Free Convection Flow along a vertical plate:

~~Abstract~~ This article investigates the free convection flow of viscous incompressible fluid along a heated vertical plate with uniform surface temperature. With the appropriate transformations, the boundary layer equations governing the flow are reduced to local nonsimilarity equations valid in free convection regim. A group of transformation is also introduced to reduce the boundary layer equations to set of ~~local~~ nonsimilar equations.

The problem of free convection flows past a vertical surfaces has been studied extensively because of its wide application in industry.

Mathematical formulation:

Consider a steady two dimensional flow of a viscous incompressible fluid along a semi-infinite vertical plate. Under the Boussinesq approximation the flows is governed by the following boundary layer equations:

Equation of continuity (Law of conservation of mass).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (1)$$

Momentum Equation (Navier-Stokes Equation).

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta_t(T - T_\infty) + g\beta_c(C - C_\infty) \longrightarrow (2)$$

Energy Equation:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \longrightarrow (3)$$

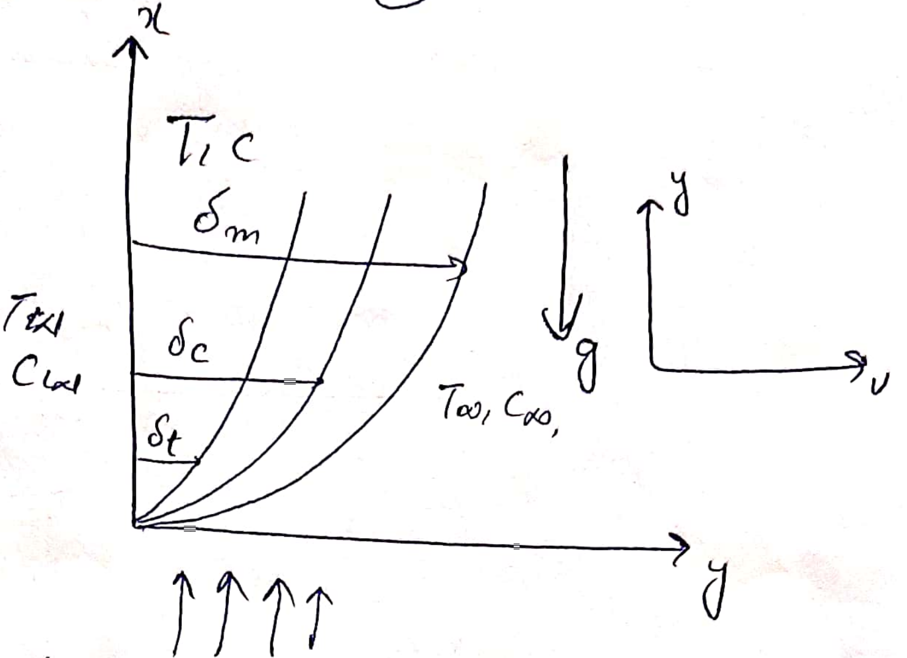
Mass concentration Equation

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2} \longrightarrow (4)$$

Subject to the boundary Conditions:

$$\begin{aligned} &u=0,\ v=0, \quad T=T_w,\ C=C_w \quad \text{at } y=0 \\ &u\to 0, \quad T\to T_\infty,\ C\to C_\infty, \quad \text{as } y\to\infty. \end{aligned} \longrightarrow (5)$$

Here, u, and v are the velocity components along x and y directions respectively. Thy symbols ν, g, β_t, and β_c are known as Kinematic viscosity, gravitation acceleration (gravitational force), thermal expansion coefficient, & concentration expansion coefficient respectively. Here, α and D_m denote thermal diffusion coefficient, and mass diffusion coefficient. where $\alpha=\frac{K}{\rho C_p}$, $\nu=\frac{\mu}{\rho}$; ρ is the density of fluid, C_p is specific heat at constant pressure, μ is dynamic viscosity.



T_w, C_w wall temperature and mass concentration. T_∞, C_∞ are ~~ar~~ free stream temperature and mass concentration. δ_t, δ_c, & δ_m are known as thermal boundary layer thickness, concentration boundary layer thickness and momentum boundary layer thickness. T and C denote the temperature and concentration of the fluid flow domain.

