Chapter 2

Finite Differences

2.1 DIFFERENCE TABLE

Suppose we have a function f(x) which is tabulated over a range of values (called tabular points) of the independent variable x. Let us denote the uniform difference (constant spacing or step-size) between any two successive values by h so that,

$$x_1 - x_0 = h = x_2 - x_1 = \ldots = x_n - x_{n-1}$$

or

$$x_1 = x_0 + h$$
$$x_2 = x_1 + h = x_0 + 2h$$
$$\ldots$$
$$x_p = x_0 + ph$$
$$\ldots$$
$$x_n = x_0 + nh$$

and

$$f(x_p) = f_p = f(x_0 + ph)$$

In many numerical processes concerned with tabulated functions certain quantities called **finite differences** are important. The procedure to compute differences is explained below:

To build up the difference table, we first write down the values of x_i's and the corresponding values of f_i's as shown below:

x_i	f_i	1st	2nd	3rd
x_0	f_0			
		$f_1 - f_0$		
x_1	f_1		$f_2 - 2f_1 + f_0$	
		$f_2 - f_1$		$f_3 - 3f_2 + 3f_1 - f_0$
x_2	f_2		$f_3 - 2f_2 + f_1$	
		$f_3 - f_2$		$f_4 - 3f_3 + 3f_2 - f_1$
x_3	f_3		$f_4 - 2f_3 + f_2$	
		$f_4 - f_3$		
x_4	f_4			

The first-order differences are obtained from the second column by subtracting each value from the next below and placing the differences to the right but halfway between the two values from which they have been obtained. In this way, the column containing all the first-order differences is formed, but each difference column contains one entry less than its predecessor column.

We are now in a position to produce a column of second-order differences from the column of the first-order differences in a similar way. In computing differences, great care should be exercised to avoid arithmetic errors in the subtractions – the fact that we subtract the upper value from the lower adds a real source of confusion. The sign of the differences is important and shows whether the function is increasing or decreasing in the range of the values obtained.

There are several uses of a difference table; a few of which are as follows:

i) A difference table provides a convenient way for examining at a glance how a particular function behaves. It is particularly applicable in determining the behaviour of the derivatives of a given function.

ii) If there are some errors in the data, the differences will also contain errors. By inspecting the difference table, often the error (or errors) can be detected and corrected.

iii) It helps in filling missing values.

iv) It helps in extending the list of values.

The word **finite** refers to the finite size of the interval used in the table as opposed to the infinitesimal intervals or increments, which are met in infinitesimal calculus. For this reason, the theory and application of finite differences is sometimes referred to as **Finite Calculus.** It plays an important role in interpolation, numerical differentiation, numerical integration, numerical solutions of difference, ordinary and partial differential equations and time series analysis.

A numerical example at this stage should help clarify some basic concepts for constructing a difference table.

Example 1 Construct the difference table for the function $f(x) = x^4$ for $x = -2$ to $x = 4$, at the interval of 1. [Usually, written as $x = -2(1)4$; the figure in brackets being the constant increment.]

Solution The values of f_i and the resulting differences are shown in the table below:

x_i	$f_i = x_i^4$	1st	2nd	3rd	4th	5th
−2	16					
		−15				
−1	1		14			
		−1		−12		
0	0		2		24	
		1		12		0
1	1		14		24	
		15		36		0
2	16		50		24	
		65		60		
3	81		110			
		175				
4	256					

An examination of the difference table reveals that all fourth-order differences are constant and thus the fifth and all higher-order differences would be zero, which is the peculiar property of an exact polynomial (i.e., when all entries in the table are exact and not rounded).

Some obvious results

a) The nth-differences of an exact polynomial of degree n are constant.

b) The $(n+1)$st differences of that polynomial are zero.

c) The above values are only true of polynomial when they are tabulated at equal intervals.

If the function does not represent an exact polynomial, the above results will not hold. In practice, we always deal with rounded numbers, where we seldom come across a column with all its differences zeros. The differences of rounded numbers are irregular and thus give rise to the irregular part of the table. In that case, the nth-order differences due to the rounding errors oscillate between $\pm 2^{n-1}$.

The reason for this is that when the tabulated values are rounded, each value has an error usually lying in the range $\pm\frac{1}{2}$, if we work in units of the last place. These errors will build up in the differences just as do mistakes, and eventually, if the true values have convergent differences, they will become greater than the true differences. In the worst case, the round-off errors will be alternately $+\frac{1}{2}$ and $-\frac{1}{2}$ and their contribution to any nth difference will be

$$\pm\frac{1}{2}\cdot\left\{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right\}=\pm 2^{n-1}$$

Example 2 Construct the difference table for the function $f(x)=\sqrt{x^2+x+1}$, rounded to 4dp, for $x = 10(1)\,16$.

x	$f(x)=\sqrt{x^2+x+1}$	1st	2nd	3rd	4th
10	10.5357				
		.9969			
11	11.5326		5		
		.9974		−2	
12	12.5300		3		3
		.9977		1	
13	13.5277		4		−2
		.9981		−1	
14	14.5258		3		−1
		.9984		−2	
15	15.5242		1		
		.9985			
16	16.5227				

Since the function is tabulated at 4dp, each difference is also to 4dp. Because of this, the decimal point and the leading zeros may be omitted in the formation of a difference table and they may then be written as integers. This makes the table easier to construct and much neater too. For instance, the first entry in the column of fourth differences is an abbreviation of 0.0003. The table shows that the fourth-order differences oscillate and are all within the range $\pm 2^{4-1} = \pm 8$.

2.2 DETECTION AND CORRECTION OF ERRORS IN A DIFFERENCE TABLE

It is likely that an error (errors) may show up while constructing differences. We observe a very characteristic kind of error propagation, which we shall illustrate in this section. An error caused by reversing the order of a pair of digits in a number is commonly made in copying down the number from the given data. It affects the other differences in the table. We may denote the error in a single entry in the difference table by the symbol, ε, which can be negative, positive, small or large. Its effect on the differences spreads out fan-wise as shown in the table below:

x	f	1st	2nd	3rd	4th
0	0				
		0			
1	0		0		ε
		0		ε	
2	0		ε		−4ε
		ε		−3ε	
3	ε		−2ε		+6ε
		−ε		3ε	
4	0		ε		−4ε
		0		−ε	
5	0		0		ε
		0		0	
6	0				

This fan-wise (triangular patterns) propagation of ε in the difference table grows quickly and makes it possible in certain cases to locate an error and also to find its numerical value, thus enabling us to rectify it with the help of tabular values. A glance at the table reveals that the coefficients of ε in the nth-order differences are binomial coefficients of x, which occur in the expansion of $(1-x)^n$. For example, the coefficients in the third-order difference column are 1, −3, 3, −1, which occur in the expansion of $(1-x)^3$ in the increasing powers of x, i.e., $1 - 3x + 4x^2 - x^3$. The corresponding coefficients for the fourth-order differences of $(1-x)^4$ are 1, −4, 6, −4, 1. The binomial coefficients in the fifth and sixth difference columns are 1, −5, 10, −10, 5, −1 and 1, −6, 15, −20, 15, −6, 1 respectively. The table shows that the higher-order differences are very sensitive to slight changes in any of the ordinates or lower-order differences. Relatively small input changes generate relatively large output changes. For the identification of gross errors, the above picture should be kept in mind.

We illustrate the procedure by means of the following example.

Example 3 The following table contains an incorrect value of f(x). Locate the error, suggest a possible cause and a suitable correction:

x	1	2	3	4	5	6	7	8	9	10
f(x)	37	74	135	226	353	531	739	1010	1341	1738

Solution **Difference Table**

x	f(x)	1st	2nd	3rd	4th
1	37				
		37			
2	74		24		
		61		6	
3	135		30		0
		91		6	
4	226		36		9
		127		15	
5	353		51		−36
		178		−21	
6	531		30		54
		208		33	
7	739		63		−36
		271		−3	
8	1010		60		9
		331		6	
9	1341		66		
		397			
10	1738				

In the above table, f(x) seems to represent an exact polynomial; thus all fourth-order differences should be zero. The error seems to have appeared in the fourth-order differences with coefficients: 1, −4, 6, −4, 1. The incorrect difference may be written as:

1(9), −4(9), 6(9), −4(9), 1(9)

This indicates that the error is 9. The next step is to locate the incorrect functional value. This can be done by moving backward to the second column. It shows that the term in error is 531 and the correct value is 531 − 9 = 522. The likely cause of the error may be due to wrongly copying the digits. The result can be checked by correcting the wrongly-placed entry and reconstructing the difference table. If the function is known analytically, it would be preferable to recalculate it at x = 6 so that the correction can be made with certainty rather just estimated.

In the above example, the functional values are exact and it was fairly easy to locate and correct the error with certainty, but this is not always the case especially when the values of f(x) have been rounded, since the errors will not then be exact multiples of the binomial coefficients. In such a case, we can only make an estimate of the error. Moreover, in a

difference table in which there are two or more errors, their fans will eventually overlap, making it more difficult to discover the errors. Some care is necessary to find out a reasonable pattern to locate the error(s) in such cases.

A table in which two errors have been made is more difficult to analyze since the binomial coefficients overlap. The following pattern shows a possible example:

f	1st	2nd	3rd	4th
		Differences		
0				
	0			
0		0		
	0		ε_1	
0		ε_1		$-4\varepsilon_1$
	ε_1		$-3\varepsilon_1$	
ε_1		$-2\varepsilon_1$		$6\varepsilon_1$
	$-\varepsilon_1$		$3\varepsilon_1$	
0		ε_1		$\varepsilon_2 - 4\varepsilon_1$
	0		$\varepsilon_2 - \varepsilon_1$	
0		ε_2		$-4\varepsilon_1 + \varepsilon_1$
	ε_2		$-3\varepsilon_2$	
ε_2		$-2\varepsilon_2$		$6\varepsilon_2$
	$-\varepsilon_2$		$3\varepsilon_2$	
0		ε_2		
	0			
0				

It may be possible to identify the error pattern in the third-order differences column but the confusion in the fourth-order difference column would probably be too great to give an opportunity to detect the error. We therefore concentrate on the problems where only one mistake is made.

Example 4 It is suspected that the following table contains an error. By differencing, locate any probable error and correct it. Check by re-differencing if any correction is made. The values of f(x) are rounded to 3dp.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
f(x)	0.905	0.819	0.741	0.677	0.607	0.549	0.497

Solution **Difference Table**

x	f(x)	1st	2nd	3rd	4th
0.1	0.905				
		−86			
0.2	0.819		8		
		−78		6	
0.3	0.741		14		−26
		−64		−20	
0.4	0.677		−6		38
		−70		18	
0.5	0.607		12		−24
		−58		−6	
0.6	0.549		6		
		−52			
0.7	0.497				

Comparing the third-order differences with the coefficients of x, we get

1	−3	3	−1
↓	↓	↓	↓
6	−20	18	−6

We may deduce that $\varepsilon = 6$, i.e., 0.006 and the corrected value of $f(0.4) = 0.677 - 0.006 = 0.671$. We can check by reconstructing the difference table, but we are not 100% sure that it is the correct value, we just estimated. However, it may prove to be a reasonable estimate. If the entries are more in the data and the above mentioned options fail to give a reasonable clue to pick-up the error, we should make use of the fifth or sixth difference column and then try again.

If there is no obvious pattern for locating an error in the difference table, we use the following formula for finding the error:

$$\text{Error} = \frac{\text{Largest value in a column}}{\text{Corresponding coefficient of } \varepsilon \text{ in that column}}$$

In this section, we have studied how to locate and correct a single error in a difference table. If there are two or more errors in the entries, it is usually not easy to separate their overlapping effects and thus locations and corrections of such errors become extremely difficult. In some cases, irregular behaviour of the differences may be caused not by errors but by irregularities in the functions.

2.3 DIFFERENCE OPERATORS

To refer to specific entries in a difference table we use some operators, called **difference operators**. An operator is not a number but it is an operation, which when applied to a function changes it to some other function. The operator technique proves to be a most useful tool when we wish to construct formulas for interpolation, numerical differentiation, numerical integration, etc. One of the biggest advantages is that we can sketch the type of formula desired in advance and then proceed directly toward the goal.

