OUTLINES

- History
- Introduction
- Types of integral equation
    - Volterra integral equation
    - Fredholm integral equation
    - Singular integral equation
    - Homogenous integral equation
- A study on various model problems as integral equations
    - Shape of an elastic thread
    - Human population
    - Mortality of equipment and rate of replacement
    - Abel's problem
- Conclusion

History:

J.Fourier (1768-1830) is the initiator of the theory of integral equations.

A term integral equation first suggested by Du Bois Reymond in 1888. Du Bois Reymond define an integral equation is understood an equation in which the unknown function occurs under one or more signs of definite integration. Late 18^{th} and early 19^{th} century Laplace, Fourier, Poisson, Liouville and Able studies some special type of integral equation. The pioneering systematic investigation goes back to late 19^{th} and 20^{th} century work of Volterra, Fredholm and Hilbert. In 1887, Volterra published a series of famous papers in which he singled out the notion of a functional and pioneered in the development of a theory of functional in theory of linear integral equation of special type. Fredholm presented the fundamentals of the Fredholm integral equation theory in a paper published in 1903 in the Acta Mathematica. This paper became famous almost overnight and soon took its rightful place among the gems of the modern mathematics.

By contrast with the differential equation, which got off a flying start with Isaac Newton's second law of motion, integral equation arrived late. They made their first appearance during the 3^{rd} and 4^{th} decade of the 19^{th} century. Even the name 'integral equation' was not suggested until the late 1880's and it was adopted only in the early 1900's. In the last four or five years of the 19^{th} century Vito Volterra and Ivar Fredholm succeeded in working out fundamental linear theories of two types which have since carried their name.

Introduction:

An integral equation is a mathematical expression that includes a required function under an integration sign.

The integral equation

$$u(x) = f(x) + \int \mathrm{K}(x,t)u(\mathrm{t})\,\mathrm{dt} \qquad (1)$$

may be written in the operational (abbreviated) form or notation as

$$u(x) = f(x) + (\mathrm{Ku})(\mathrm{x})$$

or $$\mathrm{u} = \mathrm{f} + K\,\mathrm{u}$$

Where ′ k′ is an integral operator for the integral in (1) that maps the function, u, as an input to an output

$$(Ku)(x) = \int \mathrm{K}(x,t)u(\mathrm{t})\,\mathrm{dt}$$

in the range of the integral operator K.

The most **general linear integral equation** in u(x) can be presented as

$$h(x)u(x) = f(x) + \int_a^{b(x)} K(x,\xi)u(\xi)\,d\xi \qquad (2)$$

or in operational notation

$$h(x)u(x) - f(x) = (\mathrm{Ku})(\mathrm{x}) , \quad \mathrm{a} \le \mathrm{x}$$

Where K defines the above integration operation on the function u in (2).

Types of Integral Equation:

Integral equation fall under two main categories

- **Volterra Integral Equation**

An integral equation with variable limits of integration is called Volterra integral equation. The equation (2) is called Volterra integral equation when b(x)=x, i.e

$$h(x)u(x) = f(x) + \int_a^x K(x,\xi)\,u(\xi)d\xi$$

Further there are two kinds of Volterra integral equation:

- When h(x)=0 it is called a Volterra integral equation of the **first kind**,

$$-f(x) = \int_a^x K(x,\xi)u(\xi)d\xi$$

- When h(x)=1 it is called a Volterra integral equation of the **second kind**,

$$u(x) = f(x) + \int_a^x K(x,\xi)u(\xi)d\xi$$

Examples:

Abel's integral equation in ϕ(y)

$$-\sqrt{2g}\,f(y) = \int_0^y \frac{\phi(\eta)}{\sqrt{y-\eta}}d\eta$$

is integral equation of the first kind.

The integral equation for the torsion of wire

$$m(t) = h\omega(t) + \int_{-\infty}^t \phi(t,\tau)\omega(\tau)d\tau$$

is the Fredholm integral equation of the second kind.