The temperature of the surface is held uniform at T_w which is higher than the ambient temperature T_∞. The species concentration at the surface is maintained uniform at C_w, which is taken to be C_w ~~zero~~ and that considered to be C_∞ away from the wall.

(4)

<u>Non-similar transformations</u>

Far from the leading edge, i.e in the free convection regime, the boundary layer is formed due to buoyancy forces. This suggests the following transformations:

$$\left.\begin{aligned}\psi &= \nu Gr_x^{1/4} F(\xi,\eta), \quad \eta = \frac{y}{x} Gr_x^{1/4}, \quad \xi = x \\ \theta &= \frac{T-T_\infty}{T_w - T_\infty}, \quad \phi = \frac{C-C_\infty}{C_w - C_\infty}, \quad Gr_{tx} = \frac{g_x \beta_t \Delta T x^3}{\nu^2} \\ & \qquad Gr_{cx} = \frac{g_x \beta_c \Delta C x^3}{\nu^2}\end{aligned}\right\} \rightarrow (6)$$

As we know that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

So,

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\nu Gr_x^{\frac{1}{4}} F(\xi,\eta)\right) = \nu Gr_x^{\frac{1}{4}} \frac{\partial}{\partial y} F(\xi,\eta)$$

$$u = \nu Gr_x^{\frac{1}{4}} \left(\frac{\partial F}{\partial \xi}\cdot \cancelto{0}{\frac{\partial \xi}{\partial y}} + \frac{\partial F}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$

$$u = \nu Gr_x^{\frac{1}{4}} F' \cdot \left(\frac{Gr_x^{1/4}}{x}\right)$$

$$\boxed{u = \frac{\nu Gr_x^{\frac{1}{2}}}{x} F'}$$

$$\frac{\partial u}{\partial x} = \nu \frac{\partial}{\partial x}\left[\frac{1}{x} Gr_x^{\frac{1}{2}} F'\right]$$

$$\eta = \frac{y}{x} Gr_x^{1/4}$$

$$\frac{\partial \eta}{\partial x} = y \frac{\partial}{\partial x}\left(x^{-1} Gr_x^{1/4}\right)$$

$$= y\left[-\frac{1}{x^2} Gr_x^{1/4} + \frac{\frac{1}{4} Gr_x^{\frac{1}{4}-1}}{x} \cdot \frac{\partial}{\partial x}(Gr_x)\right]$$

$$= y\left[-\frac{Gr_x^{1/4}}{x^2} + \frac{Gr_x^{\frac{1}{4}}}{4 Gr_x x} \cdot \frac{\partial}{\partial x}\left(\frac{g\beta \Delta T x^3}{\nu^2}\right)\right]$$

$$= y\left[-\frac{Gr_x^{\frac{1}{4}}}{x^2} + \frac{Gr_x^{\frac{1}{4}}}{x\, 4 Gr_x} \times \frac{g\beta\Delta T \cdot 3x^2}{\nu^2}\right]$$

$$= y\left[-\frac{Gr_x^{1/4}}{x^2} + \frac{x \cdot Gr_x^{\frac{1}{4}}}{x\, 4 Gr_x\, x} \cdot \frac{g\beta\Delta T 3x^2}{\nu^2}\right]$$

$$= y\left[-\frac{Gr_x^{1/4}}{x^2} + \frac{3 Gr_x^{1/4}}{4x^2 Gr_x} \cdot \frac{g\beta\Delta T x^3}{\nu^2}\right]$$