The following operators are commonly used:

- Δ Forward-difference operator (usually read as delta)
- ∇ Backward-difference operator (read as del or nabla).
- δ Central difference operator (read as sigma).
- μ Average operator (read as mu).
- E Shift operator.

Let us define them one by one. It must be emphasized that these operators assume equally-spaced data points.

2.3.1 Forward Difference Operator

The difference operator Δ is defined by the following relation:

$$\Delta f_r = f_{r+1} - f_r$$

where r is an integer, and $\Delta f_r = \Delta f(x_r)$.

Also, $\Delta f_{r+1} = \Delta f(x_r + h)$ and $\Delta f_{r+\frac{1}{2}} = \Delta f\left(x_r + \frac{h}{2}\right)$.

In words, when Δ operates on a function, we first shift r by r + 1 and then subtract the original function from the shifted function. This produces the difference function Δf_r.

Thus,
$$\Delta f_0 = f_1 - f_0$$
$$\Delta f_1 = f_2 - f_1$$
$$\vdots$$
$$\Delta f_{-1} = f_0 - f_{-1}, \text{ etc.}$$

Δf_{-1}, Δf_0, Δf_1, are called **first-order forward differences**. The differences, of the first-order differences, are called **second-order differences** and are computed as follows:

$$\begin{aligned} \text{Thus, } \Delta^2 f_r &= \Delta(\Delta f_r) \\ &= \Delta(f_{r+1} - f_r) \\ &= \Delta f_{r+1} - \Delta f_r \\ &= (f_{r+2} - f_{r+1}) - (f_{r+1} - f_r) \\ &= f_{r+2} - 2f_{r+1} + f_r \end{aligned}$$

The higher-order differences are obtained in the same way.

$$\begin{aligned} \text{Thus, } \Delta^3 f_r &= \Delta(\Delta^2 f_r) \\ &= \Delta\{f_{r+2} - 2f_{r+1} + f_r\} \\ &= \Delta f_{r+2} - 2\Delta f_{r+1} + \Delta f_r \end{aligned}$$

$$= (f_{r+3} - f_{r+2}) - 2(f_{r+2} - f_{r+1}) + (f_{r+1} - f_r)$$
$$= f_{r+3} - 3f_{r+2} + 3f_{r+1} - f_r$$
$$\Delta^4 f_r = f_{r+4} - 4f_{r+3} + 6f_{r+2} - 4f_{r+1} + f_r$$

In general, nth-order differences are given by:

$$\Delta^n f_r = \Delta^{n-1} f_{r+1} - \Delta^{n-1} f_r$$

where $\Delta^n f_r \neq (\Delta f_r)^n$, and $n \geq 1$.

The following difference table shows how the forward differences of all orders can be formed:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

We observe from the table that differences with the same above subscripts all lie on a downward sloping diagonal.

While experimenting with differences, we observe that if x^n is a polynomial of degree n, then Δx^n is a polynomial of degree (n − 1). In other words, differencing behaves like differentiation in the sense of reducing the degree of a polynomial.

Thus, $\Delta x^n = (x + 1)^n - x^n$

$$= nx^{n-1} + n(n-1)\, x^{n-2} + \ldots$$

If the above process is continued for n times, the polynomial x^n is reduced to degree zero, i.e., constant. This is exactly what was shown by Example 1, that the nth-order differences of a polynomial of degree n are constant and all higher-order differences are zero.

Algorithm for Generating Differences Using Forward Scheme

In general, for a function tabulated at n points, the corresponding forward difference table can be represented by a matrix of size (n − 1) * (n − 1). Note that only the elements in the columns from 1 to n − i, where the row i = 1, 3, ..., n − 1, are of interest.

The algorithm to generate forward differences table may look like the following:

Steps

```
For   J = 1 TO n - 1 by 1 DO
      FOR  I = 1 TO n - J by 1 DO
           IF (J = 1) THEN
      SET  DIJ = F(XIJ) - DI,J-1
           ELSE
      SET  DIJ = DI+1,J-1 - DI,J-1
      Print "all differences, DI,J"
```

This algorithm will compute the forward differences of all orders that can be computed from the given function table. The data with equi-spaced abscissas are initialized in the program.

Example 5 Computerize the algorithm for generating forward differences. Use the following test data:

x	1.2	1.4	1.6	1.8	2.0
y	5.64642	6.44218	7.17356	7.83327	8.41471

Solution

Program No. 1: **Difference Table**

```
# include<iostream.h>
# include<stdio.h>
# include<conio.h>

void main(void)
  {
     clrscr( );
     float interval, array[20][20]={0.0};
     int no, col, x,y;

     cout<<"\tDIFFERENCE TABLE";
     cout<<"\n\n\tENTER THE FIRST VALUE : ";cin>>array[0][0];
     cout<<"\n\tENTER THE INTERVAL : ";cin>>interval;
     cout<<"\n\tENTER TOTAL NO. OF  X : ";cin>>no;

     for(int i=1; i<no; i++)
        array[i][0]=array[i-1][0]+interval;

     cout<<"\n\n\tENTER FUNCTIONAL VALUES : \n";
     for(i=0;i<no;i++)
```

```
    {
      cout<<"\tX("<<i<<") = ";cin>>array[i][1];
    }

cout<<"\n\tHOW MANY COLUMNS ARE REQUIRED : ";cin>>col;
for(i=2;i<=(col+2);i++)
   for(int j=0;j<=(no-i);j++)
       array[j][i]=array[j+1][i-1]-array[j][i-1];

clrscr( );
cout<<"\t\t\DIFFERENCE TABLE\n\n";
cout<<" X              F(X) ";
for(i=1;i<=col;i++)
   cout<<"    col    "<<i;

cout<<"\n\n";

for(i=0;i<no;i++)
  cout<<" "<<array[i][0]<<"\n\n";

x=8; y=5;
for(i=1;i<=(col+1);i++)
   {
     gotoxy(x,y);
     for(int j=0;j<=(no-i);j++)
       {
         cout<<array[j][i];
         y+=2;
         gotoxy(x,y);
       }
     x+=9; y=i+5;
   }
}
```

DIFFERENCE TABLE

ENTER THE FIRST VALUE : 1.2

INTER THE INTERVAL : 0.2

ENTER TOTAL NO. OF X : 5

ENTER FUNCTIONAL VALUES:
X(0) = 5.64642
X(1) = 6.44218
X(2) = 7.17356

X(3) = 7.83327
X(4) = 8.41471

HOW MANY COLUMNS ARE REQUIRED : 4

Computer Output

DIFFERNENCE TABLE

X	F(X)	col 1	col 2	col 3	col 4
1.2	5.64642				
		0.79576			
1.4	6.44218		– 0.06438		
		0.73138		–0.07167	
1.6	7.17356		–0.07167		0.00069
		0.65971		–0.00660	
1.8	7.83327		–0.07827		
		0.58144			
2.0	8.41471				

2.3.2 Backward Difference Operator

The difference operator ∇ is defined by the following relation:

$$\nabla f_r = f_r - f_{r-1}$$

Hence, we shift r backward by one step, the function becomes f_{r-1} and subtract this function from the original f_r.

Thus, $\nabla f_1 = f_1 - f_0$

$$\nabla f_0 = f_0 - f_{-1}$$

$$\nabla f_2 = f_2 - f_1$$

The above differences are called first-order backward differences. In a similar manner, we can define backward differences of higher-orders. Thus, we obtain:

$$\begin{aligned}\nabla^2 f_r &= \nabla(\nabla f_r)\\ &= \nabla(f_r - f_{r-1})\\ &= \nabla f_r - \nabla f_{r-1}\\ &= (f_r - f_{r-1}) - (f_{r-1} - f_{r-2})\\ &= f_r - 2f_{r-1} + f_{r-2}\end{aligned}$$

Similarly, $\nabla^3 f_r = f_r - 3f_{r-1} + 3f_{r-2} - f_{r-3}$

In general, nth-order differences are given by:

$$\nabla^n f_r = \nabla^{n-1} f_r - \nabla^{n-1} f_{r-1} ;\ n \geq 1.$$

With the help of this operator, we can construct the table for backward differences:

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

We observe from the above table that differences with the same subscripts all lie on an upward sloping diagonal.

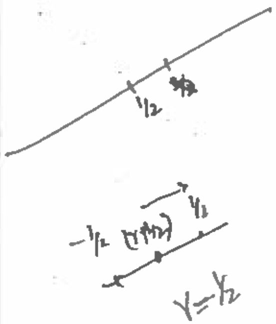
Algorithm to Generate Differences Using Backward Scheme

In general, for a function tabulated at n points, the corresponding backward difference table can be represented by a matrix of size $n \times n$. Note that only the elements in the columns from 1 to $n - i$, where the row $I = 2, 3, \ldots, n$, are of interest.

The algorithm to generate backward differences table may look like the following:

Steps

```
For    J = 1 TO n - 1 by 1 DO
       FOR   I = J +1 TO n by 1 DO
             IF (J = 1) THEN
       SET   D_IJ = F(X_I) - F(XD_I-1, J-1
             ELSE
       SET   D_IJ = D_I, J-1 - D_I-1, J-1
       PRINT "all differences, D_IJ"
```



2.3.3 Central Difference Operator

The central difference operator δ is defined as:

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}$$

Thus, $\delta f_{r+\frac{1}{2}} = f_{\left(r+\frac{1}{2}\right)+\frac{1}{2}} - f_{\left(r-\frac{1}{2}\right)+\frac{1}{2}} = f_{r+1} - f_r$

Similarly, $\delta^2 f_r = \delta(\delta f_r)$

$$= \delta\left(f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}\right)$$

$$= \delta f_{r+\frac{1}{2}} - \delta f_{r-\frac{1}{2}}$$

$$= (f_{r+1} - f_r) - (f_r - f_{r-1})$$

$$= f_{r+1} - 2f_r + f_{r-1}$$

In general, nth-order differences are given by:

$$\delta^n f_r = \delta^{n-1} f_{r+\frac{1}{2}} - \delta^{n-1} f_{r-\frac{1}{2}}$$

The differences table for δ is given below:

x	f	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
x_0	f_0				
		$\delta f_{\frac{1}{2}}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{\frac{3}{2}}$		$\delta^3 f_{\frac{3}{2}}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_4$
		$\delta f_{\frac{5}{2}}$		$\delta^3 f_{\frac{5}{2}}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{\frac{7}{2}}$			
x_4	f_4				

We note that all differences with the same subscripts lie on the same horizontal line and all even-order differences have integer subscripts. The central difference notation is preferable for many purposes but has the disadvantage of requiring fractional suffixes.

It is to be remembered that whatever notation we use, there is only one difference table and hence each entry in the table has one of the three names, for instance,

$$f_{r+1} - f_r = \Delta f_r = \nabla f_{r+1} = \delta f_{r+\frac{1}{2}}$$

Also, $\Delta f_0 = \nabla f_1 = \delta f_{\frac{1}{2}}$

$$\Delta^2 f_0 = \nabla^2 f_2 = \delta^2 f_1$$

$$\Delta^3 f_2 = \nabla^3 f_5 = \delta^3 f_{\frac{7}{2}}$$

$$\Delta^4 f_{-2} = \nabla^4 f_2 = \delta^4 f_0, \text{ etc.}$$

2.3.4 Shift Operator

The shift operator (also called the step operator) E is defined by,

$$Ef_r = f_{r+1}$$

$$E^{-1}f_r = f_{r-1}$$

$$E^2 f_r = E(Ef_r) = Ef_{r+1} = f_{r+2}$$

In general, $E^n f_r = f_{r+n}$

2.3.5 Mean Operator

The mean operator μ is defined by,

$$\mu f_r = \frac{1}{2}\left(f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}}\right)$$

$$\text{Thus, } \mu f_{r+\frac{1}{2}} = \frac{1}{2}\left\{f_{r+\frac{1}{2}+\frac{1}{2}} + f_{r+\frac{1}{2}-\frac{1}{2}}\right\} = \frac{1}{2}(f_{r+1} + f_r)$$

2.4 RELATIONSHIPS BETWEEN OPERATORS

Various relationships exist between operators. For example,

$$\Delta f_r = f_{r+1} - f_r$$

$$\Delta f_r = Ef_r - f_r = (E-1)f_r$$

or, $\Delta = E - 1$

or, $E = 1 + \Delta$

Similarly, $\nabla f_r = f_r - f_{r-1}$

$$= f_r - E^{-1}f_r$$

or, $\nabla = 1 - E^{-1}$

and $E = (1-\nabla)^{-1}$

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}} = E^{\frac{1}{2}}f_r - E^{-\frac{1}{2}}f_r$$

$$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)f_r$$

or $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

$$\text{Also, } \mu f_r = \frac{1}{2}\left(f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}}f_r + E^{-\frac{1}{2}}f_r\right)$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)f_r$$

or $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

The relationships between various operators are given in the following table:

	E	Δ	∇	δ
E	E	$1+\Delta$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$
Δ	$E-1$	Δ	$\nabla(1-\nabla)^{-1}$	$\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}$
∇	$1-E^{-1}$	$\Delta(1+\Delta)^{-1}$	∇	$-\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-\frac{1}{2}}$	$\nabla(1-\nabla)^{-\frac{1}{2}}$	δ
μ	$\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$	$\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-\frac{1}{2}}$	$\left(1-\frac{\nabla}{2}\right)(1-\nabla)^{-\frac{1}{2}}$	$\sqrt{1+\frac{\delta^2}{4}}$

The above relationships can easily be proved and we leave this as an exercise to the student to fill in the details of the above results.