- ## Fredholm Integral Equation

An integral equation with fixed limits of integration is called Fredholm integral equation. The equation (2) is called Fredholm integral equation if b(x)=b, a constant,

$$h(x)u(x) = f(x) + \int_a^b K(x,\xi)u(\xi)d\xi$$

Further there are two types of Fredholm of integral equation:

- When h(x)=0 it is called Fredholm integral equation of the **first kind,**

$$-f(x) = \int_a^b K(x,\xi)u(\xi)d\xi$$

- When h(x)=1 it is called Fredholm integral equation of the **second kind,**

$$u(x) = f(x) + \int_a^b K(x,\xi)u(\xi)d\xi$$

Examples:

The Fourier transform

$$U(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}u(x)dx$$

is the Fredholm integral equation of first kind.

The integral equation

$$y(x) = \omega^2 \int_0^l F(x,\xi)\rho(x)y(\xi)d\xi$$

is the Fredholm integral equation of the second kind.

Homogeneous Integral Equation:

In the case either Volterra or the Fredholm integral equation, the integral equation is termed homogenous when f(x)=0 in equation (2),

$$h(x)u(x) = \int_a^x K(x,\xi)u(\xi)d\xi$$

$$h(x)u(x) = \int_a^b K(x,\xi)u(\xi)d\xi$$

Examples:

Homogenous Volterra integral equation is the Bernoulli equation

$$kf(x) = \int_0^x f(\xi)d\xi$$

While the deflection of the rotating shaft in y(x),

$$y(x) = \omega^2 \int_0^l F(x,\xi)\rho(x)y(\xi)d\xi$$

is a homogenous Fredholm equation.

Singular integral equation:

An integral equation is termed as singular if the range of the integration is infinite or the kernel **K(x,ξ)** becomes infinite in the range of integration.

The Fourier integral in u(x),

$$U(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}u(x)dx$$

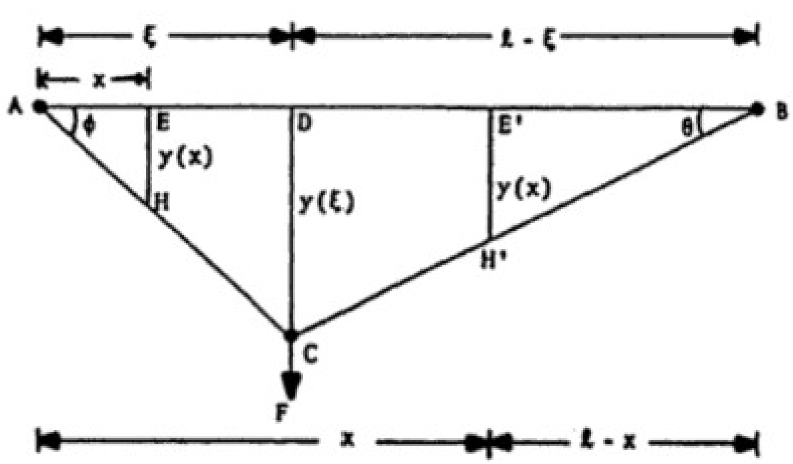
is singular.

A study on various model problems as integral equations:

- **Shape of an elastic thread (The hanging chain)**

An example of a physical problem that results naturally in an integral equation is to find how a variable density ρ(x) must be distributed along an elastic thread in order that the thread assumes a given shape f(x).

First we consider an elastic string under an initial constant tension T_0 and a vertical force F acting at a one point. Then we derive the equation for the case of distributed forces along the string, for example, the variable gravitational force due to a variable linear density of the string.



Displace due to a single vertical force:

Consider the string AB of length 'l' under initial constant tension T_0. We take y(x) to be positive in the downward direction of gravity. Let F be a constant vertical force acting on the distance x=ξ to displace it by a small vertical distance y(ξ) which is very small compared to ξ. If we equate the vertical forces assuming that the tension is constant (T_0) along the string, we have

$$F=T_0\sin\phi+T_0\sin\theta \qquad (1)$$

when θ *and* ϕ *are very small*

$$\sin\theta \approx \tan\theta \quad and \quad \sin\phi \approx \tan\phi$$

$$\sin\phi = \frac{y(\xi)}{\xi} \quad , \quad \sin\theta = \frac{y(\xi)}{l-\xi}$$

equation (1) *becomes*

$$F = \frac{T_0 y(\xi)}{\xi} + \frac{T_0 y(\xi)}{l-\xi}$$

$$F = T_0 y(\xi)\frac{(l-\xi+\xi)}{\xi(l-\xi)}$$

$$F = \frac{T_0 l}{\xi(l-\xi)}\mathrm{y}(\xi)$$

$$y(\xi) = \frac{F}{T_0 l}\xi(l-\xi) \qquad (2)$$

In similar triangles ratios of lengths of corresponding sides are equal. Thus, we consider the similar triangles ACD and AHE

