = f \tag{2.10}$$

where N is a nonlinear operator on a Hilbert space H and f is a known element of H. We assume that for a given f a unique solution u of (2.10) exists. The decomposition scheme (see [13–20] for more details on the method) assumes a series solution given by

$$u = \sum_{n=0}^{\infty} u_n \tag{2.11}$$

where the nonlinear operator N is decomposed into

$$N(u) = \sum_{n=0}^{\infty} A_n \tag{2.12}$$

where the A_n's are the Adomian polynomials (see [13]) of $u_0,\ u_1, \ldots, u_n$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \qquad n = 0, 1, 2, \ldots. \tag{2.13}$$

Substituting Eqs. (2.11) and (2.12) into the functional equation (2.10) yields

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f. \tag{2.14}$$

Upon matching both sides of the equation, we obtain the following recursive scheme:

$$\begin{aligned} u_0 &= f \\ u_1 &= A_0 \\ u_2 &= A_1 \\ &\cdots \\ u_n &= A_{n-1}. \end{aligned} \tag{2.15}$$

Thus, one can recurrently determine every term of the series $\sum_{n=0}^{\infty} u_n$.

If the operator $N(u)$ is a nonlinear function of u, say $f(u)$, then the first four Adomian Polynomials (see [13]) are given by

$$\begin{cases} A_0 = f(u_0) \\ A_1 = u_1 f^{(1)}(u_0) \\ A_2 = u_2 f^{(1)}(u_0) + \dfrac{1}{2!}u_1^2 f^{(2)}(u_0) \\ A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \dfrac{1}{3!}u_1^3 f^{(3)}(u_0) \\ A_4 = u_4 f^{(1)}(u_0) + \left(u_1 u_3 + u_2^2/2\right) f^{(2)}(u_0) + \dfrac{1}{2}u_1^2 u_2 f^{(3)}(u_0) + \dfrac{1}{24}u_1^4 f^{(4)}(u_0) \\ \ldots. \end{cases} \tag{2.16}$$

How do we interpret and solve the class of nonlinear boundary-value problem (1.1)–(1.2) in this setting? We first proceed by rewriting problem (1.1) in the form

$$x^{-m}\left(x^m y'\right)' = f(x, y(x)). \tag{2.17}$$

Following the decomposition analysis, we define the linear operator

$$L = x^{-m}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^m \frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{2.18}$$

This operator is defined in this manner in order to surmount the singularity at $x = 0$. Consequently, Eq. (1.1) can be rewritten in terms of this linear operator as follows:

$$L[y] = f(x, y). \tag{2.19}$$

Based on the definition of L in (2.18), the inverse operator of L, namely L^{-1}, is given by the following twofold indefinite integral operator:

$$L^{-1}[.] := \int_0^x x^{-m} \int_0^x x^m [.]\ \mathrm{d}x\, \mathrm{d}x. \tag{2.20}$$

Operating on both sides of (2.19) with L^{-1} yields

$$L^{-1}L[y] = L^{-1}[f(x, y(x))] \tag{2.21}$$

which simplifies to

$$y(x) - y(0) - y'(0)x = L^{-1}[f(x, y(x))]. \tag{2.22}$$

The algorithm consists of expressing the solution as an infinite series as follows:

$$y = \sum_{n=0}^{\infty} y_n \tag{2.23}$$

where the terms y_n are to be recursively computed. The nonlinear term $f(x, y)$ is decomposed in terms of the Adomian polynomials $A_n(x)$ given in (2.16), as follows:

$$f(x, y(x)) = \sum_{n=0}^{\infty} A_n(x). \tag{2.24}$$

If we substitute Eqs. (2.23) and (2.24) into (2.22) we have

$$\sum_{n=0}^{\infty} y_n = y(0) + y'(0)x + L^{-1}[A_n(x)]. \tag{2.25}$$

Matching both sides of Eq. (2.25) yields the following iterative scheme:

$$\begin{cases} y_0 = y(0) + y'(0)x \\ y_1 = L^{-1}[A_0(x)] \\ y_2 = L^{-1}[A_1(x)] \\ \cdots \\ y_{n+1} = L^{-1}[A_n(x)]. \end{cases} \tag{2.26}$$

3. Numerical examples

In this section, the mixed decomposition-spline method is implemented for tackling the singular differential equation (1.1). To illustrate the effectiveness of this novel method we shall consider three singular test examples. Comparisons with exact solutions and existing numerical methods shall also be made. We will consider three physical model problems from the literature, namely, oxygen diffusion and a non-linear heat conduction model of the human head (see [1–4]). All the numerical computations were executed on a Pentium-III PC using *Maple*.

