$$\rho c \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \dot{q} + \Phi$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla \cdot (k\nabla T) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right)$$

$$\Phi = 2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)^2 + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi} + v_r + v_\theta \cot\theta\right)^2 + \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2\right.$$
$$\left. + \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2 + \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2\right\}$$

A dimensionless number which measures the relative magnitude of the inertia effects in a fluid compared to viscous effects is the Reynolds number. This quantity is defined as

$$Re = \rho \frac{VL}{\mu},$$

where ρ is the fluid density, V is the fluid velocity, L represents a characteristic dimension in the fluid flow field, and μ is the fluid viscosity.

Creeping flows are characterized by small Reynolds numbers, whereas most practical flows are often characterized by Reynolds numbers which are large compared to unity. For example, experiments have indicated that the theory of slow motions is able to predict the drag force exerted on a sphere moving at constant speed relative to a fluid when the Reynolds number (utilizing the sphere diameter as the characteristic dimension) is less than about 1–2.

L. Prandtl in 1904 [8] made a significant advance in fluid mechanics (and, therefore, in heat transfer) when he introduced the boundary-layer approximations which allowed flows at high Reynolds numbers to be studied mathematically. The use of these approximations in studying various fluid flows results in the so-called boundary-layer theory which will be discussed in Chap. 3.

For the reasons mentioned above, the full equations cannot be considered for further analysis in all their generality, owing to their complexity, and the impossibility of postulating realistic boundary, initial and inlet conditions for them. However, two important observations are worth mentioning. First, the flow field depends upon the variation of viscosity and density with temperature, more generally with position too. Therefore, the two fields, i.e., the velocity and temperature fields, are coupled. Secondly, it is possible that the temperature field under certain conditions can become similar to the velocity field. As can be seen from Eqs. (2.34) and (2.59) that the terms which arise from the pressure gradient ∇p, Φ and **f** prevent the similarity between these two equations. Further, the viscosity μ and the thermal conductivity k may be different functions of temperature. If the pressure gradient ∇p, Φ and **f** are zero and if the Prandtl number $Pr = \nu/\alpha = 1$, the solutions for the velocity and temperature fields will be similar if their corresponding boundary conditions are also similar.

2.6 SIMILARITIES IN FLUID FLOW AND HEAT TRANSFER

For liquids within which the temperature differences are not too large, and for gases within which the temperature differences and the differences in flow speed are not too great, an enormously simplifying approximation of constant density is applicable. In the discussions to follow, we shall further assume