For $x \le \xi$

$$\frac{y(x)}{x} = \frac{y(\xi)}{\xi}$$

$$y(x) = \frac{x}{\xi}y(x)$$

from equation (2)

$$y(x) = \frac{xF}{\xi T_0 l}\xi(l-\xi)$$

$$y(x) = F\frac{x(l-\xi)}{T_0 l} \quad , \qquad 0 \le x \le \xi$$

For $\xi \le x \le l-\xi$

$$\frac{y(x)}{y(\xi)} = \frac{(l-x)}{(l-\xi)}$$

$$y(x) = y(\xi)\frac{(l-x)}{(l-\xi)}$$

From equation (2)

$$y(x) = \frac{(l-x)}{(l-\xi)}\frac{F}{T_0 l}\xi(l-\xi)$$

$$y(x) = F\frac{\xi(l-x)}{T_0 l}$$

Thus

$$y(x) = FG(x,\xi)$$

$$y(x) = F\begin{cases} \frac{x(l-\xi)}{T_0 l}, & 0 \le x \le \xi \\ \frac{\xi(l-x)}{T_0 l}, & \xi \le x \le l \end{cases} \qquad (3)$$

It is important to note the two branches of the function G(x,ξ), where the first branch satisfies the boundary condition y(0)=0, for the first end of the elastic string at x=0 to be fixed.

While the second branch satisfies the boundary condition y(l)=0, for a fixed end at x=l.

Displacement due to distributed vertical force:

We now consider the vertical force not at one point x=ξ only, but distributed continuously along the string, for example the gravitational force due to the variable linear density ρ(ξ) of a string. For such a string the gravitational force acting on the element △ξ of the string is $\Delta F(\xi) = g\rho(\xi)\Delta\xi$. According to equation (3)

$$\Delta y(x) = \Delta F(\xi)G(x,\xi) = G(x,\xi)g\rho(\xi)\Delta\xi \qquad (4)$$

$$where\ G(x,\xi)\ is\ \ given\ by\ \ equation\ (3).$$

The total displacement due to the gravity force along the whole string is obtained by superimposing all these displacements of the elements of the string or in other words integrating from ξ=0 to ξ=l,

$$y(x) = g\int_0^l G(x,\xi)\rho(\xi)d\xi$$

This is a Fredholm integral equation of the first kind in ρ(x) that relates how the linear density must be distributed along the string so that the string may assume the prescribed shape y(x).

Human population:

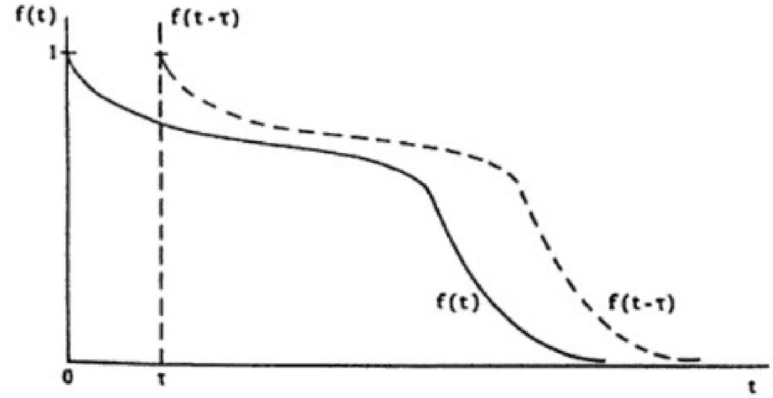
The study of population growth includes the forecasting of any future surge in birthrates, which is of great importance for future planning throughout the world.