PROBLEMS

1. Construct the difference tables for the following functions:
   a) $f(x) = x^4 - x - 1$, over the range, $x = -3(1)5$.
   b) $f(x) = 2x^3 + 2x^2 + 2x - 1$, over the range, $x = -1(1)7$.
   c) $f(x) = 2x^3 - 3x + 4$, over the range, $x = -1(.5)1$.
   d) $f(x) = 2^x$ for $x = 0(1)6$. Will there ever be a column of constant differences in this case?
   e) $f(x) = \sin x$ for $x = 1.0(0.1)1.6$.
   f) $f(x) = 2x^3 + 3x + 1$ for $x = 0.1(0.1)0.5$. What can you say about fourth order difference column? What is the reason for your observation?

g) $f(x) = 3x^3 + 4x^2 + 1$ for values $x = 0(1)5$. What do you conclude from the third-order differences column of the difference table based on this function?

2. (a) It is suspected that there is an error in one of the values of f(x) in the following table:

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f(x)	−38	−46	−59	−76	−92	−118	−140	−161	−180

Construct the differences-table, detect and correct the error.

(b) Consider the following table of values:

x	1	2	3	4	5	6	7	8	9	10	11
f(x)	7	10	17	33	63	121	185	287	423	598	817

It is suspected that one of the values may have been recorded in error. Assuming that the data follow a polynomial, determine which one, if any, of the functional values is in error and what it should be.

(c) Locate the error and estimate the correct value for the following table:

x	0	1	2	3	4	5	6	7	8
f(x)	1.0000	1.1002	1.2013	1.3045	1.4105	1.5210	1.6366	1.7586	1.8881

Construct the differences-table, detect and correct the error.

3. Locate and correct mistakes in each of the following tables:

x	1	2	3	4	5	6	7	8	9	10
f(x)	7	12	21	34	51	70	97	126	159	196
z(x)	.500	.520	.540	.560	.579	.589	.618	.637	.655	.674

4. The table of values for two quadratic polynomials y(x) and z(x) are given to 3 sf as follow:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y(x)	1.00	1.13	1.35	1.76	2.10	2.69	3.46
z(x)	4.00	4.87	5.91	7.15	8.60	10.3	12.3

Locate and correct the errors, other than those attributed to rounding off, in each table.

5. Compute the missing entries in the following table:

a.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x				
	x			
x		−4		
	3		−1	
9		−5		x
	x		x	
x		0		
	x			
x				

b.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x				
	x			
x		−3		
	2		x	
7		−3		x
	x		3	
x		0		
	x			
x				

c.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x				
	x			
1		12		
	x		x	
x		x		24
	60		60	
x		108		
	x			
241				

6. (a) The following table of values contains an error. Locate the incorrect value and find an estimate of correct value:

x	−1	0	1	2	3	4	5	6
f(x)	1.51	1.17	1.51	2.35	4.23	6.61	9.67	13.41

Reconstruct the difference table with the correct value. Comment on the nature of the function f(x).

(b) The table below contains an error. Locate and correct the error.

x	3.60	3.61	3.62	3.63	3.64	3.5	3.66	3.67	3.68
f(x)	.112046	.120204	.128350	.136462	.144600	.152702	.160788	.168857	.176908

(c) Use difference table method to locate and correct the error in the following table of values:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
f(x)	10.30	10.70	11.04	11.26	11.30	11.01	10.60	9.74	8.46	6.70

7. (a) From the difference table for the function given below. Find the values of a, b, and c, so that $\Delta^4 f(a) = \nabla^4 f(b) = \delta^4 f(c) = -0.0428$:

x	0	1	2	3	4	5	6
f(x)	.3679	.7358	.9197	.9810	.9963	.9994	.9999

(b) Tabulate the function $f(x) = x(x-1)(x-2)$ for $x = -0.2(0.1)0.2$, correct to 3 dp. What do you say on the value of $\delta^4 f$?

(c) Prove that the sum of the numbers in any column of a difference table is equal to the difference between the last and first numbers in the preceding column.

Set up a table showing the first and second differences for the following data to check the arithmetical work:

$$0.0000, -0.0104, -0.0206, -0.0307, -0.0404, -0.0496.$$

(d) Construct the difference table for the following functional values:

x	−2	−1	0	1	2	3	4
f(x)	15	1	1	3	19	85	261

If the origin $x_0 = 1$, determine the values of Δf_0, ∇f_{-1}, $\delta f_{3/2}$, $\delta^2 f_1$, $\Delta^3 f_0$, $\nabla^3 f_2$, $\Delta^2 f_1$ and $\delta^4 f_0$.

(e) Given the difference table,

f	f(x)	1st	2nd	3rd	4th
−2	−14				
		14			
0	0		0		
		14		96	
2	14		96		0
		110		96	
4	124		192		
		320			
6	426				

If the origin $x_0 = 2$, express using forward, backward and central differences in the entries, 110, 302 and 192.

8. (a) The values of y shown in the following table are alleged to be derived from a fourth degree polynomial. Test this and correct the value, where necessary.

x	0	1	2	3	4	5	6	7	8	9	10
f(x)	0	2	20	90	272	605	1332	2450	4160	6642	10100

(b) Suggest appropriate correction for the following table of values:

x	10	11	12	13	14	15	16	17	18
f(x)	2.1544	2.2240	2.2894	2.3513	2.4121	2.4662	2.5198	2.5713	2.6207

(c) The following table contains an error. Identify the error and estimate the correct value of the function:

x	1	2	3	4	5	6	7	8	9
f(x)	10	12	15	21	32	50	79	115	166

(d) Locate the incorrect entry in the following tables and estimate the correct value of each function:

i)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f(x)	−9.800	−9.061	−8.341	−7.594	−6.671	−5.776	−4.530	−2.945	−0.899	1.736	5.100

ii)

x	0:1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
f(x)	0.905	0.819	0.741	0.677	0.607	0.549	0.497	0.449	0.407

(e) Locate the incorrect entry in the following table and find its correct value:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
f(x)	0.000	0.012	0.072	0.252	0.672	1.500	2.952	5.922	8.832	13.932	21.000	30.492	42.912

9. Prove the following relationships:

(a) $\Delta = \sqrt{E}\,.\,\delta$ (b) $\Delta = E\nabla$ (c) $\delta^2 = \Delta - \nabla = \Delta\nabla = \nabla\Delta$ (d) $E = 1 + \mu\delta + \frac{\delta^2}{2}$

(e) $\mu^2 = 1 + \frac{\delta^2}{4}$ (f) $E = 1 + \delta\sqrt{E}$ (g) $\nabla = E^{-1}\Delta$ (h) $2\mu\delta = \Delta + \nabla$

(i) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (j) $E^{\frac{1}{2}} = \mu + \frac{1}{2}\delta$ (k) $E^{-\frac{1}{2}} = \mu - \frac{1}{2}\delta$ (l) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

10. (a) Find Δy_n, $\Delta^2 y_n$ and $\Delta^3 y_n$ in the following cases.

(i) $y_n = n^2$
(ii) $y_n = n^3 + 3n^2$
(iii) $y_n = n^3 - n^2 + 17n - 1$
(iv) $y_n = n(n-1)(n-2)(n-3)(n-4)$

(b) Prove that $y_n = 3^n(A + Bn)$ satisfies the equation,

$$y_{n+2} - 6y_{n+1} + 9y_n = 0$$

(c) If $f(x) = \sin(\pi x)$, prove that $\Delta f = -2f$.

(d) If $f(x) = x^3$, compute the following:

(i) $\left(\frac{\Delta^2}{E^2}\right)f(x)$ (ii) $\frac{\Delta^2 f(x)}{E^2 f(x)}$

(e) Obtain the following results:

(i) $\Delta\left(\frac{f_n}{g_n}\right)$ (ii) $\Delta\left(\frac{1}{g_n}\right)$ (iii) $\Delta(\log f_n)$ (iv) $\Delta(f_n\,.\,g_n) = f_n\Delta g_n + g_{n+1}\Delta f_n$

(v) $\Delta\sqrt{f_r} = \frac{\Delta f_r}{\sqrt{f_r} + \sqrt{f_{r+1}}}$

(f) Find $\Delta^2 x^4$

(g) Find $x\Delta(x\Delta - 1)x^2$

11. (a) Show that

(i) $\Delta f_i = \nabla f_{i+1} = \delta f_{i+1/2}$ (ii) $\Delta^2 f_i = \nabla^2 f_{i+2} = \delta^2 f_{i+1}$ (iii) If $f(x) = x^4$, then $\Delta^2 f(x) = 24$

(iv) If $f(x) = 2^x$, then $\Delta f(x) = f(x)$.

(b) Show that

(i) $\nabla^3 f_i = \Delta^3 f_{i-3}$ (ii) $\Delta^4 f_i = \nabla^4 f_{i+4}$ (iii) $\Delta^3\nabla f_i = \Delta^4 f_{i-1}$ (iv) $\delta^2 f_i = \Delta^2 f_{i-1}$

Chapter 3

Interpolation

3.1 INTRODUCTION

Suppose, we are given a table based on certain values of x and the corresponding values of a function f(x):

x	0	1	2	3	...	100
f(x)	10	85	90	98	...	125

The values in the table can be obtained by an experiment or generated if we know the function f(x). The process of computing an approximate value of the function at some point within the range (0, 100), but not in the table of data, is called **interpolation**. If the value of x lies outside the range, the process of estimating the value is called **extrapolation**.

Error of extrapolation increases as the point of interest goes farther from the data points. If a higher order interpolation is used for extrapolation without theoretical basis, errors may increase rapidly as the order of polynomial increases. Application of extrapolation may be seen in various sections of this book: for instance, see the Newton-Cotes open integration formulas, the Romberg's integration method and the predictor-corrector methods.

In most of this chapter, we limit the interpolating function to be a **polynomial**. Interpolation has many applications in approximation theory, numerical differentiation, numerical integration, numerical solutions of ordinary and partial differential equations, and for making computer drawn curves to pass through specified points.

We are now going to describe several methods; in each case some kind of advice is given as to the circumstances under which the method should be applied.

3.1.1 Choice of a Suitable Interpolation Formula

The following points are considered in choosing a method for interpolating polynomials:

a) Whether the given points x_i are equally or unequally spaced.

b) Whether the interpolation is desired towards the beginning, centre or end of a difference table.

3.1.2 Checking The Interpolated Value

The next is the question of checking the interpolated value. A single interpolation is not easy to check. One possibility is to repeat the interpolation using a different formula, but this will be more than double the labour, since the first-interpolation is usually done by the

easiest formula. When possible a functional relationship such as $e^{-x} = \frac{1}{e^x}$ is a better check. This still requires two-interpolations but since they involve different tables, the formula may be used for both.

