Heat Transfer

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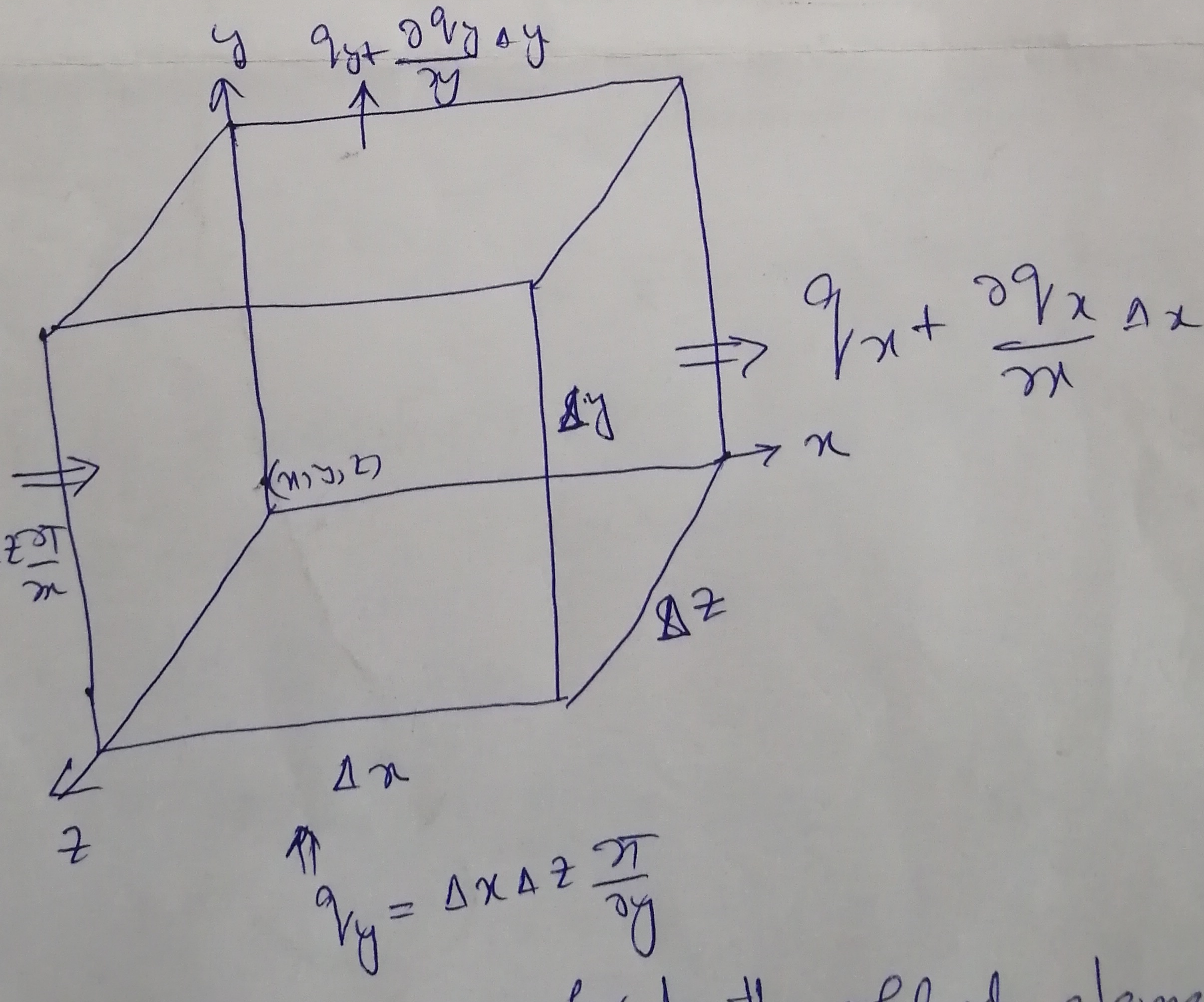
Energy Equation (Derivation of Energy Equation)

First law of thermodynamics states that the rate of heat transfer to the element minus the rate of work done by the element is equal to the rate of increase of energy of the element, which mathematically says that

$$dE = \delta Q - \delta W \quad (1)$$



Heat transfer to the fluid element.

In keeping view (1), we have the following components:

The rate of heat transfer to the ~~component~~ element

δQ:

$$\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]\Delta x\,\Delta y\,\Delta z \quad (2)$$

The net rate of work done by the fluid element agaist the surface and body forces.