5

$$\frac{\partial u}{\partial x} = \nu\left[-\frac{1}{x^2}Gr_x^{\frac{1}{2}}F' + \frac{1}{x}\cdot\frac{1}{2}\frac{Gr_x^{\frac{1}{2}}}{Gr_x}\cdot\frac{g\beta\Delta T}{\nu^2}\cdot 3x^2 F' + \frac{1}{x}Gr_x^{\frac{1}{2}}\left(\frac{\partial F'}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial F'}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)\right]$$

$$\frac{\partial u}{\partial x} = \nu\left[-\frac{Gr_x^{\frac{1}{2}}}{x^2}F' + \frac{3Gr_x^{\frac{1}{2}}}{2x^2}\cdot\frac{1}{Gr_x}\frac{g\beta\Delta T x^3}{\nu^2}F' + \frac{1}{x}Gr_x^{1/2}\left(\frac{\partial F}{\partial \xi} - \frac{F''\eta}{gx}\right)\right]$$

$$\frac{\partial u}{\partial x} = \nu\left[-\frac{Gr_x^{\frac{1}{2}}F'}{x^2} + \frac{3Gr_x^{\frac{1}{2}}}{2x^2}F' + \frac{Gr_x^{1/2}}{x}\left(\frac{\partial F}{\partial \xi} - \frac{\eta F''}{gx}\right)\right]$$

$$\frac{\partial u}{\partial x} = \frac{\nu}{x^2}Gr_x^{1/2}\left[-F' + \frac{3}{2}F' + x\frac{\partial F}{\partial \xi} - \frac{\eta F''}{g}\right]$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\nu}{x^2}Gr_x^{\frac{1}{2}}\left[\frac{1}{2}F' + \xi\frac{\partial F'}{\partial \xi} + \frac{\eta F''}{g}\right]}$$

Now $$u\frac{\partial u}{\partial x} = \frac{\nu\cdot Gr_x^{\frac{1}{2}}F'}{x}\cdot\frac{\nu}{x^2}Gr_x^{\frac{1}{2}}\left(\frac{1}{2}F' + \xi\frac{\partial F'}{\partial \xi} + \frac{\eta F''}{4}\right)$$

$$u\frac{\partial u}{\partial x} = \frac{\nu^2\cdot Gr_x}{x^3}\left(\frac{1}{2}F'^2 + \xi\frac{\partial F'}{\partial \xi}F' + \frac{\eta F'F''}{g}\right)$$

$$\boxed{u\frac{\partial u}{\partial x} = \frac{\nu^2 Gr_x}{x^3}\left(\frac{1}{2}F'^2 + \xi\frac{\partial F'}{\partial \xi}F' + \frac{\eta F'F''}{g}\right)} \rightarrow (7)$$

$$\frac{\partial \eta}{\partial x} = y\left[-\frac{Gr_x^{\frac{1}{4}}}{x^2} + \frac{3}{4x^2}Gr_x^{\frac{1}{4}}\right]$$

$$\frac{\partial \eta}{\partial x} = -\frac{y\,Gr_x^{\frac{1}{4}}}{x^2} + \frac{3y\,Gr_x^{1/4}}{4x^2}$$

$$\frac{\partial \eta}{\partial x} = \left[-\frac{1}{x}\cdot\frac{y\,Gr_x^{1/4}}{x} + \frac{3}{4x}\cdot\frac{y\,Gr_x^{1/4}}{x}\right]$$

$$\frac{\partial \eta}{\partial x} = \left[-\frac{\eta}{x} + \frac{3\eta}{4x}\right]$$

$$\frac{\partial \eta}{\partial x} = \frac{-4\eta + 3\eta}{4x}$$

$$\boxed{\frac{\partial \eta}{\partial x} = +\frac{\eta}{gx}} \quad \boxed{= \frac{\eta}{2x}}$$

$$\boxed{\frac{\partial \eta}{\partial y} = \frac{Gr_x^{1/4}}{x}}$$

$$\frac{\partial u}{\partial y} = \frac{\nu Gr_x^{\frac{1}{2}}}{x}\left(\frac{\partial F'}{\partial \xi}\cdot\cancel{\frac{\partial \xi}{\partial y}}^{0} + \frac{\partial F'}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$ (6)