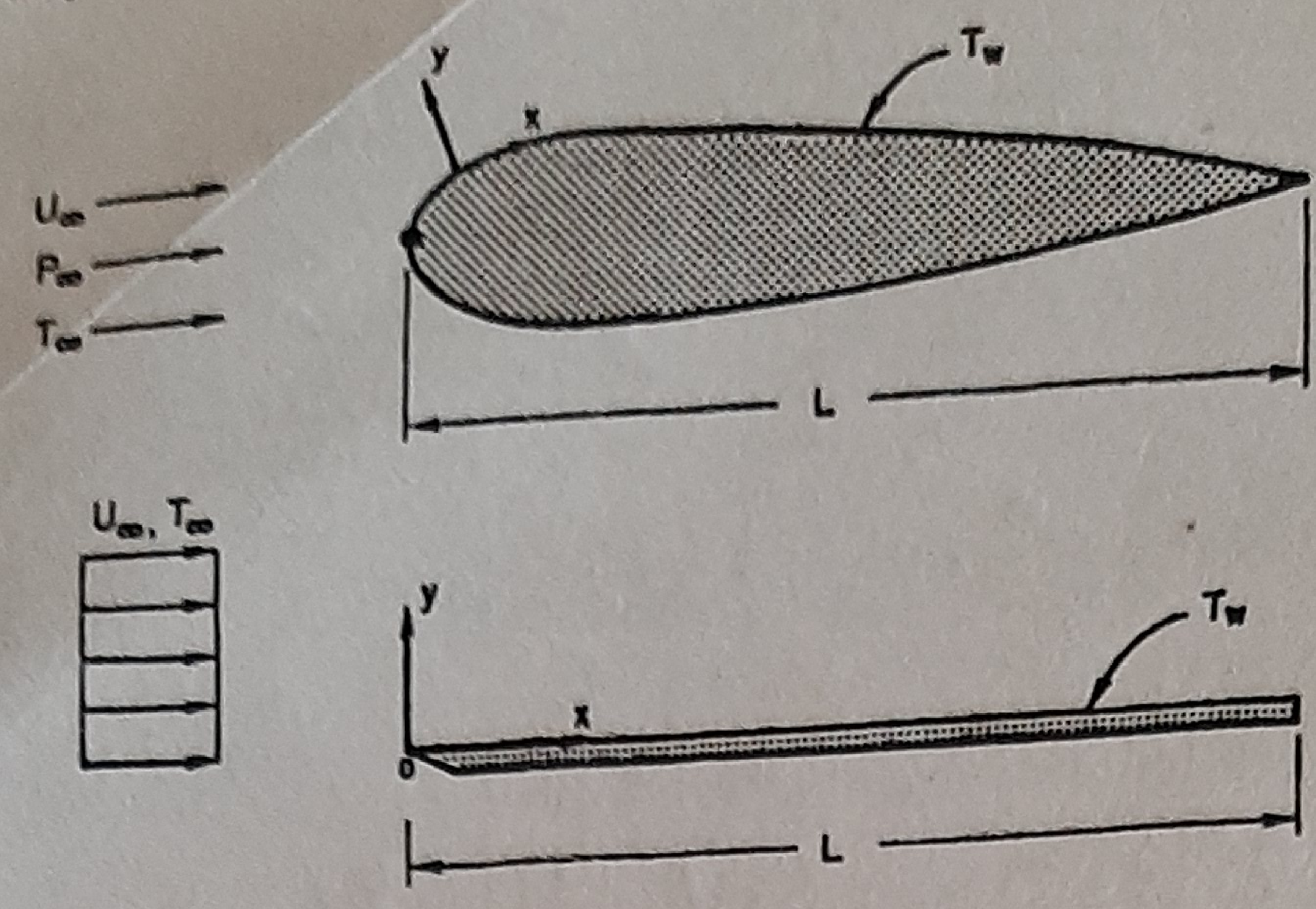

Figure 2.6 Flow over external surfaces.

that the other properties are also constant and neglect the effects of body forces and viscous dissipation. Under these conditions we shall determine the conditions for similarity in fluid flow and heat transfer.

With these assumptions, for a steady and two-dimensional flow of a viscous fluid the continuity equation, the equations of motion, and the energy equation in rectangular coordinates can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{2.64}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\,\nabla^2 u , \tag{2.65}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\,\nabla^2 v , \tag{2.66}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\,\nabla^2 T . \tag{2.67}$$

Consider now a two-dimensional body, with the characteristic dimensions L and constant surface temperature T_w, to be immersed in an essentially infinite extent of fluid that moves toward the body with the uniform and steady velocity U_∞, pressure p_∞ and temperature T_∞ far from the body as illustrated in Fig. 2.6. Introduce the following dimensionless quantities:

$$\bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}, \quad \bar{p} = \frac{p - p_\infty}{\rho\, U_\infty^2}, \quad \bar{T} = \frac{T - T_w}{T_\infty - T_w},$$

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad Re_L = \frac{\rho\, U_\infty L}{\mu}, \quad Pr = \frac{c_p \mu}{k},$$

where Pr is the so-called *Prandtl number*. In terms of these dimensionless quantities, the continuity equation (2.64), the equations of motion (2.65) and (2.66), and the energy equation (2.67) may be rewritten as

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0\,, \tag{2.68}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re_L}\bar{\nabla}^2\bar{u}\,, \tag{2.69}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{Re_L}\bar{\nabla}^2\bar{v}\,, \tag{2.70}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{1}{Re_L Pr}\bar{\nabla}^2\bar{T}\,, \tag{2.71}$$

where $\bar{\nabla}^2$ is the two-dimensional dimensionless Laplacian opeator given by

$$\bar{\nabla}^2 \equiv \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\,. \tag{2.72}$$

