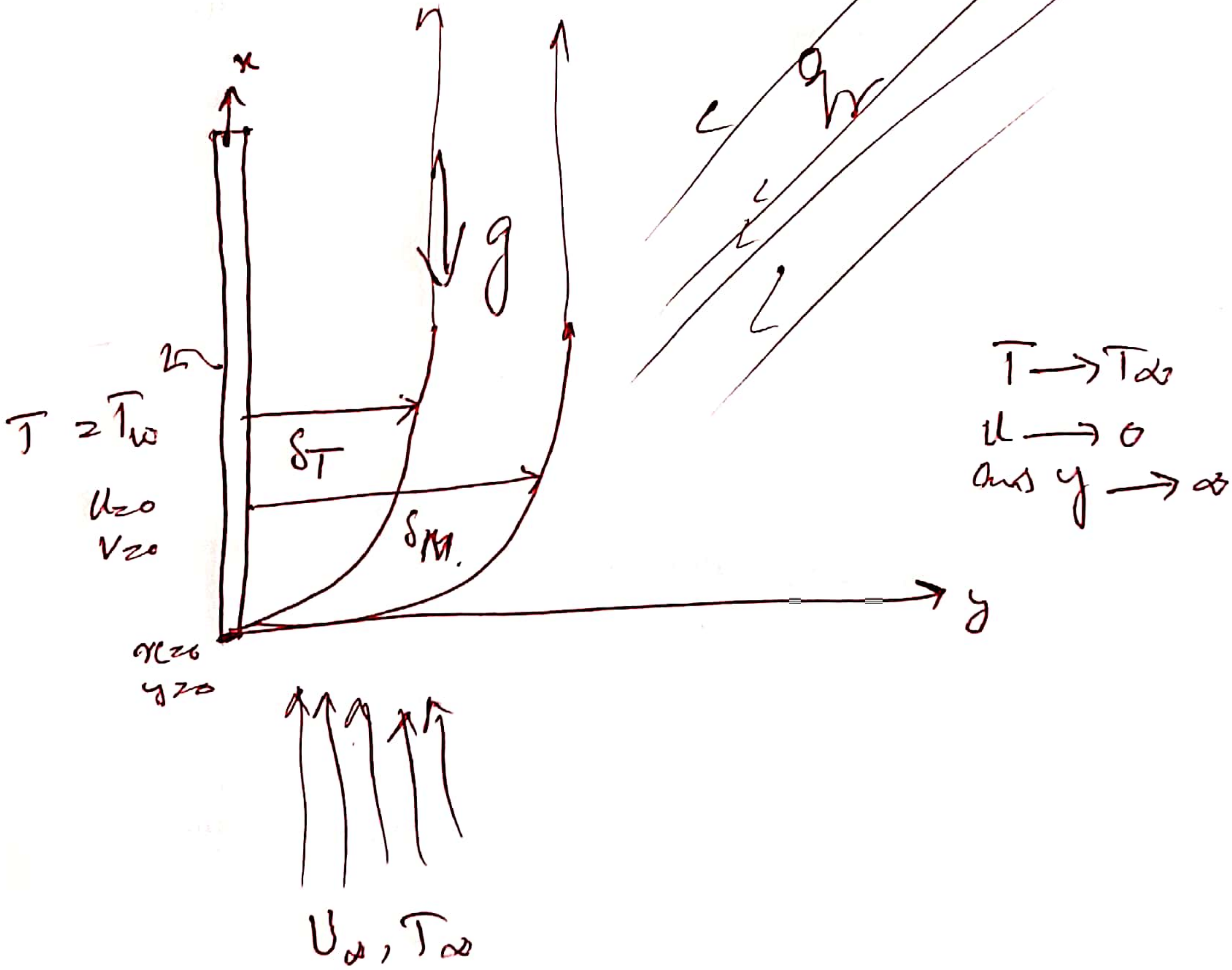
Non-similar Solution for NATURAL Convection Heat Transfer Boundary layer flow over a vertical plate



⇒ Geometric Interpretation of the Heat Transfer Model

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~~Heat transfer from a sphere with Radiation effects~~ (a vertical surface)

Non-similar Solution ~~of~~ or natural convection Boundary layer flow over a vertical plate.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \longrightarrow (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta_t(T-T_\infty) \longrightarrow (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} - \frac{K}{K\rho c_p}\frac{\partial q_r}{\partial y} \longrightarrow (3)$$

$$u=0,\ v=0,\ q_w = -K\frac{\partial T}{\partial y} \text{ at } y=0$$

$$u \to 0,\ T \to T_\infty \text{ as } y \to \infty. \longrightarrow (4)$$

Here $q_r = -\dfrac{4\sigma_1}{3K_1}\dfrac{\partial T^4}{\partial y} \longrightarrow (6)$

This is Rosseland approximation of radiative heat flux q_r.