$$\frac{\partial u}{\partial y} = \frac{\nu Gr_x^{\frac{1}{2}}}{x} F''\cdot\left(\frac{Gr_x^{\frac{1}{4}}}{x}\right) = \frac{\nu Gr_x^{3/4}}{x^2} F''$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\nu Gr_x^{\frac{3}{4}}}{x^2} F''}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\nu\cdot Gr_x^{3/4}}{x^2}\cdot F'''\cdot\frac{Gr_x^{\frac{1}{4}}}{x} = \frac{\nu\cdot Gr_x}{x^3} F'''$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{\nu Gr_x}{x^3} F'''} \longrightarrow (8)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(\nu Gr_x^{\frac{1}{4}} F(\xi,\eta)\right)$$

$$v = -\nu\frac{\partial}{\partial x}\left(Gr_x^{\frac{1}{4}} F(\xi,\eta)\right)$$

$$v = -\nu\left[Gr_x^{\frac{1}{4}}\left(\frac{\partial F}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial F}{\partial \eta}\cdot\frac{\eta}{\partial x}\right) + \frac{1}{4}\frac{Gr_x^{\frac{1}{4}}}{Gr_x}\cdot\frac{x}{x}\frac{g\beta_t \Delta T\, 3x^2}{\nu^2}\cdot F\right]$$

$$v = -\nu\left[Gr_x^{\frac{1}{4}}\left(\frac{\partial F}{\partial \xi} + \frac{\eta}{gx} F'\right) + \frac{3\, Gr_x^{1/4} F}{4x}\right]$$

$$\boxed{v = -\nu\left[Gr_x^{1/4}\left(\frac{\partial F}{\partial \xi} + \frac{\eta F'}{gx}\right) + \frac{3}{4x} Gr_x^{1/4} F\right]}$$

Now

7

$$v\frac{\partial u}{\partial y} = -\nu\left[\frac{Gr_x^{\frac{1}{4}}}{x}\left(x\frac{\partial F}{\partial \xi}+\frac{\eta F'}{4}\right)+\frac{3}{4x}Gr_x^{1/4}F\right]\times\frac{\nu Gr_x^{3/4}}{x^2}F''$$

$$v\frac{\partial u}{\partial y} = -\frac{\nu^2 Gr_x}{x^3}\left[\xi\frac{\partial F}{\partial \xi}F''+\frac{\eta F'F''}{2}+\frac{3}{4}FF''\right]$$

$$\boxed{v\frac{\partial u}{\partial y} = -\frac{\nu^2 Gr_x}{x^3}\left(\xi F''\frac{\partial F}{\partial \xi}+\frac{\eta F'F''}{2}+\frac{3}{4}FF''\right)} \rightarrow (9)$$

$$\theta = \frac{T-T_\infty}{T_w-T_\infty} \Rightarrow \theta(T_w-T_\infty) = T-T_\infty$$

$$\Rightarrow \boxed{\theta\Delta T + T_\infty = T} = \boxed{\theta\Delta T = T-T_\infty} \rightarrow (10)$$

$$\phi = \frac{C-C_\infty}{C_w-C_\infty} \Rightarrow \phi(C_w-C_\infty) = C-C_\infty$$

$$\Rightarrow \boxed{\phi\Delta C = C-C_\infty} \rightarrow (11)$$

~~put 7—11 in equation~~

Now $\cancel{\frac{\partial u}{\partial y}=}$

put equation 7 – 11 in equation (2), we have

$$\frac{\nu^2 Gr_x}{x^3}\left[\frac{1}{2}F'^2+\xi F'\frac{\partial F'}{\partial \xi}+\cancel{\frac{\eta}{2}F'F''}^{0}-\xi F''\frac{\partial F}{\partial \xi}-\cancel{\frac{\eta F'F''}{2}}^{0}-\frac{3}{4}FF''\right]$$

$$= \nu\cdot\frac{\nu\cdot Gr_x}{x^3}F''' + g\beta_t\Delta T\theta + g\beta_c\Delta C\phi$$

