In continuition of previous lecture.

Xing both sides momentum equations (9), (10) and (11) by u, v and w respectively, we obtain

$$\rho u \frac{Du}{Dt} = u\left(\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z}\right) \quad -(14)$$

$$\rho v \frac{Dv}{Dt} = v\left(\rho f_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) = (15)$$

$$\rho w \frac{Dw}{Dt} = w\left(\rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \rightarrow (16)$$

By adding (14) — (16), and by collecting like coefficients in terms of shear and normal stress of u, v & w.

$$\rho u \frac{Du}{Dt} + \rho v \frac{Dv}{Dt} + \rho w \frac{Dw}{Dt}$$

$$= u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)$$

$$+ \rho(u f_x + v f_y + w f_z) \Rightarrow (17)$$

product rule of derivative ↑

$$\rho \frac{D}{Dt}\left[\frac{u^2 + v^2 + w^2}{2}\right] = \text{other term from R.H.S of (17)}$$

$$\rho \frac{D}{Dt}\left[\frac{u^2+v^2+w^2}{2}\right] = u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)$$

$$+ \rho\left(u f_x + v f_y + w f_z\right) \Longrightarrow (18)$$

which is an energy equation obtained directly from the laws of mechanics and is appropriately called <u>mechanical energy equation</u>.

By subtracting equation (18) from total energy equation (5), we obtain

$$\rho \frac{D\mathcal{U}}{Dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$+ \sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{zz}\frac{\partial w}{\partial z} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$+ \tau_{yz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \Longrightarrow (19)$$

↳ which is termed as <u>energy equation</u>

Consider the terms from equation (19).

$$\sigma_{xx}\frac{\partial u}{\partial x}+\sigma_{yy}\frac{\partial v}{\partial y}+\sigma_{zz}\frac{\partial w}{\partial z}=-p\,\nabla\cdot V-\frac{2}{3}\mu(\nabla\cdot V)^2$$
$$+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right] \rightarrow (20)$$

where

$$\left.\begin{aligned}\sigma_{xx}&=-p+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\\ \sigma_{yy}&=-p+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial v}{\partial y}\\ \sigma_{zz}&=-p+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}\end{aligned}\right\}(21)$$

where, $\lambda=-2/3\,\mu$; $\frac{D\rho}{Dt}+\rho\,\nabla\cdot V=0$
↳ equation of continuity

~~Similarly the terms from equation (19)~~

$$\frac{D\rho}{Dt}=-\rho\,\nabla\cdot V$$

~~σ_{xx}~~

$$\frac{1}{\rho}\frac{D\rho}{Dt}=\nabla\cdot V.$$

(21) ⇒

$$\sigma_{xx}\frac{\partial u}{\partial x}+\sigma_{yy}\frac{\partial v}{\partial y}+\sigma_{zz}\frac{\partial w}{\partial z}=\frac{p}{\rho}\frac{D\rho}{Dt}-2/3\,\mu(\nabla\cdot V)^2$$
$$+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]\longrightarrow(22)$$

Similarly, some terms of (19)

$$\tau_{xy}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$$

chose $\tau_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$

$$\tau_{yz}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2$$

chose $\tau_{yz}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$

$$\tau_{zx}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2$$

$$\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$\Rightarrow$ (23)

By using (22) & (23) in (19), we have

$$\rho\frac{Du}{Dt}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_\rho\frac{D\rho}{Dt}+\phi.$$

$$\phi=2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-\frac{1}{3}(\nabla\cdot V)^2\right]$$

$\hookrightarrow$ which is called dissipation function

$\hookrightarrow$ and is the rate at which the viscous force do irreversible work.

~~Strain function~~