By expanding T^4 in a Taylor series about a free stream temperature T_∞, and neglecting high order terms to yield.

$$T^4 = 4T_\infty^3 T - 3T_\infty^4 \longrightarrow (7)$$

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using Equation (6) and (7) in last term of Equation (3). we obtain.

$$q_r \cancel{\frac{\partial q_r}{\partial y}} = -\frac{4\sigma_1}{3K_1}\frac{\partial}{\partial y}\left(4T_\infty^3 T - 3T_\infty^4\right)$$

$$q_r = -\frac{4\sigma_1 \, 4 \cdot T_\infty^3}{3K_1} \cdot \frac{\partial(T)}{\partial y} = -\frac{16\sigma_1 T_\infty^3}{3K_1}\frac{\partial T}{\partial y}$$

$$q_r = -\frac{16\sigma_1 T_\infty^3}{3K_1} \cdot \frac{\partial T}{\partial y} \rightarrow (8)$$

It is convenient to transform Eqs(1)–(5) by using the following non similarity transformation.

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}(Gr)^{1/5}, \quad \psi = \nu\xi(Gr)^{4/5} f(\xi, \eta)$$

$$\theta(\xi,\eta) = \frac{K\, Gr^{1/5}(T - T_\infty)}{a q_w}, \quad C(\xi,\eta) = \frac{D(Gr)^{1/5}(C - C_\infty)}{a m_w} \rightarrow (9)$$

where $Gr = \dfrac{g\beta_t q_w a^4}{K \nu^2}$ is Grashof number

and ψ is the dimensional stream function defined in the usual way $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$. So the equation of continuity is identically satisfied and K is thermal conductivity

(3)

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\nu \xi\, Gr^{1/5} f(\xi,\eta)\right)$$

$$u = \nu \xi\, Gr^{1/5}\left(\frac{\partial f}{\partial \xi}\cdot\cancel{\frac{\partial \xi}{\partial y}}^{\,0} + \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$

$$u = \nu \xi\, Gr^{1/5} f' \cdot \frac{Gr^{1/5}}{a}$$

$$\boxed{u = \frac{\nu \xi}{a} Gr^{2/5} f'}$$

$$\eta = \frac{y}{a} Gr^{1/5}$$

$$\frac{\partial \eta}{\partial y} = \frac{Gr^{1/5}}{a}$$

$$\frac{1}{5}+\frac{1}{5} = \frac{2}{5}$$

$$\frac{\partial u}{\partial x} = \nu\, Gr^{2/5} \frac{\partial}{\partial x}(\xi f')$$

$$\frac{\partial u}{\partial x} = \frac{\nu\, Gr^{2/5}}{a}\left[\xi\left(\frac{\partial f'}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \cancel{\frac{\partial f'}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}}^{\,0}\right) + \frac{\partial \xi}{\partial x}\cdot f'\right]$$

$$\eta = \frac{y}{a} Gr^{1/5}$$

$$\frac{\partial \eta}{\partial x} = 0.$$

$$\frac{\partial u}{\partial x} = \frac{\nu\, Gr^{2/5}}{a}\left[\xi \frac{\partial f'}{\partial \xi}\cdot\frac{1}{a}\right] + \frac{1}{a} f'\Big]$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\nu\, Gr^{2/5}}{a^2}\left[\xi \frac{\partial f'}{\partial \xi} + f'\right]}$$

$$\frac{2}{5}+\frac{2}{5} = \frac{4}{5}$$

$$u\frac{\partial u}{\partial x} = \frac{\nu\cdot\xi}{a^3} Gr^{2/5} f' \cdot \frac{\nu\, Gr^{2/5}}{a}\left[\xi \frac{\partial f'}{\partial \xi} + f'\right]$$

$$\boxed{u\frac{\partial u}{\partial x} = \frac{\nu^2 Gr^{4/5}\,\xi}{a^3}\left[\xi f' \frac{\partial f'}{\partial \xi} + f'^2\right]} \longrightarrow (10)$$