(8)

$$\frac{1}{2}F'^2 + \xi F'\frac{\partial F'}{\partial \xi} - \xi F''\frac{\partial F}{\partial \xi} - \frac{3}{4}FF'' = \frac{\nu^2 Gr_x}{x^3}\cdot\cancel{\frac{x^3}{\nu^2 Gr_x}} F''' + \frac{g\beta_t \Delta T\, x^3}{\nu^2 Gr_x}\theta + \frac{g\beta_c \Delta C}{\nu^2 Gr_x}x^3$$

$$\xi\left(F'\frac{\partial F'}{\partial \xi} - F''\frac{\partial F}{\partial \xi}\right) + \frac{1}{2}F'^2 - \frac{3}{4}FF'' = F''' + \frac{\cancel{g\beta_t \Delta T\, x^3}}{\cancel{\nu^2 \cdot g\beta_t \Delta T x^3 / \nu^2}}\theta + \frac{\cancel{g\beta_c \Delta C\, x^3}}{\cancel{\nu^2 \cdot g\beta_c \Delta C x^3/\nu^2}}\phi$$

$$\xi\left(F'\frac{\partial F'}{\partial \xi} - F''\frac{\partial F}{\partial \xi}\right) + \frac{1}{2}F'^2 - \frac{3}{4}FF'' = F''' + \theta + \phi$$

$$\boxed{F''' + \frac{3}{4}FF'' - \frac{1}{2}F'^2 + \theta + \phi = \xi\left(F'\frac{\partial F'}{\partial \xi} - F''\frac{\partial F}{\partial \xi}\right)} \quad (12)$$

This is non-similar transformed momentum equation.

Now we move towards to energy Equation.

$$\frac{\partial T}{\partial x} = \Delta T\left(\frac{\partial \theta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) + \cancel{\frac{\partial T_\infty}{\partial x}}^{\,0}$$

$$\frac{\partial T}{\partial x} = \Delta T\left(\frac{\partial \theta}{\partial \xi} + \frac{\eta}{2x}\theta'\right)$$

∵ T_∞ is constt

∵ $\frac{\partial \xi}{\partial x} = 1$

∵ $\xi = x$

$$\boxed{\frac{\partial T}{\partial x} = \Delta T\left(\frac{\partial \theta}{\partial \xi} + \frac{\eta}{2x}\theta'\right)}$$

$$u\frac{\partial T}{\partial x} = \frac{\nu Gr_x^{\frac{1}{2}}\Delta T}{x}\left(F'\frac{\partial \theta}{\partial \xi} + \frac{\eta}{2x}\theta'\right)$$

$$u\frac{\partial T}{\partial x} = \frac{\nu Gr_x^{1/2}}{x}\Delta T\left(x F'\frac{\partial \theta}{\partial \xi} + \frac{\eta}{2}\theta'\right)$$

9

$$\boxed{u\frac{\partial T}{\partial x} = \frac{\nu Gr_x^{\frac{1}{2}}\Delta T}{x^2}\left(\xi F'\frac{\partial \theta}{\partial \xi} + \frac{\eta\theta'}{g}\right)} \longrightarrow 13$$

$$\frac{\partial T}{\partial y} = \Delta T\left(\frac{\partial \theta}{\partial \xi}\cdot\cancel{\frac{\partial \xi}{\partial y}}^{0} + \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial y}\right) + \cancel{\frac{\partial T}{\partial y}}^{0}$$

$$\frac{\partial T}{\partial y} = \Delta T\ \theta' \cdot \frac{Gr_x^{1/4}}{x}$$

$$\boxed{\frac{\partial T}{\partial y} = \frac{\Delta T\ Gr_x^{\frac{1}{4}}}{x}\theta'}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\Delta T\ Gr_x^{1/4}}{x}\theta'' \cdot \frac{Gr_x^{1/4}}{x}$$

$$\boxed{\frac{\partial^2 T}{\partial y^2} = \frac{\Delta T\ Gr_x^{1/2}}{x^2}\theta''} \longrightarrow 14$$

$$v\frac{\partial T}{\partial y} = -\nu\left[Gr_x^{\frac{1}{4}}\left(\frac{\partial F}{\partial \xi} + \frac{\eta F'}{2x}\right) + \frac{3}{4x}Gr_x^{1/4}F\right] \times \frac{\Delta T\ Gr_x^{1/4}}{x}\theta'$$

$$v\frac{\partial T}{\partial y} = -\frac{\nu}{x^2}Gr_x^{1/2}\left[\theta' x\frac{\partial F}{\partial \xi} + \frac{\eta F'\theta'}{2} + \frac{3}{4}F\theta'\right]$$

$$\boxed{v\frac{\partial T}{\partial y} = -\frac{\nu Gr_x^{\frac{1}{2}}}{x^2}\left[\xi\theta'\frac{\partial F}{\partial \xi} + \frac{\eta F'\theta'}{g} + \frac{3}{4}F\theta'\right]} \longrightarrow 15$$