3.1.3 Type of Interpolation Formulas for Equally-Spaced Data Points

The following three types of interpolation formulas are used for equally-spaced data points:

a) **Newton's forward difference interpolation formula.** It uses differences near the beginning of the difference table.

b) **Newton's backward difference interpolation formula.** It uses differences near the end of the difference table.

c) **Central difference interpolation formulas.** These formulas employ differences in the centre of the difference table. The following central difference formulas are commonly used:

   i) Stirling's formula
   ii) Bessel's formula
   iii) Everett's formula
   iv) Gauss forward and backward formulas

3.1.4 Type of Interpolation Formulas for Unequally-Spaced Data Points

The following formulas may be used for unequally-spaced data points:

i) Newton's divided difference interpolation formula;
ii) Lagrange's formula;
iii) Aitken's formula; and
iv) Hermite's formula.

We shall describe only Lagrange and Aitken formulas, because they are suitable for both, equally and unequally-spaced data points. The above formulas can also be employed for extrapolation; however, the error may increase rapidly the farther we extrapolate from the given values. With the widespread use of computers tabular interpolation has lost much of its importance. The methods under the present category have been widely use.

3.2 NEWTON'S FORWARD DIFFERENCE INTERPOLATION FORMULA

The most basic formula for interpolation with equidistant points is Newton's forward difference interpolation (sometimes also called the Gregory-Newton) formula.

Given a set of n pairs of values:

$$(x_0, f_0), (x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n)$$

We shall derive this formula with the help of two difference operators, E and Δ.

The function to be estimated is written as:

$$f_p = E^p f_0 = (1+\Delta)^p f_0, \qquad \ldots \quad (3.1)$$

Expanding $(1+\Delta)^p$, we have

$$f_p = \left\{1 + p\Delta + \frac{1}{2!}p(p-1)\Delta^2 + \frac{1}{3!}p(p-1)(p-2)\Delta^3 + \cdots + \frac{1}{n!}p(p-1)(p-2)\cdots(p-n+1)\Delta^n\right\}f_0$$

$$= f_0 + p\Delta f_0 + \frac{1}{2!}p(p-1)\Delta^2 f_0 + \frac{1}{3!}p(p-1)(p-2)\Delta^3 f_0 + \cdots + \frac{1}{n!}p(p-1)(p-2)\cdots(p-n+1)\Delta^n f_0 \qquad \ldots \quad (3.2)$$

where $p = \frac{(x_p - x_0)}{h}$, obtained from $x_p = x_0 + ph$

The coefficient of $\Delta^n f_0$ will contain p^n which is an nth degree polynomial.

Remarks

i) This formula is used for interpolation near the beginning of a difference table, but in odd cases, it may also be applied in other parts of the table by suitably shifting the origin. Shifting the origin does not affect the result, but on the other hand it may result in a simpler formula, which is less prone to error.

ii) This formula is usually applicable for $0 < p < 1$. When working with differences, we can select any value of x in the table to be labeled as x_0. This is mostly done to keep p within the range.

Example 1 a) Compute the difference table for the following set of data-points:

x	.00	.25	.50	.75	1.00
f(x)	.0000	.2763	.5205	.7112	.8427

b) Use Newton's forward difference formula to pass a fourth degree polynomial through the above data.

c) Use the above polynomial to interpolate for f(0.125).

Solution a) The forward differences are computed as follows:

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
0.00	<u>.0000</u>				
		<u>2763</u>			
0.25	.2763		<u>−321</u>		
		2442		<u>−214</u>	
0.50	.5205		−535		<u>157</u>
		1907		−57	
0.75	.7112		−593		
		1315			
1.00	.8427				

b) $h = x_1 - x_0 = .25 - .00 = .25$

$x_p = x_0 + ph = 0.125$

$$p = \frac{(x_p - x_0)}{h} = \frac{(.125 - .00)}{.25} = = 0.5$$

Since the calculated value of p lies in the range (0, 1), it makes the forward difference formula applicable. The values to be used in formula (3.2) are underlined in the above table.

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0$$

$$= .000 + p \times 0.2763 + \frac{(p^2-p)}{2} \times -0.0321 + \frac{(p^3-3p^2+p)}{6} \times -0.0214 + \frac{(p^4 - 6p^3 + 11p^2 - 6p}{6} \times 0.0157$$

$$= .2763p - .0125p^2 + .0008p^3 - .0006p^4$$

c) Inserting $p = 0.50$ in the above polynomial, we get

$$f_p = .2763 \times .50 - .0125 \times (.50)^2 + .0008 \times (.50)^3 - .0006 \times (.50)^4$$

$$= .13815 - .00313 + .0001 - .00004 = 0.1351$$

The students should be careful not to think of the answer 0.1351 as the correct answer. It is an estimate of the correct answer based on the assumption that f(x) is a fourth-degree polynomial.

Example 2 Use Newton's forward difference formula to interpolate the value for f(1.75) from the following data:

(0.5, 0.000), (1.0, 1.375), (1.5, 2.000), (2.0, 2.625), and (2.5, 4.000).

Solution The difference table is as follows:

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
0.5	0.000				
		1375			
1.0	1.375		−750		
		625		750	
1.5	2.000		0		0
		625		750	
2.0	2.625		750		
		1375			
2.5	4.000				

$x_p = 1.75$; $x_0 = 0.5$; $x_1 = 1.0$;
$h = x_1 - x_0 = 0.5$

$$p = \frac{(1.75-0.5)}{0.5} = 2.5$$

As $p(=2.5)$ does not lie between 0 and 1, we cannot use the origin to be $x_0 = 0.5$. Let us shift the origin to 1.0. Then, $p = \frac{(1.75-1)}{0.5} = 1.5$. We cannot use $x_0 = 1$ as the origin because still $p > 1$. Let us shift the origin to $x_0 = 1.5$. So, we can use $x_0 = 1.5$ as the origin because the calculated value of $p < 1$.

The entries used in this case are underlined in the difference table. The reduced form of Newton's formula is as follows:

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2}\Delta^2 f_0$$

Inserting the values in the above formula, we get,

$$f_p = 2.000 + 0.5 \times 0.625 + \frac{0.5(0.5-1)}{2} \times 0.750$$

$$= 2.000 + 0.313 - 0.094 = 2.219$$

Example 3 Write a computer program to implement Newton's forward difference interpolation formula. Use the following data for testing:

x	2	4	6	8	10	12	14
f(x)	23	93	259	569	1071	1813	2843

Estimate f(2.58).

Solution For computer program, see Example 4. However, the computer output for this example is given below:

Computer Output:

X	F(X)	1ST	2ND	3RD	4TH
2.00	23.00				
		70.00			
4.00	93.00		96.00		
		166.00		48.00	
6.00	259.00		144.00		.00
		310.00		48.00	
8.00	569.00		192.00		.00
		502.00		48.00	
10.00	1071.00		240.00		.00
		742.00		48.00	
12.00	1813.00		288.00		
		1030.00			
14.00	2843.00				

ANSWER = 36.23

3.3 NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

We shall derive Newton's backward difference formula from the two difference operators E and ∇.

We know that, $f_p = E^p f_0 = (1-\nabla)^{-p} f_0$ ... (3.3)

Expanding $(1-\nabla)^{-p}$, we obtain,

$$f_p = \left\{1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots + \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n\right\} f_0$$

$$= f_0 + p\nabla f_0 + \frac{p(p+1)}{2!}\nabla^2 f_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 f_0 + \cdots$$

$$+ \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n f_0 \quad \text{... (3.4)}$$

This is called **Newton's backward difference interpolation** (also the **Gregory-Newton) formula.**

Remarks

i) This formula is used towards the end of the difference table but can also be applied in other parts of the table by suitably shifting the origin. This situation occurs whenever a table is being extended, for example, when the solution to a differential equation is being obtained by a step by step method.

ii) The formula is valid for $0 < p < 1$

Example 4 (a) Using Newton's backward difference formula, compute f(11.8) from the following data:

x	2	4	6	8	10	12	14
f(x)	23	93	259	569	1071	1813	2843

(b) Write a computer program to implement Newton's backward difference interpolation formula.

Solution (a) The backward differences are computed in the following table:

x	f(x)	∇	∇^2	∇^3	∇^4
2	23				
		70			
4	93		96		
		166		48	
6	259		144		0
		310		<u>48</u>	
8	569		<u>192</u>		0
		<u>502</u>		48	
10	<u>1071</u>		240		0
		742		48	
12	1813		288		
		1030			
14	4843				

$x_p = 11.8$

Taking $x_0 = 14$, $p = \frac{(11.8 - 14)}{2} = -1.1$. As the calculated value is outside its acceptable range, we cannot accept the origin to be at $x_0 = 14$. The suitable origin may be $x_0 = 10$, which gives $p = 0.9$. The entries used for the backward difference formula are underlined in the above difference table. Substituting these values in formula (3.4), we get,

$$f_p = 1071 + 0.9 \times 502 + \frac{.9(.9+1)}{2} \times 192 + \frac{.9(.9+1)(.9+2)}{6} \times 48$$

$$= 1071 + 451.8 + 164.16 + 39.67 = 1727$$

(b) **Program No. 2. Interpolation**

This program can be used for Newton's forward and backward interpolation formulas. It is done via a main menu. Menu Choice 1 is for the forward difference formula, while menu Choice 2 is for the backward difference formula.

Computer Program

```
# include<iostream.h>
# include<conio.h>
# include<process.h>

float interval, x0, p, array[20][20] = {0.0};
int no, col, x,y;
void difftable( )
  {
    cout<<"\tDIFFERENCE TABLE";
    cout<<"\n\n\tENTER THE FIRST VALUE  : "; cin>>array[0][0];
    cout<<"\n\tENTER THE INTERVAL  :  "; cin>>interval;
```



```
cout<<"\n\tENTER TOTAL NO. OF X  :  "; cin>>no;

for(int i=1;i<no;i++)
  {
array[i][0]=array[i-1][0]+interval;
  }

cout<<"\n\tENTER FUNCTIONAL VALUES  : \n";
for (i=0; i<no; i++)
  {
    cout<<"\tX("<<i<<") = "; cin>>array[i][1];
  }

cout<<"\n\tHOW MANY COLUMNS ARE REQUIRED  :  "; cin>>col;
for (i=2; i<=(col+2); i++)
  {
    for (int j=0; j<=(no-i); j++)
      {
        array[j][i]=array[j+1][i-1]-array[j][i-1];
      }
  }

clrscr ( );
cout<<"\t\tDIFFERENCE TABLE\n";
cout<<"  X    F(X)  ";
for (i=1; i<=col;i++)
  {
    cout<<"      col  "<<i;
  }
cout<<"\n";

for (i=0; i<no; i++)
  {
  cout<<"   "<<array[i][0]<<"\n\n";
  }

x=8; y=3;
for(i=1; i<=(col+1);i++)
  {
    gotoxy(x,y);
    for(int j=0; j<=(no-i); j++)
      {
        cout<<array[j][i];
        y+=2;
        gotoxy (x,y);
      }
    x+=9; y=i+3;
```

```
}

void findx( )
  {
     float xp;
     cout<<"\n\tXP FOR WHICH VALUE OF F(X) IS REQUIRED  :  "; cin>>xp;
     int i=0;
     while(((xp-array[i][0])/interval>1)&&(i<no))
       {
         i++;
       }
     xo=i;
     p=(xp-array[xo][0])/interval;
  }

void nford ( )
  {
     findx ( );

     cout<<"\n\n\tanswer = ";
     cout<<(array[xo][1]+(p*array[xo][2])+(p* (p-1)/2*array[xo][3])
     +(p* (p-1)*(p-2) /6*array[xo][4])+(p* (p-1) * (p-2) * (p-3) /24*array[xo][5]);
  }

void nback( )
  {
     findx( );
     cout<<"\n\n\tanswer=  ";
     cout<<(array[x0][1]+p*array [x0-1][2])+(p*(p+1) /2 * array[x0-2][3])+(p*(p+1)
     *(p+2) /6 * array[x0-3][4])+(p*(p+1)*(p+2)*(p+3) /24 * array[x0-4][5]);
  }

void main (void)
  {
     clrscr ( ); difftable ( ); getch ( );

     int choice;
     while (1)
       {
        clrscr ( );
        cout<<"\n\n\t\tMAIN MENU";
        cout<<"\n\n\tFORWARD DIFFERENCE INTERPOLATION FORMULA  --- 1";
        cout<<"\n\n\tBACKWARD DIFFERENCE INTERPOLATION FORMULA --- 2";
        cout<<"\n\n\tTO EXIT -----------------------------------------------3";
        cout<<"\n\n\n\tENTER YOUR CHOICE  : ";
        cin>>choice;
```



```
        switch(choice)
          {
            case 1:clrscr ( );nford();getch( );break;
            case 2:clrscr( );nback();getch( );break;
            case 3:exit(0);
          }
       }
  }
```

