26-03-2020
Heat Transfer

Cauchy equation of Motion

Newton's Second law of motion can be stated as

$$\sum F = Ma$$

Where the left hand side is the sum of all external forces acting on the control volume, and consists solely of surfaces and body forces.

$$\sum F = \sum F_s + \sum F_b \Rightarrow (1)$$

Where body forces F_b acting on the center of mass of the control volume.

Examples are graviational, bouyant and electromagnetic forces. The graviational body force, G, on the fluid volume V is

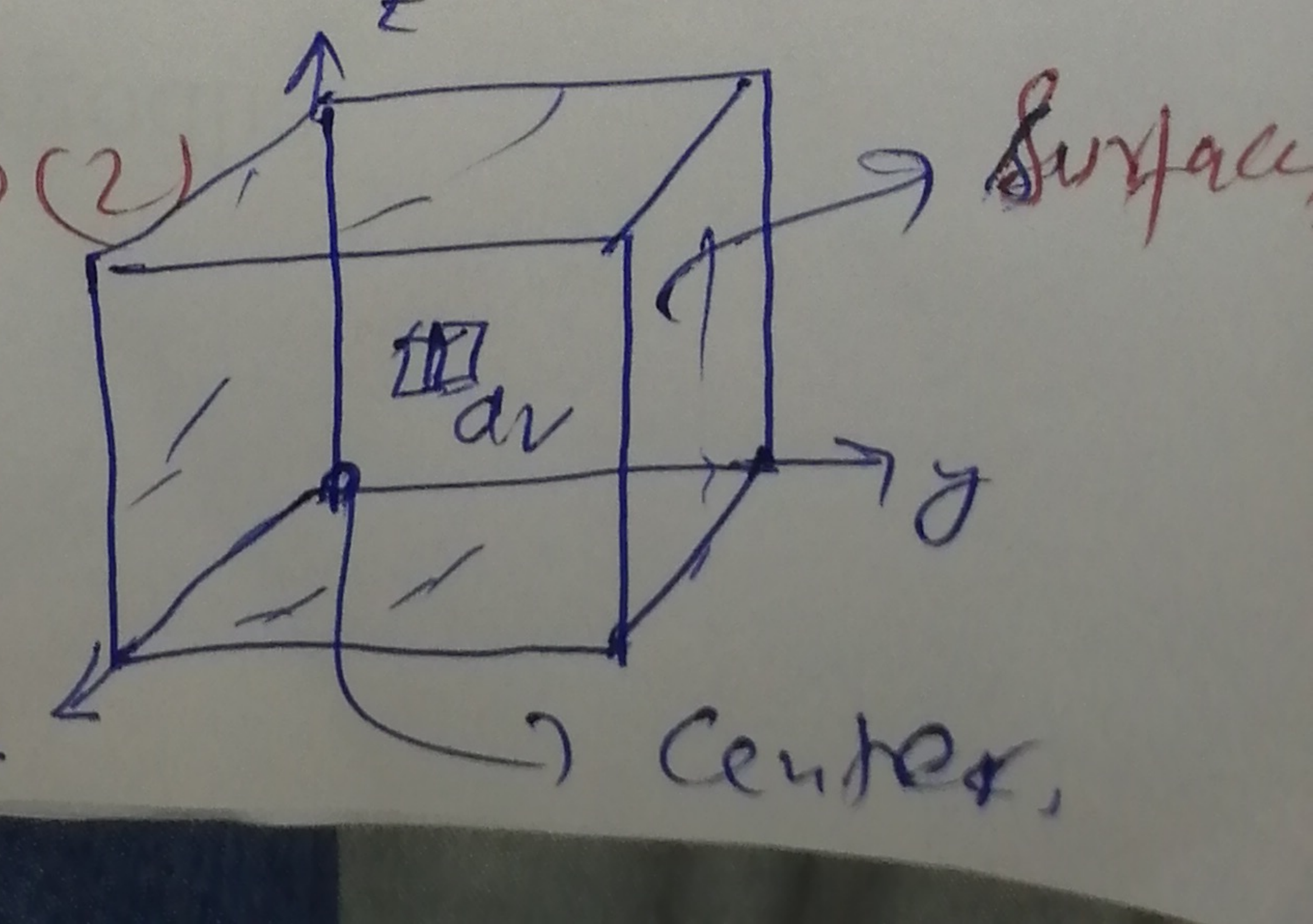
$$G = \int_V \rho g \, dV \Rightarrow (2)$$

$\rho = \frac{M}{V}$

$m = \rho V$

$W = mg$

weight of the



fluid element acting vertically downward.
g is the acceleration due to gravity.

Since the x, y axis lie in horizontal plane relative to the surface of the earth, assuming the earth is homogenous sphere, we using the relationship between mass M and density ρ and expressing the surface force in terms of stress τ, we obtain,

$$\Sigma F_s + \Sigma F_b = Ma$$

$$\int_A \tau \cdot dA + \int_V \rho g \, dV = \int_V \rho a \, dV$$

$$\int_V (g - a)\rho \, dV + \int_A \tau \cdot dA = 0 \Rightarrow (3)$$

$P = F/A$
$F = PA$
$F = \tau A$

The area integral can be expressed in terms of a volume integral through Gauss's theorem

$$\int_A \tau \cdot dA = \int_V (\nabla \cdot \tau) \, dV$$

$$\int_{V} (\dot{\xi} - a)\rho \, dV + \int_{V} (\nabla \cdot \tau)\, dV = 0$$

$$\int_{V} [(\dot{\xi} - a)\rho + \nabla \cdot \tau]\, dV = 0 \Rightarrow (4)$$

Since the volume V is completely arbitrary and the integrand is continuous throughout the volume, the integrand of equation (4) must also be zero

$$(\dot{\xi} - a)\rho + \nabla \cdot \tau = 0 \Rightarrow (5)$$

↳ is called ~~of~~ vector differential form of Cauchy ~~of~~ equation of motion.

physical significances:

of represents a unique balance of body and surface force with interbed force.

here,

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

and g_x, g_y and g_z are components of acceleration due gravity $\vec{g}$. So the above (5) in its components wise

x-comp

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x + \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

y-comp

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)$$

z-comp

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z + \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)$$

$\Rightarrow$ (6)

now

$$\left.\begin{aligned} \sigma_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} \\ \sigma_{yy} &= -p + 2\mu\frac{\partial v}{\partial y} \\ \sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} \tau_{xy} &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \tau_{xz} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \tau_{yz} &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \end{aligned}\right\} \Rightarrow (8)$$

By subsititutiny derivatives of (7) and (8) into (6).

x- comp

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial}{\partial x}(D)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{\partial}{\partial y}(D)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \mu/\rho\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}(D)\right)$$

$$\Rightarrow (9)$$

$$\boxed{\nu = \mu/\rho}$$

↳ is called kinematic viscosity

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

→ equation of continuity

So (9) becomes

x-comp

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \delta_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y-comp

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \delta_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$+ \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z-comp

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \delta_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$+ \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$\Longrightarrow$ (10)

is called Navier Stokes equation of motion.

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = g - \frac{1}{\rho}\nabla P + \nu \nabla^2 V$$

↳ which is called vector differential form of Navier Stokes equation.

Physical significances:

$\frac{\partial V}{\partial t}$ = local acceleration

$(V \cdot \nabla)V$: Convective acceleration

g : acceleration due to gravity.

$-\frac{1}{\rho}\nabla P$: is the pressure acceleration due to the "pumping" action of the flow.

$\nu \nabla^2 V$: is the viscous deceleration due to the fluid's frictional resistance.