Put eq (13) — (15) in Eq. (3), We obtain

$$\frac{\nu Gr_x^{\frac{1}{2}}\Delta T}{x^2}\left(\xi F'\frac{\partial\theta}{\partial\xi}+\frac{\eta\theta' F'}{2}\right)-\frac{\nu Gr_x^{\frac{1}{2}}}{x^2}\left(\xi\theta'\frac{\partial F}{\partial\xi}+\frac{\eta F'\theta'}{2}+\frac{3}{4}F\theta'\right)$$

$$= \cancel{\frac{\Delta T\, Gr_x^{1/2}}{x^2}}\alpha\,\theta'' \times \frac{x^2}{\nu Gr_x^{1/2}\Delta T}$$

$$\xi\left(F'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial F}{\partial\xi}\right)+\cancel{\frac{\eta\theta' F'}{2}}-\cancel{\frac{\eta F'\theta'}{2}}-\frac{3}{4}F\theta'=\frac{\alpha}{\nu}\theta''$$

$$\xi\left(F'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial F}{\partial\xi}\right)-\frac{3}{4}F\theta' = \frac{\alpha}{\nu}\theta''$$

$$\boxed{\frac{1}{Pr}=\frac{\alpha}{\nu} \quad \therefore\ Pr=\frac{\nu}{\alpha}}$$

$$\xi\left(F'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial F}{\partial\xi}\right)=\frac{1}{Pr}\theta''+\frac{3}{4}F\theta'$$

$$\boxed{\frac{1}{Pr}\theta''+\frac{3}{4}F\theta'=\xi\left(F'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial F}{\partial\xi}\right)} \rightarrow (16)$$

This is non-similar transformed form of energy equation.

Now, we transform mass concentration equation.

$$\phi=\frac{C-C_\infty}{C_w-C_\infty} \qquad \text{as } \Delta C = C_w - C_\infty.$$

$$\phi\,\Delta C = C-C_\infty$$

$$\phi\,\Delta C + C_\infty = C$$

$$\frac{\partial\phi}{\partial x}=\Delta C\left(\frac{\partial\phi}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial\phi}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}\right)+\cancel{\frac{\partial C_\infty}{\partial x}}^{\,0}$$

Scanned with CamScanner

$$\frac{\partial \phi}{\partial x} = \Delta C\left(\frac{\partial \phi}{\partial \xi} + \frac{\eta}{2x}\phi'\right)$$

$$\frac{\partial \phi}{\partial x} = \Delta C\left(\frac{\partial \phi}{\partial \xi} + \frac{\eta \phi'}{2x}\right)$$

$$u\frac{\partial \phi}{\partial x} = \frac{\nu G_{rx}^{\frac{1}{2}}}{x} F' \Delta C\left(\frac{\partial \phi}{\partial \xi} + \frac{\eta \phi'}{2x}\right)$$

$$= \frac{\nu \Delta C}{x^2} G_{rx}^{\frac{1}{2}}\left(\cancel{x} F'\frac{\partial \phi}{\partial \xi} + \eta \phi' F'\right)$$

$$\boxed{u\frac{\partial \phi}{\partial x} = \frac{\nu \Delta C}{x^2} G_{rx}^{1/2}\left(\xi F'\frac{\partial \phi}{\partial \xi} + \eta \phi' F'\right)} \rightarrow (17) \quad \xi = x$$

$$\frac{\partial \phi}{\partial y} = \Delta C\left(\frac{\partial \phi}{\partial \xi}\cdot\cancel{\frac{\partial \xi}{\partial y}}^{\to 0} + \frac{\partial \phi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) + \cancel{\frac{\partial \phi_\infty}{\partial y}}^{\to 0}$$