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$$\frac{\partial u}{\partial y} = \frac{\nu \xi}{a} Gr^{2/5} \frac{\partial}{\partial y} f' = \frac{\nu \xi Gr^{2/5}}{a} \cdot \left(\cancel{\frac{\partial f'}{\partial \xi} \cdot \frac{\partial \xi}{\partial y}}^{0} + \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right)$$

$$\frac{\partial u}{\partial y} = \frac{\nu Gr^{2/5} \xi}{a} f'' \cdot \frac{Gr^{1/5}}{a} = \frac{\nu \xi Gr^{3/5}}{a^2} \cdot f''$$

$$\frac{2}{5} + \frac{1}{5} = 3/5$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\nu \xi Gr^{\frac{3}{5}}}{a^2} f''}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\nu \xi Gr^{3/5}}{a^2} \cdot \left(\cancel{\frac{\partial f''}{\partial \xi} \cdot \frac{\partial \xi}{\partial y}}^{0} + \frac{\partial f''}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\nu \xi Gr^{3/5}}{a^2} \cdot f''' \cdot \frac{Gr^{\frac{1}{5}}}{a} = \frac{\nu \xi Gr^{\frac{4}{5}}}{a^2}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{\nu \xi Gr^{4/5}}{a^3} f'''} \rightarrow (4)$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$v = -\frac{\partial}{\partial x} \left(\nu \xi Gr^{\frac{1}{5}} f(\xi, \eta) \right)$$

$$v = -\nu Gr^{1/5} \frac{\partial}{\partial x} \left(\xi f(\xi, \eta) \right)$$

$$v = -\nu Gr^{\frac{1}{5}} \left[\xi \left(\frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \right) + f \frac{\partial \xi}{\partial x} \right]$$

$$\xi = \frac{x}{a}$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{a}$$

$$v = -\nu Gr^{\frac{1}{5}}\left[\xi\left(\frac{\partial f}{\partial \xi}\frac{1}{a}\right) + f\,\cancel{\partial \xi}\frac{1}{a}\right]$$

$$\boxed{v = -\frac{\nu Gr^{\frac{1}{5}}}{a}\left(\xi\frac{\partial f}{\partial \xi} + f\right)}$$

$$v\frac{\partial u}{\partial y} = -\frac{\nu Gr^{\frac{1}{5}}}{a}\left(\xi\frac{\partial f}{\partial \xi} + f\right) \times \frac{\nu \xi Gr^{3/5}}{a^2} f''$$

$$\boxed{v\frac{\partial u}{\partial y} = -\frac{\nu^2 \xi Gr^{4/5}}{a^3}\left(\xi f''\frac{\partial f}{\partial \xi} + ff''\right)} \rightarrow (12)$$

Now putting these derivatives in eq (2), we obtain.

$$\frac{\nu^2 Gr^{4/5}}{a^3}\xi\left(\xi f'\frac{\partial f'}{\partial \xi} + f'^2\right) - \frac{\nu^2 Gr^{4/5}\xi}{a^3}\left(\xi f''\frac{\partial f}{\partial \xi} + ff''\right)$$

$$= \frac{\nu \cdot \nu \xi Gr^{4/5}}{a^3} f''' + \frac{g\beta a q_w}{K Gr^{1/5}}\theta$$

$$\left(\xi f'\frac{\partial f'}{\partial \xi} + f'^2 - \xi f''\frac{\partial f}{\partial \xi} - ff''\right) \times \frac{\nu^2 Gr^{4/5}\xi}{a^3} = \frac{\nu^2 \xi Gr^{4/5} f'''}{a^3} + \frac{g\beta q_w a}{K Gr^{1/5}}\theta$$

(b)

$$\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) + f'^2 - ff'' = \frac{\nu^2 \cancel{Gr^{4/5}\xi}}{\cancel{a^3}} \times \frac{\cancel{a^3}}{\nu^2 \cancel{Gr^{4/5}\xi}} f''' + \frac{a^3}{\nu^2 Gr^{4/5}\xi} \cdot \frac{g\beta q_w a}{k Gr^{1/5}}\theta$$

$$\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) + f'^2 - ff'' = f''' + \frac{a^4 . g\beta_t q_w}{\nu^2 . \xi Gr\, k}\theta .$$

$$\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) + f'^2 - ff'' = f''' + \frac{a^4 . g\beta_t q_w}{\cancel{\nu^2} \cdot \xi \cancel{k} \cdot \frac{g\beta_t q_w a^4}{k \nu^2}}\theta .$$

$$\boxed{\begin{aligned} \theta &= \frac{k\, Gr^{1/5}(T - T_\infty)}{a q_w} \\ \left(a q_w / k Gr^{1/5}\right)\theta &= T - T_\infty \end{aligned}}$$

$$= f''' + \frac{1}{\xi}\theta$$

$$\Rightarrow \quad \xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) = f''' + ff'' - f'^2 + \frac{1}{\xi}\theta$$

$$\boxed{f''' + ff'' - f'^2 + \frac{1}{\xi}\theta = \xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right)}$$