Example 1. Consider the following special case of Eq. (1.1):

$$y'' + \frac{1}{x}y' = -\mathrm{e}^y \tag{3.27}$$

and subject to one of the following two cases of boundary conditions:

Case A.

$$y'(0) = 0, \qquad y(1) = \mu = 0 \tag{3.28}$$

Case B.

$$y'(0) = 0, \qquad y(1.5) = \mu = 2\ln\left(\frac{4-2\sqrt{2}}{7.75-4.50\sqrt{2}}\right). \tag{3.29}$$

Problem (3.27)–(3.28) has the exact solution $y(x) = 2\ln\left(\frac{C+1}{Cx^2+1}\right)$, where $C = 3-2\sqrt{2}$.

For this case, the nonlinear function in (1.1) is given by $f(x,y) = -e^{y}$. Further, we have $m = 1$. Assuming a solution of the form $y(x) = \sum_{n=0}^{\infty} y_n$, then upon using (2.16) and (2.26) the various iterates for problem (3.27)–(3.28) can be determined as follows:

$$y_0 = y(0) + y'(0)x = \eta. \tag{3.30}$$

We used the fact that $y'(0) = 0$ and we set $y(0) = \eta$, where η is not given and will be determined later. For the other higher iterates, we have

$$\begin{aligned} y_1 &= L^{-1}[A_0] = L^{-1}[f(x,y_0)] = -L^{-1}\left[e^{y_0}\right] = -L^{-1}[e^{\eta}] \\ &= -\int_0^x x^{-1}\int_0^x x\, e^{\eta}\, dx\, dx = -\frac{x^2}{4}\, e^{\eta}. \end{aligned} \tag{3.31}$$

In a similar manner,

$$\begin{aligned} y_2 &= L^{-1}[A_1] = L^{-1}\left[f_y(x,y_0)\, y_1\right] = -L^{-1}\left[e^{y_0}\, y_1\right] = -L^{-1}\left[e^{\eta}\left(-\frac{x^2}{4}\, e^{\eta}\right)\right] \\ &= \int_0^x x^{-1}\int_0^x x\left(\frac{x^2}{4}\, e^{2\eta}\right) dx\, dx = \frac{1}{64}x^4 e^{2\eta} \end{aligned} \tag{3.32}$$

$$\begin{aligned} y_3 &= L^{-1}[A_2] = L^{-1}\left[f_y(y_0)\, y_2 + \frac{1}{2}y_1^2\, f_{yy}(y_0)\right] \\ &= -L^{-1}\left[e^{y_0}\, y_2 + \frac{1}{2}y_1^2\, e^{y_0}\right] = -L^{-1}\left[e^{\eta}\left(\frac{x^4}{64}\, e^{2\eta}\right) + \frac{1}{2}\left(-\frac{x^2}{4}\, e^{\eta}\right)^2\, e^{\eta}\right] \\ &= -\int_0^x x^{-1}\int_0^x x\left(\frac{3x^4}{64}\, e^{3\eta}\right) dx\, dx = -\frac{1}{768}x^6 e^{3\eta} \end{aligned} \tag{3.33}$$

$$\begin{aligned} y_4 &= L^{-1}[A_3] = L^{-1}\left[f_y(y_0)\, y_3 + y_1 y_2 f_{yy}(y_0) + \frac{1}{6}y_1^3 f_{yyy}(y_0)\right] \\ &= \frac{1}{8192}x^8 e^{4\eta} \end{aligned} \tag{3.34}$$

and

$$y_5 = L^{-1}[A_4] = -\frac{1}{81920}\, x^{10}\, e^{5\eta}. \tag{3.35}$$

Upon summing these iterates, the approximate solution is

$$y(x) \approx y_{S_1} = \sum_{i=0}^{5} y_i = \eta - \frac{x^2}{4}\, e^{\eta} + \frac{1}{64}x^4 e^{2\eta} - \frac{1}{768}x^6 e^{3\eta} + \frac{1}{8192}x^8 e^{4\eta} - \frac{1}{81920}\, x^{10}\, e^{5\eta}. \tag{3.36}$$

