Energy Equation (continue) :-

$$\rho \frac{DU}{Dt} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \frac{P}{\rho}\frac{D\rho}{Dt} + \phi \quad \Rightarrow (1).$$

where we have introduced

$$\phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{1}{3}(\nabla \cdot V)^2\right]$$

→ is called the dissipation function and is the rate at which the viscous force do irreversible work on the fluid particles per unit volume.

The equation (1) may also be written in terms of the fluid enthalpy defined by $i = U + P/\rho$

Enthalpy: Thermodynamic quantity equivalent to the total heat content of a system. It is equal to the internal energy of the system plus the product of pressure and volume.

$$i = u + P/\rho$$

$$\rho \frac{Di}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \frac{Dp}{Dt} + \phi \implies (2)$$

$$\rho \frac{Di}{Dt} = \nabla\cdot(k\nabla T) + \frac{Dp}{Dt} + \phi$$

$$\nabla\cdot(k\nabla T) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

For a perfect gas

$$Di = C_p\, DT \implies (3)$$

where C_p is the specific heat at constant pressure. Hence, for a perfect gas the energy equation (2)

$$\implies \rho C_p \frac{DT}{Dt} = \nabla\cdot(k\nabla T) + \frac{Dp}{Dt} + \phi \implies (4)$$

physical significances:

1- $\rho c_p \frac{DT}{Dt}$ represent the convective term.

2. $\nabla \cdot K(\nabla T)$; the rate of heat diffusion to the fluid particles,

3. $\frac{Dp}{Dt}$: The rate of reversible work done on the fluid particles by compression

Φ: rate of ~~work~~ viscous dissipation per unit volume.

For low speed flows with constant thermal conductivity the energy equation (4) for perfect gases becomes

$$\frac{Dp}{Dt} = 0$$

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (K \nabla T) + \Phi$$

$$\rho c_p \frac{DT}{Dt} = K \, \nabla \cdot (\nabla T) + \Phi$$

$$\frac{DT}{Dt} = \frac{k}{\rho c_p} \nabla \cdot (\nabla T) + \frac{1}{\rho c_p} \phi$$

$$\alpha = \frac{k}{\rho c_p}$$

$$\frac{DT}{Dt} = \alpha \nabla \cdot (\nabla T) + \frac{1}{\rho c_p} \phi \Rightarrow (5)$$

α is the thermal diffusivity of the fluid, and

$$\nabla^2 T = \nabla \cdot \nabla T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

For an incompressible fluid

$$d u = c\, dT$$

where $c = c_v = c_p$

$$\rho c \frac{DT}{Dt} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \phi \Rightarrow (6)$$

$\phi = \text{---}$

when thermal conductivity is constant, the energy equation (6) can be written as

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{1}{\rho c} \phi$$

For a steady flow of an incompressible fluid with constant thermal conductivity the energy equation becomes

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{1}{\rho c} \phi$$