This is non-similar form of momentum Equation

Now we transform Energy equation

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(\frac{a q_w}{k Gr^{1/5}} . \theta + T_\infty\right)$$

$$\frac{\partial T}{\partial x} = \frac{a q_w}{k Gr^{1/5}} \frac{\partial}{\partial x}(\theta) + \cancel{\frac{\partial}{\partial x}(T_\infty)}^{\,0}$$

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$$\frac{\partial T}{\partial x} = \frac{a q_w}{K Gr^{1/5}}\left(\frac{\partial \theta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \theta}{\partial \eta}\cdot\cancel{\frac{\partial \eta}{\partial x}}^{0}\right)$$

$$\frac{\partial T}{\partial x} = \frac{a q_w}{K Gr^{1/5}}\cdot\frac{\partial \theta}{\partial \xi}\frac{1}{a} \Rightarrow \boxed{\frac{\partial T}{\partial x} = \frac{a q_w}{K Gr^{\frac{1}{5}} a}\cdot\frac{\partial \theta}{\partial \xi}}$$

$$u\frac{\partial T}{\partial x} = \frac{\nu \xi Gr^{2/5}}{\cancel{a}}\cdot\frac{a q_w}{K Gr^{1/5} a\ a}\cdot f'\ \frac{\partial \theta}{\partial \xi}$$

$$\boxed{u\frac{\partial T}{\partial x} = \frac{\nu \xi Gr^{1/5} q_w}{K\ a} f'\frac{\partial \theta}{\partial \xi}} \rightarrow 13$$

$$\boxed{\frac{2}{5}-\frac{1}{5} \quad \frac{2-1}{5}=\frac{1}{5}}$$

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{a q_w}{K Gr^{1/5}}\theta + T_\infty\right)$$

$$\frac{\partial T}{\partial y} = \frac{a q_w}{K Gr^{1/5}}\left(\frac{\partial \theta}{\partial \xi}\cdot\cancel{\frac{\partial \xi}{\partial y}}^{0} + \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) + \left(\text{---}\cdot\cancel{\frac{\partial T_\infty}{\partial y}}^{0}\right)$$

$$\frac{\partial T}{\partial y} = \frac{\cancel{a} q_w}{K \cancel{Gr^{1/5}}}\ \theta'\cdot\frac{\cancel{Gr^{1/5}}}{\cancel{a}}$$

$$\boxed{\frac{\partial T}{\partial y} = \frac{q_w}{K}\theta'}$$

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$$v\frac{\partial T}{\partial y} = -\frac{\nu\, Gr^{1/5}}{a}\left(\xi\frac{\partial f}{\partial \xi} + f\right) \times \frac{q_w}{k}\,\theta'$$

$$\boxed{v\frac{\partial T}{\partial y} = \frac{-\nu q_w\, Gr^{1/5}}{a k}\left(\xi\,\theta'\frac{\partial f}{\partial \xi} + f\theta'\right)} \rightarrow (14)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{q_w}{k}\cdot\left(\frac{\partial \theta'}{\partial \xi}\cdot\cancel{\frac{\partial \xi}{\partial y}}^{\,0} + \frac{\partial \theta'}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{q_w}{k}\,\theta''\cdot\frac{Gr^{\frac{1}{5}}}{a}$$

$$\boxed{\frac{\partial^2 T}{\partial y^2} = \frac{q_w}{k a}\, Gr^{1/5}\,\theta''} \longrightarrow (15)$$

Now.

$$\frac{\partial q_r}{\partial y} = \frac{-16\sigma_1 T_\infty^3}{3k_1}\; \frac{q_w}{k}\frac{\partial}{\partial y}\theta' = \frac{-16\sigma_1 T_\infty^3 q_w}{3k_1 k}\,\theta''\,\frac{\partial \eta}{\partial y}$$

$$\boxed{\frac{\partial q_r}{\partial y} = -\frac{16\sigma_1 T_\infty^3}{3k_1 k}\, q_w\, \theta''\, \cancel{\frac{\partial}{\partial y}(\theta')}\; \frac{Gr^{\frac{1}{5}}}{a}} \longrightarrow \cancel{(16)}$$

$$\Rightarrow \boxed{\frac{\partial q_r}{\partial y} = \frac{-16\sigma_1 T_\infty^3 q_w Gr^{\frac{1}{5}}}{3k_1 k\, a}\cdot\theta''} \rightarrow 16$$

put eqs (13) — (16) in eq. (3)

we obtain

As Energy Equation is

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}-\frac{1}{K\rho c_p}\frac{\partial q_r}{\partial y}$$