To determine the value of η we consider two approaches. The first approach (I) is suitable for small intervals mainly of length at most 1 and with boundary conditions as that given in case A above. In this approach, the second boundary condition in (3.28), namely $y(1) = 0$, is manipulated to determine the value of η in (3.36). Requiring the approximate solution in (3.36) to satisfy this boundary condition we obtain

$$\eta = 0.316704855297798.$$

Substituting this value of η into (3.36) yields the approximate solution on the interval [0, 1]:

$$\begin{aligned} y_I(x) &= 0.316704855297798 - 0.343149349321440x^2 + 0.0294378689849319x^4 - 0.00336719519586307x^6 \\ &\quad + 0.000433294065187008x^8 - 0.0000594738306135055x^{10}. \end{aligned} \tag{3.37}$$

Table 2
Numerical errors of Example 1 on [0, 1] with $\delta = 0.5$.

x	Approach I			Approach II			Method in [7]	
	$n = 5$	$n = 10$	$n = 20$	$n = 5$	$n = 10$	$n = 20$	n	Max. Err.
0.0	1.05 (−5)	1.05 (−5)	1.05 (−5)	3.22 (−5)	8.06 (−6)	2.00 (−6)	20	3.16 (−5)
0.1	1.05 (−5)	1.05 (−5)	1.05 (−5)	3.21 (−5)	8.03 (−6)	1.99 (−6)		
0.2	1.03 (−5)	1.03 (−5)	1.03 (−5)	3.18 (−5)	7.95 (−6)	1.97 (−6)		
0.3	1.02 (−5)	1.02 (−5)	1.02 (−5)	3.13 (−5)	7.81 (−6)	1.94 (−6)		
0.4	9.93 (−6)	9.93 (−6)	9.93 (−6)	3.05 (−5)	7.63 (−6)	1.83 (−6)		
0.5	9.62 (−6)	9.62 (−6)	9.62 (−6)	2.96 (−5)	7.40 (−6)	1.78 (−6)	40	1.55 (−6)
0.6	2.73 (−6)	6.07 (−6)	6.93 (−6)	2.69 (−5)	6.72 (−6)	1.67 (−6)		
0.7	6.67 (−7)	3.65 (−6)	4.75 (−6)	2.16 (−5)	5.41 (−6)	1.34 (−6)		
0.8	1.58 (−6)	2.02 (−6)	2.93 (−6)	1.48 (−5)	3.70 (−6)	9.20 (−7)		
0.9	1.08 (−6)	8.76 (−7)	1.37 (−6)	7.36 (−6)	1.84 (−6)	4.57 (−7)		
1.0	0	0	0	0	0	0	90	1.55(−6)

Table 3
Maximum numerical errors of Example 1 on [0, 1.5].

n	5	10	15	20
Max. Error	2.37×10^{-5}	6.18×10^{-6}	3.03×10^{-6}	1.56×10^{-6}

The domain $\overline{\Omega} = [0, 1]$ is divided into two subintervals as $\overline{\Omega} = \Omega_1 \cup \Omega_2 = [0, \delta] \cup [\delta, 1]$. The modified decomposition method is employed in the vicinity of the singularity $x = 0$, that is within the subdomain Ω_1, and outside it in Ω_2 the spline collocation method is applied. From Eq. (3.37), the boundary condition at $x = \delta = 0.5$ is found to be

$$y(\delta) = y_I(0.5) = 0.2327064068. \tag{3.38}$$

Numerical results using this approach (I), condition (3.38) and the second boundary condition in (3.28), for different numbers of mesh points are presented in Table 2. This approach, for boundary conditions given in case B, fails to provide a series solution with good accuracy. This is as expected since we have a wider interval as compared with case A.