$$\cancel{\frac{\partial \phi}{\partial y} = \Delta C\left(\frac{\partial \phi}{\partial \xi} + \frac{\eta}{2x}\phi'\right)}$$

$$\cancel{\boxed{\frac{\partial \phi}{\partial y} = \Delta C\left(\frac{\partial \phi}{\partial \xi} + \frac{\eta \phi'}{2x}\right)} = \Delta C}$$

$$\cancel{\frac{\partial^2 \phi}{\partial y^2} = \Delta C} \quad \frac{\partial \phi}{\partial y} = \Delta C\, \phi' \cdot \frac{G_{rx}^{1/4}}{x}$$

$$\boxed{\frac{\partial \phi}{\partial y} = \frac{\Delta C\, G_{rx}^{1/4}}{x}\phi'}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\Delta C\, Gr_x^{1/4}}{x}\cdot\phi''\cdot\frac{Gr_x^{1/4}}{x} = \frac{\Delta C\, Gr_x^{1/2}}{x^2}\phi''$$

$$\boxed{\frac{\partial^2\phi}{\partial y^2} = \frac{\Delta C\, Gr_x^{\frac{1}{2}}}{x^2}\phi''} \longrightarrow (18)$$

$$\boxed{v\frac{\partial\phi}{\partial y} = -\frac{\nu Gr_x^{\frac{1}{2}}}{x^2}\left[\xi\phi'\frac{\partial F}{\partial\xi} + \frac{\eta F'\phi'}{2} + \frac{3}{4}F\phi'\right]} \longrightarrow (19)$$

Put eqs (17) — (19) into eq. (4), we get

$$\frac{\nu Gr_x^{\frac{1}{2}}\Delta C}{x^2}\left(\xi F'\frac{\partial\phi}{\partial\xi} + \frac{\eta F'\phi'}{2}\right) - \frac{\nu Gr_x^{1/2}}{x^2}\left(\xi\phi'\frac{\partial F}{\partial\xi} + \frac{\eta F'\phi'}{2} + \frac{3}{4}F\phi'\right)$$

$$= D_m\frac{\Delta C\, Gr_x^{1/2}}{x^2}\phi''$$

$$\xi\left(F'\frac{\partial\phi}{\partial\xi} - \phi'\frac{\partial F}{\partial\xi}\right) + \cancel{\frac{\eta F'\phi'}{2}} - \cancel{\frac{\eta F'\phi'}{2}} - \frac{3}{4}F\phi' = \frac{D_m}{\nu}\cdot\frac{\Delta C\, Gr_x^{\frac{1}{2}}}{x^2}\times\frac{x^2\phi''}{Gr_x^{\frac{1}{2}}\Delta C}$$

$$\xi\left(F'\frac{\partial\phi}{\partial\xi} - \phi'\frac{\partial F}{\partial\xi}\right) - \frac{3}{4}F\phi' = \frac{1}{Sc}\phi''$$

$$\boxed{\frac{1}{Sc}\phi'' + \frac{3}{4}F\phi' = \xi\left(F'\frac{\partial\phi}{\partial\xi} - \phi'\frac{\partial F}{\partial\xi}\right)}$$

where Pr, Sc are prandtl number; $Sc = \frac{\nu}{D_m}$ and Schmidt number, respectively.

B.C's Transformed boundary Conditions are

$$F(\xi,0) = F'(\xi,0) = 0, \quad \theta(\xi,0) = 1, \quad \phi(\xi,0) = 1, \quad at\ \eta = 0$$

$$F'(\xi,\infty) = 0, \quad \theta(\xi,\infty) = 0, \quad \phi(\xi,\infty) = 0, \quad as\ \eta \to \infty.$$

Here to note That:

F is dimensionless Stream function.

x Coordinate measuring distance along the plate.

y coordinate measuring distance normal to plate.

α effective thermal diffusivity ($K/\rho C_p$).

θ is dimensionless temperature.

ϕ is dimensionless mass concentration.

η is similarity variable

ξ scaled streamwise coordinate