We put the $u\frac{\partial T}{\partial x}$, $v\frac{\partial T}{\partial y}$, $\frac{\partial^2 T}{\partial y^2}$ and $\frac{\partial q_r}{\partial y}$

in it, the we obtain.

$$\frac{\nu\xi Gr^{\frac{1}{5}} q_w}{Ka} f'\frac{\partial\theta}{\partial\xi}-\frac{\nu q_w Gr^{1/5}}{Ka}\left(\xi\theta'\frac{\partial f}{\partial\xi}+f\theta'\right)$$

$$=\frac{\alpha . q_w Gr^{1/5}}{Ka}\theta''+\frac{1}{\rho c_p}.\frac{16\sigma_1 T_\infty^3}{3K_1K}\frac{q_w}{a}Gr^{\frac{1}{5}}\theta''$$

$$\frac{\nu Gr^{1/5} q_w}{Ka}\left(\xi f'\frac{\partial\theta}{\partial\xi}-\xi\theta'\frac{\partial f}{\partial\xi}-f\theta'\right)=\frac{\alpha q_w Gr^{\frac{1}{5}}}{Ka}\theta''+\frac{16\sigma_1 T_\infty^3}{3\rho c_p K_1 K}\frac{q_w Gr^{1/5}}{a}\theta''$$

$$\xi\left(f'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial f}{\partial\xi}\right)-f\theta'=\frac{\alpha}{\nu}\theta''+\frac{K\,16\sigma_1 T_\infty^3}{3K_1K\rho c_p\,\nu}.\theta''$$

$$\xi\left(f'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial f}{\partial\xi}\right)-f\theta'=\frac{\alpha}{\nu}\theta''+\frac{K}{\rho c_p}.\frac{4.4\sigma_1 T_\infty^3}{3K_1K\,\nu}\theta''$$

$$=\frac{\nu}{\nu}\theta''+\frac{\alpha.4.4\sigma_1 T_\infty^3}{3K_1K\nu}.\theta''$$

$$\xi\left(f'\frac{\partial\theta}{\partial\xi} - \theta'\frac{\partial f}{\partial\xi}\right) - f\theta' = \frac{\alpha}{\nu}\left[\theta'' + \frac{4}{3}\cdot\frac{4\sigma_1 T_\infty^3}{k_1 k}\cdot\theta''\right]$$

$$= \frac{1}{Pr}\theta''\left(1 + \frac{4}{3R_d}\right)$$

$$\xi\left(f'\frac{\partial\theta}{\partial\xi} - \theta'\frac{\partial f}{\partial\xi}\right) - f\theta' = \frac{1}{Pr}\left(1 + \frac{4}{3R_d}\right)\theta''$$

$$\boxed{\frac{1}{Pr}\left(1 + \frac{4}{3R_d}\right)\theta'' + f\theta' = \xi\left(f'\frac{\partial\theta}{\partial\xi} - \theta'\frac{\partial f}{\partial\xi}\right)}$$

Where $R_d = \dfrac{kk_1}{4\sigma_1 T_\infty^3}$

a radiation parameter.

Transformed boundary conditions.

$f = 0,\ f' = 0,\ \theta = 1$ at $\eta = 0$

$f' \to 0,\ \theta \to 0,$ as $\eta \to \infty$

η_t	f''	θ'	ϕ'
10.0			

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Transformed boundary conditions

$u=0,$

$$\frac{\nu\xi}{a}\,Gr^{1/5}\xi f'=0 \Rightarrow f'=0,$$

$$v=0,\quad -\frac{\nu\,Gr^{1/5}}{a}\left(\xi\frac{\partial f}{\partial \xi}+f\right)=0$$

$$\Rightarrow \xi\frac{\partial f}{\partial \xi}+f=0$$

~~T~~ Dw

$$q_w=-K\cdot\frac{q_w}{K}\theta'$$

$$\Rightarrow \boxed{\theta'=-1}$$

and $f'\longrightarrow 0,\quad \theta=0$

So

$$f'=0,\quad f+\xi\frac{\partial f}{\partial \xi}=0,\qquad \theta'=-1 \qquad \text{at } \eta=0$$

$$f'\rightarrow 0,\quad \theta\longrightarrow 0, \qquad \text{as } \eta\rightarrow\infty$$