A second approach (II) is implemented for case B, which can be manipulated to improve the approximate solution for case A. In this approximation, Eq. (3.36) is used to estimate $y(\delta)$ and $y'(\delta)$ which will be functions of η. Then the spline collocation, Eq. (2.6), is applied on the interval $[\delta, b]$, for $j = 1, 2, \ldots, n$, ignoring the approximate solution at δ ($j = 0$). The $n \times (n + 3)$ outcome system requires three additional equations which are given as follows:

$$y(\delta) = y_{S_1}(\delta), \qquad y'(\delta) = y'_{S_1}(\delta), \qquad y(b) = \mu. \tag{3.39}$$

Thus, in this case it is not necessary to have the solution to be twice differentiable at $x = \delta$. In Table 2, numerical results using the two approaches (I and II) are presented and compared with the numerical solutions obtained in reference [7]. Our proposed method using approach (II) yields better results than in [7] with less mesh points used. In Table 3, numerical solutions of (3.27) are shown for the boundary conditions given in case B. In Fig. 1, the true solution and the numerical decomposition-spline solution are plotted for the choice of $\delta = 0.5$ and $n = 10$ for case B.

From Tables 2 and 3 we notice that the numerical solutions converge to the exact solution. Using the double mesh principle $p \approx \left|\frac{Err(n)}{Err(2n)}\right|$, the order of convergence is verified to be 2.

Regarding the choice of δ, the following algorithm was implemented to calculate the δ:

Assume $y(0) = \eta$ for a given choice of δ. We consider r terms of the approximate solution obtained by the decomposition approach, namely $y_r = \Sigma_{i=0}^{r} y_i$ on $[0, \delta]$. The numerical scheme is repeated for different choices of the terminal point δ such that

$$|\eta_{r+1} - \eta_r| < \tau$$

for a prescribed choice of the tolerance τ, where η_r is the solution of the boundary condition

$$\alpha y_r(b) + \beta y_r'(b) = \gamma.$$

In other words, the break point δ is chosen by noticing that the computational value of the boundary condition at $x = \delta$ (namely, $\alpha y(\delta) + \beta y'(\delta) = \gamma$) stabilizes. The intention of the algorithm is to minimize and control the error in the boundary condition at $x = \delta$, which will affect the accuracy of the solution obtained by the spline approach on the second subinterval $[\delta, 1]$. Regarding the value of h used for the spline method on $[\delta, 1]$, it can be chosen to be independent of the value of δ.

Example 2. Consider the following special case of Eq. (1.1):

$$y'' + \frac{2}{x} y' = -le^{-lky}, \quad l > 0, \; k > 0 \tag{3.40}$$

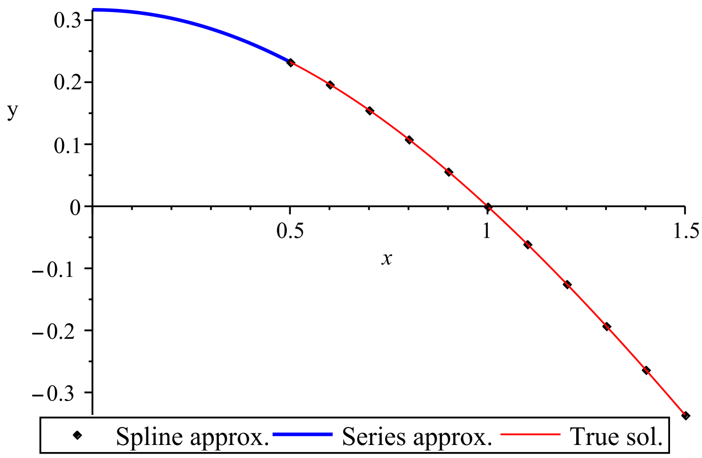

Fig. 1. Numerical solutions of Example 1 for case B.