Computer Output

DIFFERENCE TABLE

ENTER THE FIRST VALUE : 2

ENTER THE INTERVAL   : 2

ENTER TOTAL NO. OF X   : 7

ENTER FUNCTIONAL VALUES :
X(0) = 23
X(1) = 93
X(2) = 259
X(3) = 569
X(4) = 1071
X(5) = 1813
X(6) = 2843

HOW MANY COLUMNS ARE REQUIRED : 4

DIFFERENCE TABLE

X	F(X)	col 1	col 2	col 3	col 4
2	23				
		70			
4	93		96		
		166		48	
6	259		144		0
		310		48	
8	569		192		0
		502		48	
10	1071		240		0
		742		48	
12	1813		288		
		1030			
14	2843				

MAIN MENU

FORWARD DIFFERENCE INTERPOLATION FORMULA --- 1

BACKWARD DIFFERNECE INTERPOLATION FORMULA -- 2

TO EXIT -- 3

ENTER YOU CHOICE : 1

XP, FOR WHICH VALUE OF F(X) IS REQUIRED : 2.58

A N S W E R = 36.233509

ENTER YOUR CHOICE : 2

XP, FOR WHICH VALUE OF F(X) IS REQUIRED : 11.8

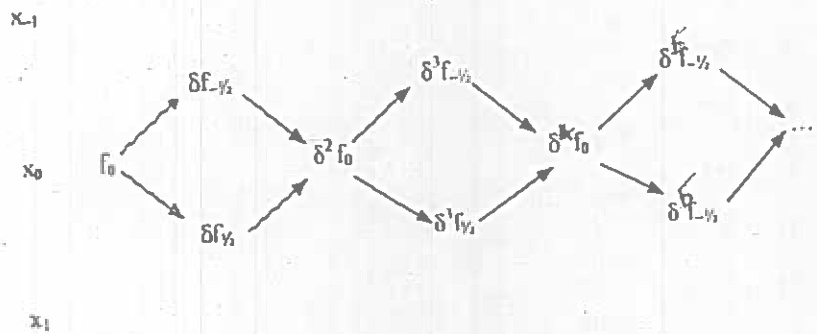
A N S W E R = 1726.63208

3.4 INTERPOLATION WITH CENTRAL DIFFERENCE FORMULAS

The two formulas by Newton are used only occasionally and almost exclusively at the beginning or at the end of a table. More important are formulas which make use of central differences, a whole series of such formulas with slightly different properties can be constructed. In this section, we shall mention without proofs some well-known central difference formulas. The structure of all these formulas can easily be demonstrated by sketching a difference scheme, where different quantities are represented by points. The column to the left stands for the function values, then we have the first differences and so on.

3.4.1 Stirling's Interpolation Formula

Stirling's formula follows the path through the difference table given below:



It is expressed as follows:

$$f_p = f_0 + \frac{1}{2}p\left(\delta f_{-\frac{1}{2}} + \delta f_{\frac{1}{2}}\right) + \frac{1}{2!}p^2\delta^2 f_0 + \frac{p(p^2-1)}{2\times 3!}\left(\delta^3 f_{-\frac{1}{2}} + \delta^3 f_{\frac{1}{2}}\right)$$
$$+\frac{p^2(p^2-1)}{4!}\delta^4 f_0 + \frac{p(p^2-1)(p^2-4)}{2\times 5!}\left(\delta^5 f_{-\frac{1}{2}} + \delta^5 f_{\frac{1}{2}}\right)$$
$$+\frac{p^2(p^2-1)(p^2-4)}{6!}\delta^6 f_0 + \cdots \qquad \ldots\ 3.5(a)$$

It can also be written in another form as:

$$f_p = f_0 + p\mu\delta f_0 + \frac{1}{2!}p^2\delta^2 f_0 + \frac{1}{3!}p(p^2-1)\mu\delta^3 f_0 + \frac{1}{4!}p^2(p^2-1)\delta^4 f_0$$
$$+\frac{1}{5!}p(p^2-1)(p^2-4)\mu\delta^5 f_0 + \frac{1}{6!}p^2(p^2-1)(p^2-4)\delta^6 f_0 + \cdots \qquad \ldots\ 3.5(b)$$

Example 5 Use Stirling's interpolation formula to find f(1.62) from the following table:

x	1.2	1.4	1.6	1.8	2.0
f(x)	5.64642	6.44218	7.17356	7.83327	8.41471

Solution The difference table is as follows:

x	f(x)	δ	δ^2	δ^3	δ^4
1.2	5.64642				
		79576			
1.4	6.44218		-6438		
		73138		-729	
x_0 = 1.6	7.17356		-7167		69
		65971		-660	
1.8	7.83327		-7827		
		58144			
2.0	8.41471				

Taking $x_0 = 1.6$, $h = x_1 - x_0 = 1.8 - 1.6 = .2$

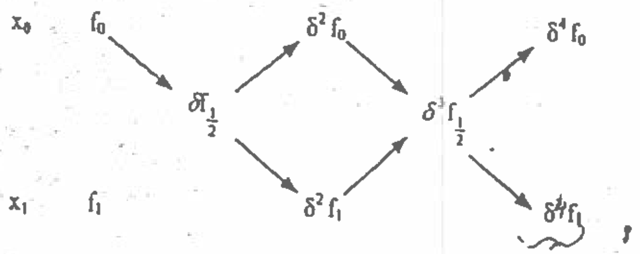
$$p = \frac{(1.62 - 1.6)}{0.2} = \frac{0.02}{0.2} = 0.1$$

Inserting the values in Stirling's formula 3.5(a), we get,

$$f_p = 7.17356 + \frac{1}{2}\times 0.1(.73138 + .65971) + \frac{1}{2}\times 0.1\times 0.1\times -.07167$$
$$+\frac{0.1(0.1\times 0.1 - 1)}{12}\times(-.00729 - .00660)$$
$$+\frac{0.1\times 0.1\ (0.1\times 0.1 - 1)}{24}\times .00069$$
$$= 7.17356 + 0.06955 - 0.00036 + 0.00011 - 0.00000$$
$$= 7.24286$$

3.4.2 Bessel's Interpolation Formula

Bessel's formula follows path through the difference table:



Bessel's formula is expressed as follows:

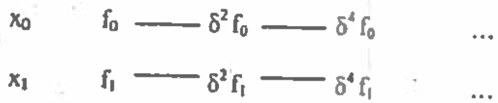
$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2.2!}(\delta^2 f_0 + \delta^2 f_1) + \frac{p(p-1)(p-\frac{1}{2})}{3!}\delta^3 f_{\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{2.4!}(\delta^4 f_0 + \delta^4 f_1) + \ldots \quad \ldots\ 3.6(a)$$

It can also be written in another form as:

$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{1}{2!}p(p-1)\mu\delta^2 f_{\frac{1}{2}} + \frac{1}{3!}p(p-1)(p-\frac{1}{2})\mu\delta^2 f_{\frac{1}{2}} + \frac{1}{4!}(p+1)p(p-1)(p-2)\mu\delta^4 f_{\frac{1}{2}} + \cdots \quad \ldots\ 3.6(b)$$

3.4.3 Everett's Interpolation Formula

Everett's formula follows the path through the difference table:



Everett's formula is expressed as follows:

$$f_p = qf_0 + \frac{q(q^2-1)}{3!}\delta^2 f_0 + \frac{q(q^2-1)(q^2-4)}{5!}\delta^4 f_0 + \cdots + pf_1 + \frac{p(p^2-1)}{3!}\delta^2 f_1 + \frac{p(p^2-1)(p^2-4)}{5!}\delta^4 f_1 + \cdots \quad \ldots\ (3.7)$$

where $q = 1 - p$.

3.4.4 Gaussian Interpolation Formulas

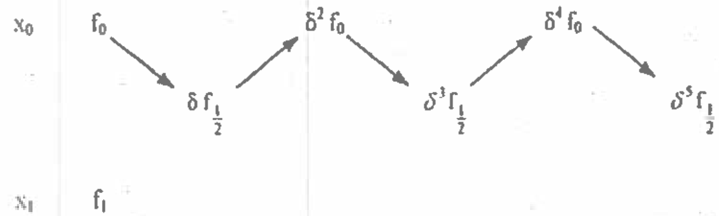
a) Gauss Forward Interpolation Formula

b) Gauss Backward Interpolation Formula

Let us discuss them one by one.

a) Gauss Forward Interpolation Formula

This formula follows the zigzag path as indicated in the difference table:

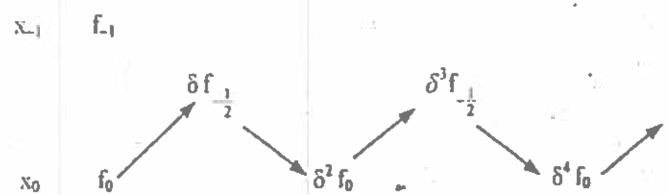


Gauss forward difference formula is expressed as follows:

$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 f_0 + \frac{p(p+1)(p-1)}{3!}\delta^3 f_{\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \cdots \quad \ldots\ (3.8)$$

b) Gauss Backward Interpolation Formula

This formula follows the zigzag path as indicated in the following difference table:



Gauss backward formula is expressed as follows:

$$f_p = f_0 + p\delta f_{-\frac{1}{2}} + \frac{p(p+1)}{2!}\delta^2 f_0 + \frac{p(p+1)(p-1)}{3!}\delta^3 f_{-\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \cdots \quad \ldots\ (3.9)$$

Note that if we take the mean of Gauss forward and backward formulas, we get Stirling's interpolation formulas.

As mentioned earlier that for most purposes formulas using central differences are to be preferred. However, some remarks about their use are in order. The two Gaussian interpolation formulas are of interest almost exclusively from theoretical standpoint. Stirling's formula is suitable for values of small values of p, for example, $-\frac{1}{4} \le p \le +\frac{1}{4}$, and Bessel's formula which is probably the most used of all interpolation formulas, is suitable for values of p not too far from $\frac{1}{2}$, for example, $\frac{1}{4} \le p \le \frac{3}{4}$. Everett's formula which is simple and fast is perhaps the one which is most generally useful and further because even differences often are used together with the function values.