Boundary conditions for the above equations can be written as

at $\bar{y} = 0$ (on the surface of the body): $\bar{u} = \bar{v} = \bar{T} = 0\,,$
as $\bar{y} \to \infty$ (far from the body): $\bar{u} = \bar{T} = 1,\ \bar{p} = 0\,.$

For the given boundary conditions the solutions of the above equations (for dimensionless velocity, pressure and temperature distributions) will depend on the dimensionless independent variables $\bar{x}$ and $\bar{y}$ and the dimensionless parameters Re_L and $Re_L Pr$. From Eqs. (2.68), (2.69) and (2.70) we conclude that

$$\bar{u} = \Psi_1(\bar{x}, \bar{y}, Re_L)\,, \tag{2.73}$$

$$\bar{v} = \Psi_2(\bar{x}, \bar{y}, Re_L)\,, \tag{2.74}$$

$$\bar{p} = \Psi_3(\bar{x}, \bar{y}, Re_L)\,. \tag{2.75}$$

We see that geometrically similar bodies at the same corresponding points will have the same dimensionless pressure, velocity and, therefore, the same shear stress distribution when the Reynolds number of the flows are the same.

We may also note that in the case of a constant density, constant property flow, the velocity field is independent of the temperature distribution and can be determined once for all, regardless of the heat transfer conditions imposed on the flow.

The results for the velocity distribution may now be substituted into the energy equation and the temperature distribution may be determined. In view of Eqs. (2.73) and (2.74), from the energy equation (2.71) we conclude that the solution for the dimensionless temperature distribution depends upon the independent variables $\bar{x}$ and $\bar{y}$, the Reynolds number, and the product of the Reynolds number and the Prandtl number, i.e., Péclêt number ($Pe = Re_L Pr = U_\infty L/\alpha$), which is the measure of the relative magnitude of heat transfer by convection to heat transfer by conduction. Since the Reynolds number is required independently for dynamical similarity, it is customary to work with the Reynolds and Prandtl numbers separately rather than the Péclêt number, and hence,

$$\overline{T} = \Psi_4(\overline{x}, \overline{y}, Re_L, Pr)\ . \tag{2.76}$$

We can therefore deduce from the above equations that the condition for complete similarity in fluid flow and heat transfer between two different cases of forced convection with geometrically similar boundaries is that the Reynolds and the Prandtl numbers should each have the same values in the two systems (in moderate velocities).

We have already defined the heat transfer coefficient by Eq. (1.60) as

$$h = \frac{-k\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_w - T_\infty}\ . \tag{2.77}$$

It becomes convenient to nondimensionalize the heat transfer coefficient as

$$Nu = \frac{hL}{k}\ , \tag{2.78}$$

which is called the *Nusselt number,* and in terms of the nondimensional parameters it becomes

$$Nu = \left(\frac{\partial \overline{T}}{\partial \overline{y}}\right)_{\overline{y}=0}\ . \tag{2.79}$$

Hence, we conclude that

$$Nu = \Psi_5(\overline{x}, Re_L, Pr)\ . \tag{2.80}$$

Alternately, the heat transfer coefficient can also be put into a nondimensional form as follows:

$$St = \frac{h}{\rho c_p U_\infty} = \frac{Nu}{Re_L Pr} = \Psi_6(\overline{x}, Re_L, Pr) \tag{2.81}$$

which is known as the *Stanton number.*

If the energy equation (2.53) or (2.59) for a fluid with constant thermal conductivity is expressed in dimensionless form, we get

$$Nu = \Psi_7(\overline{x}, Re_L, Pr, Ec)\ , \tag{2.82}$$

where the dimensionless parameter *Ec* is defined as

$$Ec = \frac{U_\infty^2}{c_p\left|T_\infty - T_w\right|}\ , \tag{2.83}$$

which is called the *Eckert number.* If the flow velocity is not very large, the Eckert number is usually small and, therefore, the effect of viscous dissipation becomes negligible.