where $0 < x \leq 1$, and subject to the following boundary conditions:

$$y'(0) = 0, \qquad 0.1y(1) + y'(1) = 0. \tag{3.41}$$

We will take $l = k = 1$. So note that for this case, the nonlinear function in Eq. (1.1) is given by $f(x, y) = -e^{-y}$ and also we have $m = 2$. Thus, if $y(x) = \sum_{n=0}^{\infty} y_n$, then (2.16) and (2.26) imply that the various iterates, for the boundary conditions (3.41), can be determined as follows:

$$y_0 = y(0) + y'(0)x = \eta \tag{3.42}$$

where $y(0) = \eta$ will be determined later. We have

$$\begin{aligned} y_1 &= L^{-1}[A_0] = L^{-1}[f(x, y_0)] = -L^{-1}\left[e^{-y_0}\right] = -L^{-1}\left[e^{-\eta}\right] \\ &= -\int_0^x x^{-2} \int_0^x x^2 \, e^{-\eta} \, dx \, dx = -\frac{x^2}{6} \, e^{-\eta}. \end{aligned} \tag{3.43}$$

In a like manner, we find

$$\begin{aligned} y_2 &= L^{-1}[A_1] = L^{-1}\left[f_y(x, y_0) \, y_1\right] = L^{-1}\left[e^{-y_0} y_1\right] = L^{-1}\left[e^{-\eta}\left(-\frac{x^2}{6} \, e^{-\eta}\right)\right] \\ &= -\int_0^x x^{-2} \int_0^x x^2 \left(\frac{x^2}{6} \, e^{-2\eta}\right) dx \, dx = -\frac{1}{120} x^4 e^{-2\eta} \end{aligned} \tag{3.44}$$

$$\begin{aligned} y_3 &= L^{-1}[A_2] = L^{-1}\left[f_y(y_0) \, y_2 + \frac{1}{2} y_1^2 f_{yy}(y_0)\right] \\ &= L^{-1}\left[e^{y_0} y_2 - \frac{1}{2} y_1^2 e^{y_0}\right] = -\frac{1}{1890} x^6 e^{-3\eta} \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} y_4 &= L^{-1}[A_3] = L^{-1}\left[f_y(y_0) \, y_3 + y_1 y_2 f_{yy}(y_0) + \frac{1}{6} y_1^3 f_{yyy}(y_0)\right] \\ &= -\frac{61}{1632960} x^8 e^{-4\eta} \end{aligned} \tag{3.46}$$

$$y_5 = L^{-1}[A_4] = -\frac{629}{224532000} \, x^{10} \, e^{-5\eta}. \tag{3.47}$$

Upon summing these iterates, we observe that the approximate solution is

$$y(x) \approx y_{S_2} = \sum_{i=0}^{5} y_i = \eta - \frac{x^2}{6} \, e^{-\eta} - \frac{1}{120} x^4 e^{-2\eta} - \frac{1}{1890} x^6 e^{-3\eta} - \frac{61}{1632960} x^8 e^{-4\eta} - \frac{629}{224532000} \, x^{10} \, e^{-5\eta}. \tag{3.48}$$

Table 4
Numerical solutions of Example 2 on [0, 1].

x	Proposed method $(h = 1/20)$	Method in [7] $(h = 1/60)$
0.0	1.14704079519111	1.14703993670271
0.1	1.14651141921222	1.14651055946170
0.2	1.14492228171805	1.14492141825538
0.3	1.14227034791619	1.14226947822689
0.4	1.13855053934243	1.13854966085306
0.5	1.13375570287390	1.13375481292594
0.6	1.12787656170330	1.12787566262296
0.7	1.12090166457601	1.12090076206338
0.8	1.11281731735712	1.11281641561478
0.9	1.10360749049872	1.10360659299888
1.0	1.09325371614752	1.09325282603337