Example 6 Given the following table of values:

x	2.2	2.6	3.0	3.4	3.8	4.2	4.6
f(x)	.374607	.438371	.500000	.559193	.615661	.669131	.719340

a) Construct the difference table including differences up to 4th order only.

b) Interpolate f(3.64) using the following formulas centred at x = 3.4:

(i) Stirling (ii) Bessel (iii) Everett and (iv) Gaussian forward and backward

Solution (a) Difference Table

x	f(x)	δ	δ^2	δ^3	δ^4
2.2	.374607				
		63764			
2.6	.438371		−2135		
		61629		−301	
3.0	.500000		−2436		12
		59193		−289	
x = 3.4	.559193		−2725		16
		56468		−273	
3.8	.615661		−2998		10
		53470		−263	
4.2	.669131		−3261		
		50209			
4.6	.719340				

b) (i) Stirling's Formula

$x_p = 3.64,\quad x_0 = 3.4,\quad h = 0.4$

$$p = \frac{(x_p - x_0)}{h} = \frac{(3.64 - 3.4)}{0.4} = 0.6$$

Substituting values in formula (3.5(a)), we get,

$$f_p = .559193 + \frac{.6}{2}(.059193 + .056468) + \frac{.6 \times .6}{2} \times -.002725$$

$$+ \frac{.6(.6 \times .6 - 1)}{12}(-.000289 - .000273) + \frac{.6 \times .6(.6 \times .6 - 1)}{24} \times .000016$$

$$= .559193 + .034698 - .000491 + .000018 - .000000$$

$$= 0.593418$$

ii) Bessel's Formula

Substituting values in (3.6(a)), we get,

$$f_p = .559193 + 0.6 \times .056468 + \frac{.6(.6-1)}{4}(-.002725 - .002998)$$

$$+ \frac{.6(.6-1)(.6-\frac{1}{2})}{6} \times -0.000273$$

$$+ \frac{(.6+1)\,.6(.6-1)\,(.6-2)}{48} \times (.000016 + .000010)$$

$$= 0.559193 + 0.033881 + .000343 + .000001 + .000000$$

$$= 0.593418$$

iii) Everett's Formula

$$q = 1 - .6 = .4$$

Substituting values in (3.7), we get,

$$f_p = .4 \times 0.559193 + \frac{.4(.4 \times .4 - 1)}{6} \times -0.002725$$

$$+ \frac{.4(.4 \times .4 - 1)\,(.4 \times .4 - 4)}{120} \times 0.000016$$

$$+ .6 \times 0.615661 + \frac{.6(.6 \times .6 - 1)}{6} \times -0.002998$$

$$+ \frac{.6(.6 \times .6 - 1)\,(.6 \times .6 - 4)}{120} \times 0.000010$$

$$= .223677 + .000153 + .000000 + .369397 + .000192 + .000000$$

$$= 0.593419$$

iv) a) Gauss Forward Formula

Substituting values in (3.8), we get,

$$f_p = .559193 + 0.6 \times .056468 + \frac{.6(.6-1)}{2} \times -.002725$$

$$+ \frac{.6(.6+1)(.6-1)}{6} \times -.000273$$

$$+ \frac{(.6+1)\,.6(.6-1)\,(.6-2)}{24} \times .000016$$

$$= 0.559193 + 0.033881 + .000327 + 0.000017 + .00000$$

$$= 0.593418$$

b) Gauss Backward Formula :

Substituting values in (3.9), we get,

$$f_p = .559193 + 0.6 \times .059193 + \frac{.6(.6+1)}{2} \times -.002725$$

$$+\frac{(.6+1).6(.6-1)}{6} \times -.000289$$

$$+\frac{.6(.6+1)(.6-1)(.6-2)}{24} \times .000016$$

$$= 0.559193 + .035516 + .001308 + .000018 + .000000$$

$$= 0.593419$$

3.5 LAGRANGE'S FORMULA

It was mentioned in the previous sections that difference table could be used for interpolation, but this was restricted to the case of function values at equidistant intervals.

To introduce the basic idea behind the Lagrange's formula, consider the following:

Given the data points $x_0, x_1, \ldots, x_n$ (may or may not be equidistant), the problem is to find an nth degree polynomial $f(x)$ using Lagrange's formula.

Lagrange's formula can be derived by writing:

$$\begin{aligned} f(x) &= A_0(x-x_1)(x-x_2)\ldots(x-x_n) \\ &+ A_1(x-x_0)(x-x_2)\ldots(x-x_n) \\ &+ A_2(x-x_0)(x-x_1)\ldots(x-x_n) \\ &\vdots \\ &+ A_i(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n) \\ &\vdots \\ &+ A_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \qquad \ldots (3.10) \end{aligned}$$

where $A_0, A_1, \ldots, A_n$ are unknown constants. If we substitute $x = x_0$ in (3.10), we get,

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

Similarly, substituting $x = x_1, x = x_2, \cdots, x = x_n$ respectively in (3.10), we get,

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

$$A_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)}$$

$$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Substituting the values of $A_0, A_1, \cdots, A_n$ in (3.10), we get the following formula due to Lagrange:

$$\begin{aligned} f(x) &= \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) \\ &+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)} f(x_2) \\ &+ \cdots \qquad \cdots \\ &+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n) \qquad \ldots (3.11) \end{aligned}$$

It is obvious that (3.11) is a polynomial of degree n and can be written as:

$$f(x) = L_0(x)\,f(x_0) + L_1(x)\,f(x_1) + L_2(x)\,f(x_2) + \cdots + L_n(x)\,f(x_n)$$

It can be concisely represented as:

$$f(x) = \sum_{i=0}^{n} L_i(x)\,f(x_i) \qquad \ldots (3.11(a))$$

where $L_i(x) = \prod_{i=0}^{n}\left(\frac{x-x_j}{x_i-x_j}\right)$

$j = 0$
$j \neq i$

Another form of this formula is:

$$f(x) = \sum_{i=0}^{n}\left(\prod_{\substack{j=0 \\ j\neq i}}^{n}\left(\frac{x-x_j}{x_i-x_j}\right)\right) f(x_i) \qquad \ldots (3.11(b))$$

The basic formula, apparently due to Waring, is associated with the name of Lagrange. This is one of the more practical and simpler method to be used on computer; but difficult for hand calculations if data points are more. Evaluation of error is also not easy.

Example 7 (a) Fit a polynomial using Lagrange's formula to the following data:

(1, 4), (3, 7), (4, 8) and (6, 11)

(b) Use the polynomial to estimate a value for x = 5.

(c) Write a computer program to implement Lagrange's formula.

Solution (a) The points are:

x	1	3	4	6
f(x)	4	7	8	11

Inserting the values in Lagrange's formula, we get:

$$f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} \times 4 + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} \times 7$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} \times 8 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} \times 11$$

$$= \frac{-2}{15}(x^3 - 13x^2 + 54x - 72) + \frac{7}{6}(x^3 - 11x^2 + 34x - 24)$$

$$\frac{-8}{6}(x^3 - 10x^2 + 27x - 18) + \frac{11}{30}(x^3 - 8x^2 + 19x - 12)$$

$$= \frac{1}{30}(2x^3 - 21x^2 + 103x + 36)$$

(b) The interpolated value at x = 5, is as follows:

$$f(5) = \frac{1}{30}(2 \times 5^3 - 21 \times 5^2 + 103 \times 5 + 36)$$

$$= \frac{1}{30} \times 276 = 9$$

b) Program No. 3 **LAGRANGE'S FORMULA**

```
# include<iostream.h>
# include<conio.h>

void main (void)
  {
   float table[10][2], xp,temp,ans=0.0;
   int no,y=0,a=7,i,j;

   cout<<"How Many Values Of X  : ";
   cin>>no;
   cout<<"\nEnter The Values Of X and f(x)\n";
   cout<<"\n\t  x       |        f(x)";
   cout<<"\n\t-----------------------";
```


f,);

```
   for(i=0;i<no;i++)                                   // Input of X & Fx
     {
       gotoxy(11,a);
       cin>>table[i][y];
       gotoxy(21,a);
       cin>>table[i][y+1];
       a++;
     }

   cout<<"\nEnter The Value of X       : ";
   cin>>xp;

   for(j=0;j<no;j++)                                   // calculation of formula
     {
       temp=1;
       for(i=0;i<no;i++)
          if(i!=j)
            temp *= ((xp-table[i][0]) / (table[j][0]-table[i][0]));

       ans += temp * table[j][1];
     }

   cout<<"\nA N S W E R =          : "<<ans;        // out put
  }
```

Computer Output

How Many Values Of X : 4

Enter The Values Of x and f(x)

x	f(x)
1	4
3	7
4	8
6	11

Enter The Value of X : 5

A N S W E R = : 9.2

3.6 ITERATIVE INTERPOLATION METHOD

Like Lagrange's method, this formula is also more suitable for computer application and its use is also not limited to uniformly spaced data. The iterative interpolation process is based on the repeated application of simple (linear) interpolation method. This method is due to Aitken.

Consider the following data points (equally or unequally-spaced):

x	x_0	x_1	x_2	x_3	...	x_n
$f(x)$	f_0	f_1	f_2	f_3	...	f_n

In order to estimate the value of the function f corresponding to any value of x, we proceed as follows:

Let $f_0 = f(x_0)$

$f_1 = f(x_1)$

$\vdots$

$f_k = f(x_k)$

$\vdots$

$f_n = f(x_n)$

also let $f(x \mid x_0, x_1, \ldots, x_n)$ denote the unique polynomial of degree n coinciding with f(x) at x_0, $x_1, \ldots, x_n$.

Hence, $f(x \mid x_0) = f(x_0)$

$f(x \mid x_1) = f(x_1)$

$\vdots$

$f(x \mid x_n) = f(x_n)$

Also, $$f(x \mid x_0, x_1) = \frac{1}{(x_1 - x_0)} \begin{vmatrix} x - x_0 & f(x|x_0) \\ x - x_1 & f(x|x_1) \end{vmatrix} = \frac{1}{x_1 - x_0} \begin{vmatrix} x - x_0 & f_0 \\ x - x_1 & f_1 \end{vmatrix}$$

$$= \frac{1}{(x_1 - x_0)} \{(x - x_0)\, f_1 - (x - x_1)\, f_0\}$$

$$f(x \mid x_0, x_2) = \frac{1}{x_2 - x_0} \begin{vmatrix} x - x_0 & f_0 \\ x - x_2 & f_2 \end{vmatrix}$$

$$= \frac{1}{(x_2 - x_0)} \{(x - x_0)\, f_2 - (x - x_2)\, f_0\}, \text{ etc.}$$

Similarly, $$f(x \mid x_0, x_1, x_2) = \frac{1}{(x_2 - x_1)} \begin{vmatrix} x - x_1 & f(x|x_0, x_1) \\ x - x_2 & f(x|x_0, x_2) \end{vmatrix}$$

$$f(x \mid x_0, x_1, x_3) = \frac{1}{(x_3 - x_1)} \begin{vmatrix} x - x_1 & f(x|x_0, x_1) \\ x - x_3 & f(x|x_0, x_3) \end{vmatrix}$$

and $$f(x \mid x_0, x_1, x_4) = \frac{1}{(x_4 - x_1)} \begin{vmatrix} x - x_1 & f(x|x_0, x_1) \\ x - x_4 & f(x|x_0, x_4) \end{vmatrix}$$

denote polynomials of degree ≤ 2 that pass through the four points (x_0, f_0), (x_1, f_1), (x_2, f_2); (x_0, f_0), (x_1, f_1), (x_3, f_3); and (x_0, f_0), (x_1, f_1), (x_4, f_4) respectively.

whereas $$f(x \mid x_0, x_1, x_2, x_3) = \frac{1}{x_3 - x_2} \begin{vmatrix} x - x_2 & f(x|x_0, x_1, x_2) \\ x - x_3 & f(x|x_0, x_1, x_3) \end{vmatrix}$$

denotes polynomial of degree ≤ 3 and so on.

Continuing the above process, we can develop the interpolating polynomials to any degree we want.

$$f(x \mid x_0, x_1, x_2, x_3, x_4) = \frac{1}{x_4 - x_3} \begin{vmatrix} x - x_3 & f(x|x_0, x_1, x_2, x_3) \\ x - x_4 & f(x|x_0, x_1, x_2, x_4) \end{vmatrix}$$

$$f(x \mid x_0, x_1, x_2, x_3, x_5) = \frac{1}{x_5 - x_3} \begin{vmatrix} x - x_3 & f(x|x_0, x_1, x_2, x_3) \\ x - x_5 & f(x|x_0, x_1, x_2, x_5) \end{vmatrix}$$

The following table illustrates the arrangement of the work needed to construct $f(x \mid x_0, x_1, \ldots, x_n)$:

x_0	$x - x_0$	$f(x \mid x_0)$				
x_1	$x - x_1$	$f(x \mid x_1)$	$f(x \mid x_0, x_1)$			
x_2	$x - x_2$	$f(x \mid x_2)$	$f(x \mid x_0, x_2)$	$f(x \mid x_0, x_1, x_2)$		
x_3	$x - x_3$	$f(x \mid x_3)$	$f(x \mid x_0, x_3)$	$f(x \mid x_0, x_1, x_3)$	$f(x \mid x_0, x_1, x_2, x_3)$	
x_4	$x - x_4$	$f(x \mid x_4)$	$f(x \mid x_0, x_4)$	$f(x \mid x_0, x_1, x_4)$	$f(x \mid x_0, x_1, x_2, x_4)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The tabular values are generated row-wise (or column-wise). Since the current values are generated from the previous values that is why this method is often called the **iterative interpolation method** and also named as Neville's formula. The rightmost value in the table is the required value of interpolation.