Let the number of people present at time t=0 by n_0. If we look at survival or insurance tables, we find that there is some sort of a survival function f(t) similar to that shown in figure, which gives the fraction of people surviving to age t. It is assumed that these people are either male or female. The surviving population $n_s(t)$ at time t is then

$$n_s(t) = n_0 f(t) \qquad (1)$$

where $n_s(0) = n_0 f(0) = n_0.$

Under normal circumstances there is a continuous addition to the population through new births. If children are born at an average rate r(t), then in a particular time interval $\Delta_i \tau$ about the time τ_i, there are $r(\tau_i)\Delta_i \tau$ added who, if they survive, will be of age $t - \tau_i$ at time t.



But according to figure only a fraction $f(t-\tau_i)$ of these children will survive to age $(t-\tau_i)$, so the final addition to the population at time t, from the children born in the interval $\Delta_i\tau$ about time τ_i is

$$f(t-\tau_i)r(\tau_i)\Delta_i\tau$$

Now if this process is repeated for all the m subintervals of the time interval (0,t), we obtain the partial sum

$$b_m(t)=\sum_{i=1}^{m} f(t-\tau_i)r(\tau_i)\Delta_i\tau \qquad (2)$$

As the number of people added through new births which, if passed to the limit (as m→∞), becomes the integral

$$b(t)=\int_0^t f(t-\tau)r(\tau)\,\mathrm{d}\,\tau. \qquad (3)$$

If this is added to $n_s(t)$ in equation (1) (the survivors of the initial population), we obtain the total population at time t as

$$n(t)=n_s(t)+b(t)=n_0 f(t)+\int_0^t f(t-\tau)r(\tau)d\tau \qquad (4)$$

It is reasonable now to assume that the rate of birthrate $r(\tau)=\dfrac{dn}{dt}$ is proportional to n(t), the number of the population present at time t,

$$r(t) = k\, n(t) \qquad (5)$$

From equations (3) and (5), it follows that

$$n(t) = n_0 f(t) + k\int_0^t f(t-\tau)\,\mathrm{n}(\tau)\,d\tau$$

Which is a Volterra integral equation of the second kind in n(t) with a difference kernel $k\, f(t-\tau)$.

Mortality of Equipment and Rate of Replacement:

Now we formulate the problem of finding the rate dr/dt at which equipment should be replaced, to keep a specified number f(t) in operating system at any time t.

We first assume that we have s(t), the function that determines the number of pieces of new equipment bought at t=0 that survives to time t. If we start with f(0) as the number of new pieces bought at time t=0, then due to loss or wear , only the fraction $f(0)s(t)$ will survive to time t.

To keep a specified number larger than $f(0)s(t)$ at time t we must continuously add equipment at the desired rate from time t=0 to time t. If the desired rate of replacement at which we must add new equipment at time τ is

$$\frac{dr(\tau)}{d\tau} \text{ ,}$$

then at time t this equipment will be of age $t-\tau$ with a survival function

$$s(t-\tau)$$

that is dependent on their age $t-\tau$ .From

$(\frac{dr}{d\tau})\Delta\tau$, what we replace in time interval $\Delta\tau$, only a fraction

$$s(t-\tau)(\frac{dr}{d\tau})\Delta\tau$$

will survive to time t.

Hence if these survivals of the continuous replacements are added along the time interval (0,t), we obtain

$$r(t)=0\int_0^t s(\mathrm{t}-\tau)\frac{dr}{d\tau}d\tau\ ,\quad t\succ 0$$

The number of pieces of equipment surviving to time t, which were purchased as replacements during the time $0\prec\tau\prec t$.

If we add this to $f(0)s(t)$, the surviving number of pieces of original equipment (new at time t=0), we obtain the desired total number of pieces of equipment in operating condition at time t,

$$f(t)=f(0)s(t)+\int_0^t s(t-\tau)\frac{dr}{d\tau}d\tau$$

Which is Volterra integral equation of the first kind in the unknown rate of replacement $\frac{dr}{dt}$.