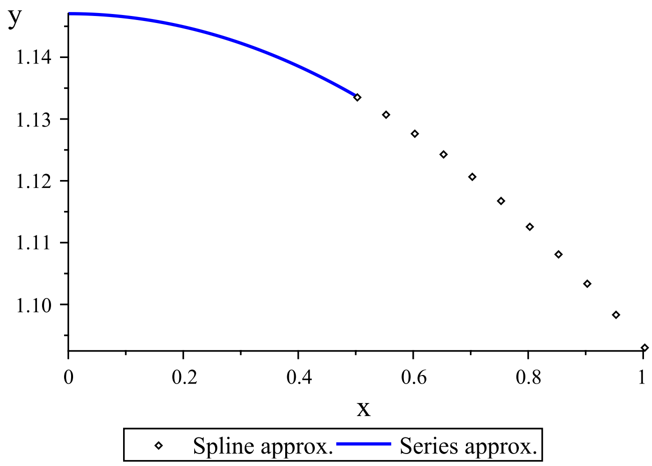

Fig. 2. Numerical solutions of Example 2 for $\delta = 0.5$ and $n = 10$.

Following approach (II) used in Example 1: for a given number of meshes n, we obtain the boundary conditions

$$y(\delta) = y_{S_2}(\delta), \qquad y'(\delta) = y'_{S_2}(\delta)$$

which are functions of η. These conditions together with the second boundary condition in (3.41), namely $0.1y(1)+y'(1) = 0$, are utilized to find the spline approximations for the case $\delta = 0.5$. The numerical results are given in Table 4 which clearly show a good agreement, up to at most 3×10^{-7}, with the numerical results given in reference [7]. The advantage in our proposed approach as compared with [7], is that fewer number of mesh points are required to obtain similar results.

In Fig. 2, the numerical decomposition-spline solution for Example 2 is plotted for $\delta = 0.5$ and $n = 10$.

Example 3. Consider the following special case of Eq. (1.1):

$$y'' + \frac{2}{x}\,y' = \frac{ny}{y+k}, \quad n > 0,\ k > 0 \tag{3.49}$$

where $0 < x \leq 1$, and subject to the following two cases of boundary conditions:

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5. \tag{3.50}$$

We will take $n = 0.76129$, $k = 0.03119$. So note that for this case, the nonlinear function in Eq. (1.1) is given by $f(x, y) = \frac{ny}{y+k}$ and also we have $m = 2$. Thus, if $y(x) = \sum_{n=0}^{\infty} y_n$, then (2.16) and (2.26) imply that the various iterates, for the boundary conditions (3.50), can be determined as follows:

$$y_0 = y(0) + y'(0)x = \eta \tag{3.51}$$

where $y(0) = \eta$ will be determined later. We have

$$\begin{aligned} y_1 &= L^{-1}[A_0] = L^{-1}[f(x, y_0)] = L^{-1}\left[\frac{0.76129\eta}{\eta + 0.03119}\right] \\ &= -\int_0^x x^{-2}\int_0^x x^2\,\frac{0.76129\eta}{\eta+0.03119}\,dx\,dx = \frac{0.126881666666666\eta}{0.03119+\eta}\,x^2 \end{aligned} \tag{3.52}$$

Table 5
Numerical solutions of Example 3 on [0, 1] for $\delta = 0.5$.

x	Proposed method		Method in [7]	
	$n = 5$	$n = 20$	$n = 40$	$n = 60$
0.0	0.82848300634573	0.82848329481355	0.82848325127828	0.82848327295802
0.1	0.82970580840464	0.82970609688790	0.82970605379256	0.82970607521884
0.2	0.83337444951368	0.83337473804308	0.83337469615261	0.83337471691089
0.3	0.83948962973406	0.83948991833986	0.83948987847273	0.83948989814383
0.4	0.84805250004880	0.84805278876051	0.84805275215744	0.84805277036165
0.5	0.85906463868421	0.85906492753032	0.85906489756682	0.85906491397434
0.6	0.87252803891014	0.87252831569855	0.87252829407669	0.87252830841853
0.7	0.88844505304223	0.88844529949702	0.88844528382436	0.88844529589927
0.8	0.90681833702984	0.90681854179965	0.90681853058322	0.90681854026297
0.9	0.92765082665921	0.92765098305256	0.92765097529878	0.92765098252660
1.0	0.95094568997062	0.95094579480523	0.95094578981933	0.95094579461056