Example 8 (a) Using Aitken's iterative scheme, find the value of log 4.5 from the following values:

x	4.0	4.2	4.4	4.6
f(x)	0.60206	0.62325	0.64345	0.66276

(b) Write a computer program to implement Aitken's method.

Solution (a) $x = 4.5$

Aitken's table is as follows:

$x_0 = 4.0$	$x - x_0 = .5$	0.60206			
$x_1 = 4.2$	$x - x_1 = .3$	0.62325	0.65504		
$x_2 = 4.4$	$x - x_2 = .1$	0.64345	0.65380	0.65318	
$x_3 = 4.6$	$x - x_3 = -.1$	0.66276	0.65264	0.65324	0.65321

$$f(x\,|\,x_0, x_1) = \frac{1}{(x_1-x_0)}\begin{vmatrix} x-x_0 & f(x|x_0) \\ x-x_1 & f(x|x_1) \end{vmatrix}$$

$$= \frac{1}{(4.2-4.0)}\begin{vmatrix} .5 & .60206 \\ .3 & .62325 \end{vmatrix}$$

$$= \frac{(.5 \times .62325 - .3 \times .60206)}{0.2}$$

$$= \frac{(.311625 - .180618)}{0.2} = 0.65504$$

$$f(x\,|\,x_0, x_2) = \frac{1}{(x_2-x_0)}\begin{vmatrix} x-x_0 & f(x|x_0) \\ x-x_2 & f(x|x_2) \end{vmatrix}$$

$$= \frac{1}{(4.4-4.0)}\begin{vmatrix} .5 & .60206 \\ .1 & .64345 \end{vmatrix}$$

$$= \frac{(.321725 - .060206)}{0.4} = 0.65380$$

$$f(x\,|\,x_0, x_3) = \frac{1}{(x_3-x_0)}\begin{vmatrix} x-x_0 & f(x|x_0) \\ x-x_3 & f(x|x_3) \end{vmatrix}$$

$$= \frac{1}{(4.6-4.0)}\begin{vmatrix} .5 & .60206 \\ -.1 & .66276 \end{vmatrix}$$

$$= \frac{(.5 \times .66276 + .1 \times .60206)}{0.6}$$

$$= \frac{(.33138 + .060206)}{0.6} = 0.65264$$

$$f(x\,|\,x_0, x_1, x_2) = \frac{1}{(x_2-x_1)}\begin{vmatrix} x-x_1 & f(x|x_0, x_1) \\ x-x_2 & f(x|x_0, x_2) \end{vmatrix}$$

$$= \frac{1}{(4.4-4.2)}\begin{vmatrix} .3 & .65504 \\ .1 & .65380 \end{vmatrix}$$

$$= \frac{(.3 \times .65380 - .1 \times .65504)}{0.2}$$

$$= \frac{(.19614 - .065504)}{0.2} = 0.65318$$

$$f(x\,|\,x_0, x_1, x_3) = \frac{1}{(x_3-x_1)}\begin{vmatrix} x-x_1 & f(x|x_0, x_1) \\ x-x_3 & f(x|x_0, x_1) \end{vmatrix}$$

$$= \frac{1}{(0.4)}\begin{vmatrix} .3 & .65504 \\ -.1 & .65264 \end{vmatrix}$$

$$= \frac{(.195792 - .065504)}{0.4} = 0.65324$$

$$f(x\,|\,x_0, x_1, x_2, x_3) = \frac{1}{(x_3-x_2)}\begin{vmatrix} x-x_2 & f(x|x_0, x_1, x_2) \\ x-x_3 & f(x|x_0, x_1, x_3) \end{vmatrix}$$

$$= \frac{1}{0.2}\begin{vmatrix} .1 & .65318 \\ -.1 & .65324 \end{vmatrix}$$

$$= \frac{(.0195324 - .065318)}{0.2} = 0.65321$$

The right most entry in each row in the table gives,

$f(4.5\,|\,x_0, x_1) = 0.65504$

$f(4.5\,|\,x_0, x_1, x_2) = 0.65318$

$f(4.5\,|\,x_0, x_1, x_2, x_3) = 0.65321$

It is seen that log 4.5 = 0.65321, which is the anticipated answer.

(b) Program No. 4 Aitken's Method

```
#include<conio.h>
#include<iostream.h>
#include<complex.h>
#include<stdio.h>
void main( )
  {
    clrscr( );
    float x[10],f[10],r[10][10],diff[10],xp;
    int i,j,l,m,n,p,k,y,z;
    double term1,term2,term3;
    cout<<"n\t\t\t Aitken Method\n\n";
    cout<<"Enter the number of X data  :  ";
    cin>>n;
    cout<<"Enter value of xp  :  ";
    cin>>xp;
    for(i=0;i<n;i++)
    {
        cout<<"Enter value of X["<<i<<"]\t";
        cin>>x(i];
        diff[i] = xp – x[i];
    }
```

```
cout<<"\n\n\nGiven the values of function\n\n\n";
for(i=0;i<=n-1;i++)
    {
        cout<<"Enter values of F("<<i<<")\t";
        cin>>f[i];
    }
for(i=0;i<=n-1;i++)
    r[i][0] = f[i];
for(i=0;i<=n-1;i++)
{
        for(j=0;j<n-1;j++)
        {
            term1 = diff[i]*r[j+1][i];
            term2 = diff[j+1]*r[i][i];
            term3 = x[j+1] - x [i];
            if(t erm3 != 0)
                r[j+1][i+1] = (term1 - term2)/term3;
        }
}
y = 13;
clrscr( );
gotoxy(3,5);
cout<<"          Implementation of Aitken's Method\n";
for(i=0;i<=n-1;i++)   //loop to print the value of x differnces
{
    y = i + 13;
    gotoxy(5,y);

cout<<setiosflags(ios::fixed)<<setiosflags(ios::showpoint)<<setprecision(5)<<x[i];
    gotoxy(15,y);
    cout<<"\t"<<setiosflags(ios::fixed)<<setprecision(5)<<diff[i];
}
p=0;
m=25;
k=13;
for(i=0;i<=n-1;i++)
{
        k = i + 13;
        for(j=p;j<=n-1;j++)
        {
            gotoxy(m,k);
cout<<setiosflags(ios::fixed)<<setprecision(5)<<setw(10)<<r[j][i];

            k = k + 1;
        }
        p = p + 1;
```



```
            m = m + 11;
    }
    z = 0;

    for(y=0;y<n-1;y++)
        z = z + 1;
    cout<<"\n\n\n\tAtXp="<<setw(15)<<setiosflags(ios::fixed)<<setprecision(3)<<xp"
    function value is\t"<<setiosflags(ios::fixed)<<setprecision(5)<<setw(15)<<r[y][z];
    getch( );
}
```

Computer Output

Aitken Method

Enter the number of X data : 4
Enter value of xp : 4.5
Enter value of X[0] 4.0
Enter value of X[1] 4.2
Enter value of X[2] 4.4
Enter value of X[3] 4.6

Given the values of function

Enter value of F[0] .60206
Enter value of F[1] .62325
Enter value of F[2] .64345
Enter value of F[3] .66276

Implementation of Aitken's Method

4.00000	0.50000	0.60206			
4.20000	0.30000	0.62325	0.65504		
4.40000	0.10000	0.64345	0.65380	0.65318	
4.60000	–0.10000	0.66276	0.65264	0.65324	0.65321

At Xp = 4.500 function value is 0.65321

3.7 ERROR ESTIMATION IN INTERPOLATION

So far, we have studied several formulas for interpolation. The basic principle in all these formulas is the approximation of a polynomial so that this polynomial passes through the set of points in a given table.

The error in an interpolation process comes from several sources.

(a) The truncation error due to terminating the series at the term in, say, the nth difference.

(b) The round-off errors in the function values and resulting errors in the differences causing oscillating $(n+1)$th differences.

(c) The round-off errors in the individual terms of the formula and their sum.

(d) Inaccuracy, usually due to rounding-off, in the given value of p.

We can estimate errors in any of the interpolation formulas from the first neglected term. We conclude this section by giving the error estimates in Newton's forward and backward difference formulas.

3.7.1 Error in Newton's Forward Difference Formula

If the function $y = f(x)$ is known explicitly, the remainder term in case of the nth-order forward difference formula is as follows:

$$E = \frac{h^{n+1}}{(n+1)!} p(p-1)\cdots(p-n) f^{(n+1)}(Z) \qquad \ldots \qquad (3.12)$$

where $x_0 < Z < x_n$.

If the function is specified by tabular values, the error is given by the following relation:

$$E = \frac{p(p-1)\cdots(p-n)}{(n+1)!} \Delta^{n+1} f_0 \qquad \ldots \qquad (3.13)$$

What can be done if the next term (i.e., (N+1)st) is not available? In this case, check if an additional point is available on the other side, namely f_{-1}. If it is available, $\Delta^{n+1} f_{-1}$ can be computed and used as an approximation for $\Delta^{n+1} f_0$.

Example 9 Given $f(x) = e^x$, compute the values of f(x) for $x = 0(0.1)\ 0.5$ correct to 4 dp.

i) Make a difference table and interpolate f(.175) using Newton's forward difference formula.

ii) Calculate the actual value of e^x for $x = .175$. Find also the error.

iii) Use the formula (3.12) and estimate the error.

iv) What discretization size should be used if the entries are given to 6 dp?

Solution i) Difference Table

x	$f(x) = e^x$	Δ	Δ^2	Δ^3	Δ^4
0.0	1.0000				
		1052			
0.1	1.1052		110		
		1162		13	
0.2	1.2214		123		−2
		1285		11	
0.3	1.3499		134		5
		1419		16	
0.4	1.4918		150		
		1569			
0.5	1.6487				

$$x_p = 0.175;\ h = 0.1;\ x_0 = 0.1$$

$$p = \frac{(0.175 - 0.1)}{0.1} = 0.75$$

Using Newton's forward difference formula (3.2), we get,

$$f_p = 1.1052 + 0.75 \times .1162 + \frac{0.75(.75-1)}{2} \times .0123$$

$$+ \frac{0.75(0.75-1)(0.75-2)}{6} \times .0011$$

$$+ \frac{0.75(0.75-1)(0.75-2)(.75-3)}{24} \times .0005$$

$$= 1.1052 + 0.08715 - 0.0012 + 0.00004 - 0.00001$$

$$= 1.1912$$

ii) True value, $e^x = e^{.175} = 1.1912$

Error = True value − Interpolated value

$$= 1.1912 - 1.1912 = 0$$

iii) $x_0 = 0.1;\ p = 0.75;\ h = 0.1$

$x_5 = 0.5,\ f(x) = e^x$

The fifth derivative is $f^{(v)}(x) = e^x$.

The maximum value, $f^{(v)}(x) = e^5 = 1.64872$

$$E = \frac{h^5}{5!} p(p-1)(p-2)(p-3)(p-4) f^{(v)}(x)$$

$$= \frac{.75(.75-1)(.75-2)(.75-3)(.75-4)}{120} \times (.1)^5 \times 1.64872$$

$$= 0.0138411 \times .00001 \times 1.64872 = 0.0000002$$

iv) To keep the accuracy less than $\frac{1}{2} \times 10^{-6}$, h should be:

$$h = \left[\frac{E \times 120}{p(p-1)(p-2)(p-3)(p-4)} \times \frac{1}{1.64872}\right]^{\frac{1}{5}}$$

$$= \left[\frac{0.0000005 \times 120}{1.64872 \times 1.7139}\right]^{\frac{1}{5}} = \left[\frac{0.00006}{2.8257412}\right]^{\frac{1}{5}}$$

$$= \left(2.12333 \times 10^{-5}\right)^{\frac{1}{5}} = 0.1163$$

3.7.2 Error in Newton's Backward Difference Formula

If the function y = f(x) is known explicitly, the remainder term in case of the nth-order backward difference formula is as follows:

$$E = \frac{h^{n+1}}{(n+1)!} p(p+1)(p+2)\cdots(p+n) f^{(n+1)}(Z) \qquad \cdots \qquad (3.14)$$

where $x_0 < Z < x_n$

If the function y = f(x) is not known but is specified only by tabular values, the error is given by the following relation:

$$E = \frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!} \nabla^{n+1} f_0 \qquad \cdots \qquad (3.15)$$

Let us illustrate this method with an example.