References

1. Kakaç, S. Shah, R.K., and Aung, W. (Eds.), *Handbook of Single Phase Convective Heat Transfer,* John Wiley, 1987.
2. Lamb, H., *Hydrodynamics,* Dover Publ., 1945.
3. Schlichting, H., Boundary-Layer Theory, Translated into English by J. Kestin, 7th ed., McGraw-Hill, 1979,

Similarity in Fluid Flow and Heat transfer

Consider the energy equation in rectangular Coordinates:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\alpha = \frac{k}{\rho c_p}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \Rightarrow (1)$$

$$\bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}$$

$$\bar{P} = \frac{P - P_\infty}{\rho U_\infty^2}, \quad \bar{T} = \frac{T - T_w}{T_\infty - T_w}$$

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L},$$

$$\Rightarrow u = \bar{u}\, U_\infty, \quad v = \bar{v}\, U_\infty$$

$$P - P_\infty = \bar{P} \rho U_\infty^2$$

$$P = \bar{P} \rho U_\infty^2 + P_\infty$$

$$T = \Delta T\, \bar{T} + T_w$$

$$x = L\bar{x}, \quad y = L\bar{y}$$

$$\Rightarrow (2)$$

$$\boxed{T_\infty - T_w = \Delta T}$$

Subsituting (2) in (1)

$$U_\infty \bar{u} \frac{\partial}{\partial (L\bar{x})}(\Delta T\,\bar{T} + T_w) + U_\infty \bar{v} \frac{\partial}{\partial (L\bar{y})}(\Delta T\,\bar{T} + T_w)$$

$$= \alpha \left(\frac{\partial^2 (\Delta T\,\bar{T} + T_w)}{\partial (L^2 \bar{x}^2)} + \frac{\partial^2 (\Delta T\,\bar{T} + T_w)}{\partial (L^2 \bar{y}^2)} \right)$$

$$\frac{U_\infty \Delta T}{L} \frac{\partial \bar{T}}{\partial \bar{x}} + \frac{U_\infty \Delta T}{L} \frac{\partial \bar{T}}{\partial \bar{y}} = \alpha \left\{ \frac{\Delta T}{L^2} \left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right) \right\}$$

$$\frac{\partial \bar{T}}{\partial \bar{x}} + \frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\alpha \cdot L}{U_\infty \Delta T} \cdot \frac{\Delta T}{L^2} \left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right)$$

$$\frac{\partial \bar{T}}{\partial \bar{x}} + \frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\alpha}{U_\infty} \cdot \frac{1}{L} \left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right)$$

$$= \frac{\alpha}{\nu} \cdot \frac{\nu}{U_\infty L} \left(\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right)$$

$$P_r = \frac{\nu}{\alpha}$$

$$Re_L = \frac{U_\infty L}{\nu}$$

So

$$\boxed{\frac{\partial \bar{T}}{\partial \bar{x}} + \frac{\partial \bar{T}}{\partial \bar{y}} = \frac{1}{Re_L} \cdot \frac{1}{P_r} \left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right)}$$

which is two dimensionless form

in the outer flow region. In the boundary-layer analysis, the free-stream velocity $U_\infty(x)$ is assumed to be available from the solution of the potential flow outside the boundary layer, and thus the pressure gradient term dp/dx is considered to be known from Eq. (3.13).

Hence, Eqs. (3.1), (3.2) and (3.3) may now be replaced by the following *Prandtl's boundary-layer equations:*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{3.14}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) , \tag{3.15}$$

with the following boundary conditions:

$$at\ y = 0 : \quad u = v = 0 , \tag{3.16a,b}$$

$$as\ y \to \infty : \quad u = U_\infty(x) . \tag{3.16c}$$

In addition, the velocity distribution at $x = 0$ must also be specified.

It may be noted that, although one of the viscous terms in Eq. (3.8) has been dropped, the order of this equation has not been reduced. Also, one of the equations of motion has been dropped completely. As a result, the number of unknowns has been reduced by one.