Surface forces → normal stresses/forces
Surface forces → shear stresses/forces

body force: weight of the body acting vertically downward.

$\delta W \Rightarrow$

$$-\left[\frac{\partial}{\partial x}\left(u\sigma_{xx}+v\tau_{xy}+w\tau_{xz}\right)+\frac{\partial}{\partial y}\left(u\tau_{yx}+v\sigma_{yy}+w\tau_{yz}\right)+\frac{\partial}{\partial z}\left(u\tau_{zx}+v\tau_{zy}+w\sigma_{zz}\right)+\rho\left(uf_x+vf_y+wf_z\right)\right]\Delta x\,\Delta y\,\Delta z \Rightarrow (3)$$

Here, $\tau_{xy}, \tau_{xz}, \tau_{yz}$ etc... are shear forces in plane xy, xz and yz, while $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are normal forces along, x, y, and z - direction, and f_x, f_y, f_z are body forces respectively. Moreover, u, v and w are x, y, and z-components of velocities.

The rate of increase of internal and kinetic energies of the element can be written as.

$$\text{ΔE}: \quad \rho \Delta x \Delta y \Delta z \frac{D}{Dt}\left[\mathcal{U} + \frac{u^2+v^2+w^2}{2} \right] \Rightarrow (4).$$

$$\text{Density} = \rho = \frac{\text{Mass}}{\text{volume}} = \frac{\text{mass}}{\Delta x \Delta y \Delta z}$$

$$\boxed{m = \rho \Delta x \Delta y \Delta z}$$

$$= I.E + K.E = \frac{1}{2} m v^2$$

where $\mathcal{U}$ is the internal energy per unit mass of the fluid.

By substituting (2)-(4) into (1) and dividing by $\Delta x \Delta y \Delta z$ both sides

$$\rho \frac{D}{Dt}\left[U + \frac{u^2+v^2+w^2}{2}\right] = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$+ \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial x}\left(u\sigma_{xx} + v\tau_{xy} + w\tau_{xz}\right)$$

$$+ \frac{\partial}{\partial y}\left(u\tau_{yx} + v\sigma_{yy} + w\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\sigma_{zz}\right)$$

$$+ \rho\left(uf_x + vf_y + wf_z\right) \Longrightarrow (5)$$

↳ This equation is known as the total energy equation and is the sum of both thermal and mechanical energies

Consider the above fluid particle of mass ($\rho\Delta x\Delta y\Delta z$) situated at the x, y and z at time 't' in a flow field. This element has the following acceleration in x, y, and z-direction.

x-direction

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \Longrightarrow (6)$$

$\frac{\partial u}{\partial t}$ ↳ local acceleration

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ → convective acceleration y- direction

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \rightarrow (7)$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \Rightarrow (8)$$

Similarly, the net force acting on this fluid particle in the x-direction is

$$F_x = \left(\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\Delta x \Delta y \Delta z$$

where $F = ma$.

$$\Rightarrow F_x = m\frac{Du}{Dt}.$$

$$m = \rho \Delta x \Delta y \Delta z$$

so

$$m\frac{DU}{Dt} = \left(\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\Delta x \Delta y \Delta z$$

↳ density (ρ)

$$\rho \Delta x \Delta y \Delta z \frac{Du}{Dt} = \left(\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\Delta x \Delta y \Delta z$$

$$\rho\frac{Du}{Dt} = \rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \Rightarrow (9).$$

Similarly in y and z - direction

$\Rightarrow$

$$\rho \frac{DV}{Dt} = \rho f_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \Rightarrow (10)$$

$$\rho \frac{DW}{Dt} = \rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \Rightarrow (11)$$

Equations (9) (11) are called momentum equation or Equation of motion. The normal and shear forces are defined by the relation

$$\left.\begin{aligned} \sigma_{xx} &= -p + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial u}{\partial x} \\ \sigma_{yy} &= -p + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial v}{\partial y} \\ \sigma_{zz} &= -p + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial w}{\partial z} \end{aligned}\right\} \Rightarrow (12)$$

here $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is the equation of continuity.

μ is the dynamic viscosity of the

λ is called second coefficient of viscosity the fluid which measures the resistance of the fluid flow.

$$\left.\begin{aligned} \tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \end{aligned}\right\} \rightarrow (13).$$

We will use these values of (10) + (13) into the total energy equation.

Further,

next lecture.