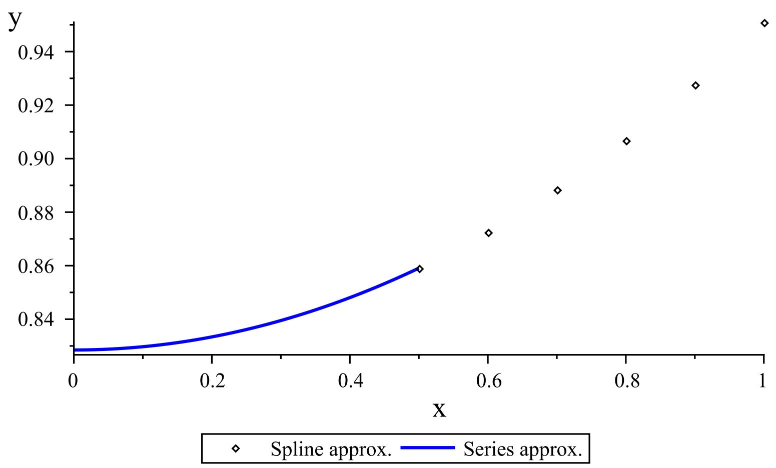

Fig. 3. Numerical solutions of Example 3 for $\delta = 0.5$ and $n = 5$.

and

$$y_2 = L^{-1}[A_1] = L^{-1}\left[f_y(x, y_0)\, y_1\right] = \frac{0.00015063794379399\eta}{(0.03119+\eta)^3}\, x^4. \tag{3.53}$$

In a like manner, we find the other iterates as in the previous two examples. Upon summing the iterates, we will obtain the following approximate solution:

$$y(x) \approx y_{S_3} = \sum_{i=0}^{3} y_i = \eta + \frac{0.12688166666666\eta}{0.03119+\eta}\, x^2 + \frac{0.00015063794379399\eta}{(0.03119+\eta)^3}\, x^4 + \frac{8.5162928752444\times 10^{-8}\,\eta - 9.1015206532481\times 10^{-8}\,\eta^2}{(0.03119+\eta)^5}\, x^6. \tag{3.54}$$

As in the previous two examples, for a given number of meshes n we use (3.54) to obtain the boundary conditions:

$$y(0.5) = y_{S_3}(0.5), \qquad y'(0.5) = y'_{S_3}(0.5).$$

These two conditions together with the second boundary condition given in (3.50) are utilized to find the spline collocation approximations. The numerical results (see Table 5) for various number of meshes are compared with the results in reference [7]. In Fig. 3, the numerical decomposition-spline solution for Example 3 is plotted for $\delta = 0.5$ and $n = 5$.

4. Conclusion

A new approach, based on a combination of a modified decomposition method and cubic B-spline collocation, has been introduced for the numerical solution of a class of singular boundary value problems arising in physiology. A special linear operator is applied to surmount the singularity at the origin followed by a series solution in the neighborhood of the singular point. The approach has been tested on some existing physical problems and it is evident from the numerical examples that the method is of second-order $\mathcal{O}(h^2)$ accuracy. The results give a better estimation to the solution than the stated existing numerical methods with the same number of knots. The method is effective and applicable for a wide range of singular problems and gives very accurate approximations using a relatively small number of mesh points.

A modified version of the decomposition approach was implemented only on a small subinterval that lies in the vicinity of the singularity. The deficiency of the decomposition scheme is that it diverges very rapidly as the applicable domain increases. To overcome this, the spline procedure was employed outside the vicinity, namely on the larger domain. The B-spline method gives an accurate global approximation regardless of the size of the interval, nevertheless it has a drawback in handling the singularity at the origin. Spline collocation may not produce an effective approximation near the singularity: the resulting system of equations might become inconsistent due to a loss of an equation that could contain terms which cannot be evaluated at the singularity.

The current approach avoids the deficiencies of the decomposition and spline methods and maximizes the advantages of both methods. In contrast with existing finite-difference methods, this method produces a series solution in the vicinity of the singularity as well a spline function which can be utilized to obtain a solution at any point in the domain and not restricted to the values at the chosen knots.

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