Abel's Problem: Sliding a Bead along a Wire

Abel's problem is one of the earliest problems modeled as an integral equation. It deals with finding the path y(x) in the vertical xy-plane along which a particle, under the influence of gravity and starting from rest at y_0, must move in order that it descends a distance y_0 in a prescribed time $t=f(y_0)$.

To simplify problem, we consider the path of the particle to be known when we know α, the angle that the tangent to the path makes with the x axis. In this case

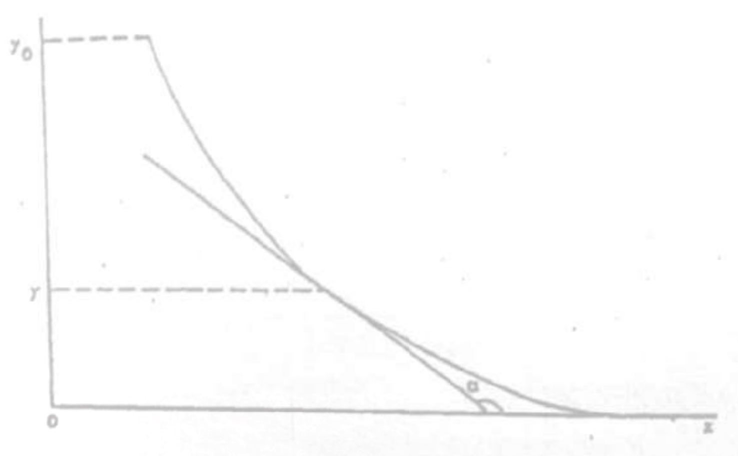
$$\frac{dy}{dx} = \tan\alpha$$

$$so \quad \frac{dy}{ds} = -\sin\alpha \text{ , } where \quad v = \frac{ds}{dt}.$$

For particle starting from rest at $y=y_0$, under gravity, the velocity v at y is governed by

$$v^2 = 2g(y_0 - y)$$

$$v = \frac{ds}{dt} = \sqrt{2g(y_0 - y)} \qquad (1)$$



Where g is the acceleration of gravity. To have the desired expression for dt, we write

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = -\sqrt{2g(y_0 - y)}\sin\alpha$$

$$dt = \frac{-dy}{\sqrt{2g(y_0 - y)}\sin\alpha} \qquad (2)$$

Realizing that α depends on y, we let

$$\frac{1}{\sin\alpha} = \phi(y) \;\; \textit{in equation } (2)\text{, then}$$

$$\mathrm{dt} = \frac{-\phi(y)\,\mathrm{dy}}{\sqrt{2g(y_0 - y)}}$$

And integrate from initial time of descent $t(y_0) = f(y_0)$ to the final time $t(y = 0) = 0$.

$$t\Big|_{t(y_0)}^{t(0)} = -\int_{y_0}^{0} \frac{\phi(y)\,\mathrm{dy}}{\sqrt{2g(y_0 - y)}}$$

$$0 - t(y_0) = -f(y_0) = -\int_{y_0}^{0} \frac{\phi(y)\,\mathrm{dy}}{\sqrt{2g(y_0 - y)}} \qquad (3)$$

Hence equation (3) is the final integral equation in φ(y) that relates the form of the path φ(y) to the predetermined time of descent $f(y_0)$ of the particle,

$$-\sqrt{2g}\,f(y_0) = \int_0^{y_0} \frac{\phi(\mathrm{y})dy}{\sqrt{y_0 - y}} \qquad (4)$$

To avoid having the variable y_0 looks like a constant, we replace the two variables y_0 and y by y and η, respectively, to write equation (4) in the form of Abel's integral equation

$$-\sqrt{2g}\,f(y) = \int_0^{y_0} \frac{\phi(\eta)d\eta}{\sqrt{y - \eta}}$$

We note that taking the final time $t(y = 0) = 0$, we are making a negative initial time $t(y_0) = f(y_0) \prec 0$.

Conclusion:

Integral equations are very important in real life. We formulated different modeled problems as integral equations of different kinds. We represented a human population, the hanging chain and the Abel's problem as integral equations. There are different methods to solve these integral equations.

There are many other problems that are modeled as integral equations that include the propagation of nervous impulse, the smoke filtration in a cigarette, and the chance to find a time gape T in order to cross a dense traffic.