A similar analysis has been carried out for the boundary layer flow along a curved wall and it has been concluded that the above equations may be applied to a curved wall as long as no large variations in curvature occur [3].

It should be noted that the boundary-layer approximations are valid for large values of the Reynolds number and the no-slip condition is assumed on the solid surface i.e., the fluid layer at $y = 0$ sticks to the solid surface.

3.3 BOUNDARY-LAYER ENERGY EQUATION

If the plate temperature is different from the fluid temperature a thermal boundary layer will also develop over the plate as illustrated in Fig. 3.1, indicating a significant temperature variation over a narrow zone in the immediate vicinity of the plate. The thermal boundary-layer thickness δ_T may be defined as that distance from the surface where $(T - T_w) = 0.99(T_\infty - T_w)$ as shown in Fig. 3.5.

For a steady, two-dimensional and incompressible viscous flow with constant thermophysical properties, the energy equation (2.61) reduces to

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] . \tag{3.17}$$

Introducing the following dimensionless variables

$$\bar{x} = \frac{x}{L}; \quad \bar{y} = \frac{y}{L}; \quad \bar{p} = \frac{p}{\rho U_o^2}; \quad \bar{u} = \frac{u}{U_o}; \quad \bar{v} = \frac{v}{U_o}; \quad \bar{T} = \frac{T - T_w}{T_\infty - T_w},$$

where $U_o = U_\infty(0)$, the energy equation (3.17) may be written in the following dimensionless form:

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{1}{RePr}\left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right) + 2\frac{Ec}{Re}\left[\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right] , \tag{3.18}$$

where we have introduced

$$Re = \frac{U_o L}{\nu}, \quad Pr = \frac{c_p \mu}{k}, \quad Ec = \frac{U_o^2}{c_p(T_\infty - T_w)} .$$

Following the discussions of the previous section, we now again conclude that in the energy equation (3.18)

$$\bar{u} = O(1), \quad \frac{\partial \bar{u}}{\partial \bar{x}} = O(1), \quad \frac{\partial \bar{u}}{\partial \bar{y}} = O(1/\bar{\delta})$$

$$\bar{v} = O(\bar{\delta}), \quad \frac{\partial \bar{v}}{\partial \bar{x}} = O(\bar{\delta}), \quad \frac{\partial \bar{v}}{\partial \bar{y}} = O(1)$$

and $Re = O(1/\bar{\delta}^2)$, where, as before, $\bar{x} = O(1)$, $\bar{y} = O(\bar{\delta})$ and $\bar{\delta} = \delta/L$.

Comparing the order of magnitudes of the dissipation terms, the energy equation (3.18) can be rewritten as

$$\underset{1\;\;1}{\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}} + \underset{\bar{\delta}\;\;1/\bar{\delta}_T}{\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}} = \frac{1}{RePr}\left(\underset{1}{\frac{\partial^2 \bar{T}}{\partial \bar{x}^2}} + \underset{1/\bar{\delta}_T^2}{\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}}\right) + \underset{1/\bar{\delta}^2}{\frac{Ec}{Re}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2} . \tag{3.19}$$

Furthermore, we note that in the thermal boundary layer $\bar{T} = O(1)$ because $\bar{T}$ varies from zero at the surface of the plate to almost unity at $y = \delta_T$, and also $\bar{y} = O(\bar{\delta}_T)$, where $\bar{\delta}_T = \delta_T/L$. Hence, we observe that

$$\frac{\partial \bar{T}}{\partial \bar{x}} = O(1) \quad \text{and} \quad \frac{\partial^2 \bar{T}}{\partial \bar{x}^2} = O(1) ,$$

$$\frac{\partial \bar{T}}{\partial \bar{y}} = O(1/\bar{\delta}_T) \quad \text{and} \quad \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} = O(1/\bar{\delta}_T^2) .$$

The order of magnitude of various terms in Eq. (3.19) is indicated under each term. Comparing the orders-of-magnitude of the conduction terms we conclude that

$$\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} << \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} .$$

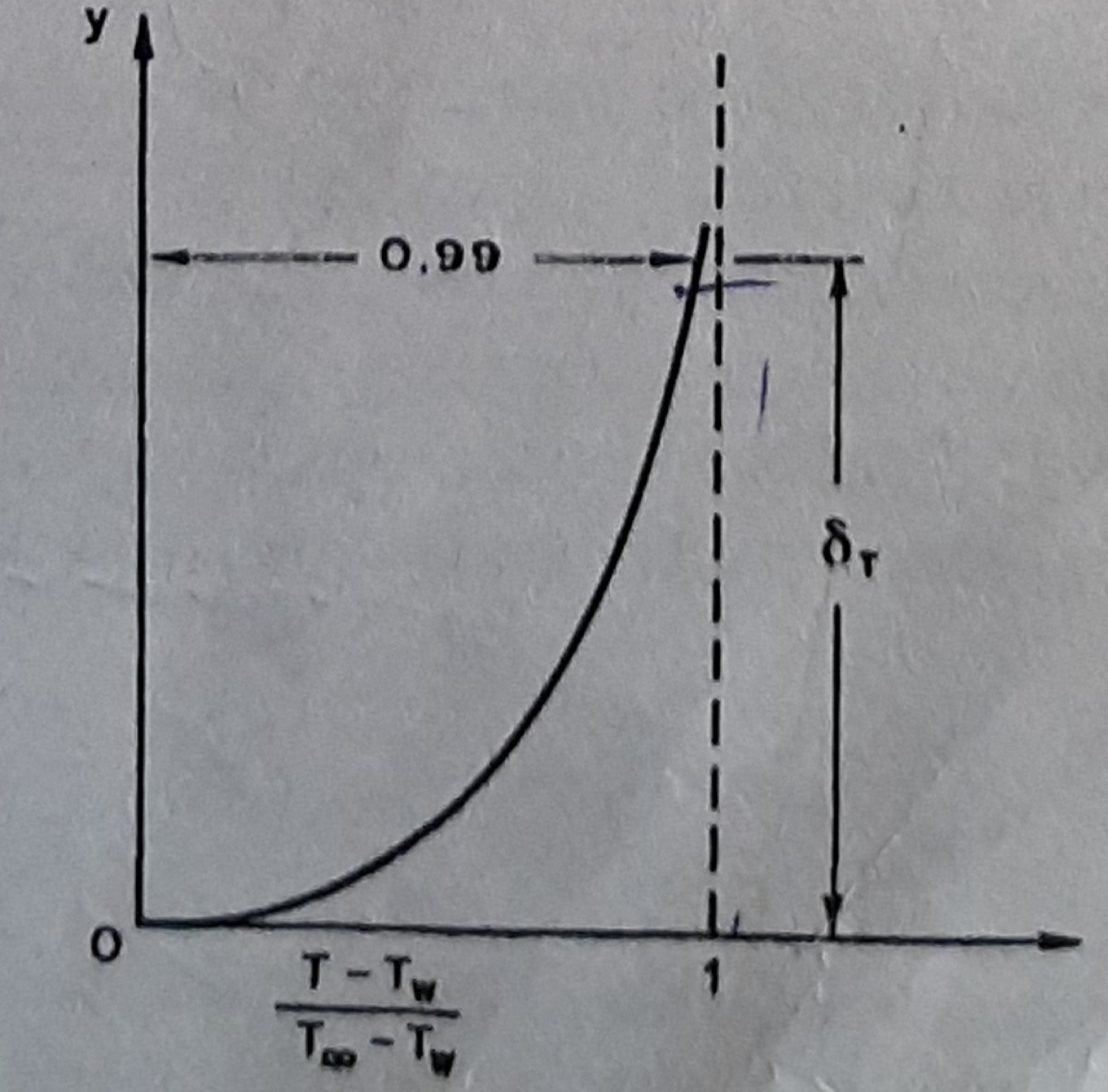

Figure 3.5 Thermal boundary layer thickness.