Example 10 Given f(x) = sin x, compute the values of f(x) for x = 0.1(0.1)0.8 correct to 4 dp.

a) Construct the difference table and interpolate f(0.75) using Newton's backward difference formula.

b) Calculate the exact value of sin x for x = 0.75. Find the error.

c) Use the formula (3.14) and estimate the error.

d) What discretization size should be used if the entries are given to 6 dp?

Solution f(x) = sin x ; n = 0. (0.1)0.8 radians

a) **Difference Table :**

x	f(x)	∇	∇^2	∇^3	∇^4	∇^5
0.1	0.0998					
		989				
0.2	0.1987		−21			
		968		−8		
0.3	0.2955		−29		1	
		939		−7		−12
0.4	0.3894		−36		−11	
		903		−18		−8
0.5	0.4797		−54		−19	
		849		1		7
0.6	0.5646		−53		−12	
		796		−11		
$x_0 = 0.7$	0.6442		−64			
		732				
0.8	0.7174					

$f^{(iv)}(x) = \sin x$

$f^{(v)}(x) = \cos x$

maximum, $f^{(v)}(x) = f^{(v)}(0.1 = 0.9950.$

$$E = \frac{h^{n+1}}{(n+1)!} p(p+1)(p+2)(p+3)(p+4) f^{(v)}(x)$$

$$= \frac{(.1)^5}{5!} \times 0.5(0.5+1)(0.5+2)(0.5+3)(0.5+4) \times 0.9950$$

$$= \frac{0.00001}{120} \times (0.5)(1.5)(2.5)(3.5)(4.5) \times 0.9950$$

$$= 0.00000245$$

d) $E = \frac{1}{2} \times 10^{-6} = 0.0000005$

From the error formula, we get,

$$h = \left(\frac{E \times 120}{p(p+1)(p+2)(p+3)(p+4) \times f^{(v)}(x)} \right)^{\frac{1}{5}}$$

$$= \left(\frac{0.0000005 \times 120}{.5(1.5)(2.5)(3.5)(4.5) \times 0.9950} \right)^{\frac{1}{5}}$$

$$= (0.000002041)^{\frac{1}{5}} = 0.073$$

PROBLEMS

1. (a) Show that a curve y = f(x), where f(x) is of the fourth degree, can be drawn through the points given by:

x	−1	0	1	2	3	4	5
f(x)	23	13	3	1	34	148	408

Use Newton's forward difference formula to find y exactly when x = 1.2.

(b) Given the following data:

x	−4	−2	0	2	4	6
f(x)	180	0	4	0	40	504

Use Newton's forward difference formula to find f(1.75).

(c) Consider the following table of values:

x	0.2	0.3	0.4	0.5	0.6
f(x)	0.2304	0.2788	0.3222	0.3617	0.3979

Find f(0.36) using Newton's forward difference formula.

2. (i) Use Newton's backward difference interpolation formula to estimate the value of f(1.45) from the following data:

x	1.0	1.1	1.2	1.3	1.4	1.5
f(x)	2.0	2.1	2.3	2.7	3.5	4.5

(ii) Use Newton's forward difference formula to find f(1.05) from the above data.

(iii) Consider the following data:

x	−1	−0.75	−0.50	−0.25	0	0.25	0.50	0.75	1
f(x)	−0.4401	0.0447	0.4311	0.6694	0.7652	0.7522	0.6714	0.5587	0.4401

a) Use Newton's forward difference interpolation formula to estimate f(−0.33).

b) Use Newton's backward difference interpolation formula to estimate f(0.62).

3.(a) The values of a low degree polynomial are given in the table below. It is suspected that there is a transposition error in one of the values. By differencing, locate and correct the error and find f(2.5):

x	2	3	4	5	6	7	8	9	10
f(x)	15	40	85	165	259	400	585	820	1111

(b) One of the functional values in the following table contains an error:

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7
f(x)	1.4142	1.4491	1.4832	1.5160	1.5492	1.5811	1.6125	1.6432

i) Detect and correct the erroneous term and then reconstruct the difference with the corrected functional value.

ii) Find f(2.05) using Newton's forward difference formula.

iii) Find f(2.65) using Newton's backward difference formula.

4. (i) Using Stirling's interpolation formula, find f(3.25) from the following data:

x	1	2	3	4	5
f(x)	0.0000	0.6931	1.0986	1.3863	1.6094

(ii) The following table gives the value of p_x of a polynomial of the fourth degree for certain values of p_x :

x	5	6	7	8	9
p_x	6.195	5.919	5.630	5.326	5.006

Estimate the polynomial using Stirling's formula when x = 7.5.

5. (a) Given the following table:

x	f(x)
0.01	98.4342
0.02	48.4392
0.03	31.7775
0.04	23.4492
0.05	18.4542

Estimate the value of f(x) corresponding to x = 0.0341 using the following formulas

(i) Stirling (ii) Bessel (iii) Everett and (iv) Gauss forward and backward.

(b) Consider the following table of values:

x	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
f(x)	.0495	.0605	.0739	.0903	.1102	.1346	.1644	.2009

Find the values of f(2.35) and f(2.2) using all the central difference formulas you have studied.

(c) Use the following table of current i against deflection, θ:

θ	.40	.45	.50	.55	.60	.65
i	1.268	1.449	1.639	1.839	2.052	2.281

to find i, when θ = .536, from Stirling and Everett formulas. Check the answer using Newton's forward difference formula.

(d) Consider the following data:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
f(x)	1.54308	1.66852	1.81066	1.97091	2.15090	2.35241	2.57746

Find the value of f(1.35) using:

(i) Stirling, (ii) Bessel, (iii) Everett and (iv) Gauss both formulas.

(e) Consider the following data:

x	1.0	1.2	1.4	1.6	1.8	2.0
f(x)	0.367879	0.301194	0.246597	0.201897	0.165299	0.135335

Find the value of f(1.675) using:

(i) Stirling, (ii) Bessel, (iii) Everett and (iv) Gauss both formulas.

(f) Kinematic viscosity of water, v, is related to temperature in the following manner:

T(°F)	40	50	60	70	80
v	1.66	1.41	1.22	1.06	0.93

Use a suitable interpolation formula to predict v at T = 62 °F.

(g) You measure the voltage drop v, across for a number of different values of current i.

The results are:

i	0.25	0.75	1.25	1.75	2.25
v	−0.23	−0.33	0.70	1.88	6.00

Use a suitable interpolation formula to estimate the voltage drop for i = 0.9.

6. (a) Find f(x) at x = 1, ~~from the following table using Lagrange's formula~~:

x	−1	0	2	3
f(x)	6	10	12	19

(b) Fit a polynomial to the following data:

(−4, 180), (−2, 0), (0, 4), (1, 0), (3, 40) and (5, 504).

Use the polynomial to find a value for f(2.4).

(c) Fit a polynomial for the function $f(x) = \frac{2^x}{x}$, for x = 2, 4 and 8. Use this polynomial to estimate f(6).

(d) Given the following table of values:

x	14	17	31	35
y	68.7	64.0	44.0	39.1

What is y(27)?

(e) Use Lagrange's interpolation formula to obtain a polynomial of least degree that assumes the following values:

x	1	2	3	4
y	7	11	28	63

Use the polynomial obtained to estimate f(4.5). Check the answer using Newton's backward difference formula.

(f) The function y = f(x) is given in the points: (7, 3), (8, 1), (9, 1) and (10, 9). Find the value of y for x = 9.5 using Lagrange's interpolation formula.

(g) Use Lagrange's formula to estimate f(2.0) and f(4.5) from the following data:

x	1.6	2.9	3.7	4.8
f(x)	0.6250	0.3448	0.2703	0.2083

(h) Given the following data:

x	1	2	4	8
f(x)	1	3	7	11

Find f(7) using Lagrange's formula.

(i) Let $f(x) = \frac{8x}{2^x}$. Fit a polynomial for the function when x = 0(1)3. Estimate the value of f(1.5).

7. (a) Estimate the interpolation polynomial for $f(x) = x^2 + \sin \pi x$ through (0, 0), (1, 1) and (2, 4), using Newton's forward difference formula at x = 0.5.

(b) What is the exact error when x = 0.5?

(c) What is the maximum error in (a) above?

(d) Find the largest value of h that will ensure 4 dp accuracy in the value of f(x), assuming quadratic interpolation is used.

8. Consider the following table of values:

x	1	2	3	4	5
f(x)	1.0000	1.4142	1.7320	2.0000	2.2361

a) Use Newton's forward difference formula to estimate f(1.5). Using third-order interpolation, estimate also the maximum error.

b) If we are known that $f(x) = \sqrt{x}$, what is the error in this case? Find also the maximum error.

c) Find the largest value of h that will give 6 dp accuracy in the value of x, assuming third-order interpolation is used.

9. (a) Use Aitken's formula to estimate f(0.2) as accurately as possible from the following rounded values of f(x):

x	.17520	.25386	.33565	.42078	.50946
f(x)	.84147	.86742	.89121	.91276	.93204

(b) Use Aitken's formula to estimate f(1.4) correct to 4 dp from the following data:

x	1.20	1.25	1.30	1.35	1.45	1.50
f(x)	0.1823	0.2231	0.2624	0.3365	0.3716	0.4055

(c) Use Aitken's formula to estimate f(5) correct to 2 dp from the following data:

x	1	4	7	9
f(x)	2	13	122	504

(d) Use Aitken's method to evaluate log 3.63 from the following table.

x	3.50	3.60	3.70	3.80
log x	1.252763	1.28093	1.30833	1.33500

(e) Consider the function: $f(x) = \frac{1}{1+20x^2}$. Calculate the values of the function correct to 4 dp, for x = 0.2(0.2) 1.0. Estimate the value of f(0.55) using Aitken's method.

10.(a) Consider the following table of values:

x	-0.1	0.1	0.3	0.5	0.7	0.9
f(x)	.7196	.8075	.8812	.9385	.9776	.9975

i) Use Newton's forward difference interpolation formula to estimate f(0.25) using upto fourth-order differences.

ii) Find the maximum error.

(b). Consider the following table:

x	.2	.4	.6	.8	1
f(x)	.19951	.39646	.58813	.72210	.94608

Find the value of f(0.3) using Lagrange and Aitken's formulas.

11. (a) Consider the following part of a difference table:

x	f(x)	δ	δ^2	δ^3	δ^4
6	1296				
		2800		1344	
8	4096		3104		384
		5904		1728	
10	10000				

Compute f(9) using Stirling's formula for interpolation.

(b) Given the following part of a difference table:

x^0	$\sin x^0$	δ	δ^2	δ^3	δ^4
25	0.422618		–3216		23
		77382		590	
30	0.500000		–3806		52

Estimate f(26.5) using Bessel's formula for interpolation.

12. Using table values of Q.5(d) above, do the following:

a) Estimate the value of f(1.425) using Newton's backward difference formula with fourth-order differences.

b) Compute the maximum error.

13. Using tabular values of Q.5(e) above, do the following:

a) Use Newton's backward difference formula to estimate f(1.90) with fourth-order differences.

b) Compute the maximum error if the tabular values are based on the function $f(x) = e^{-x}$.

c) Compute the largest value of h that will give 7 dp accuracy in the value of x, assuming the same order of interpolation as used in (a) above.

Chapter 4

Numerical Differentiation

4.1 INTRODUCTION

Numerical differentiation is useful in estimating the derivatives of a function f(x) when either f(x) is difficult to differentiate easily, or, it is not known as an explicit expression in x, but the values of the function are given in a tabular form. We use numerical differentiation only when there is no better alternative method available to compute derivatives analytically or when the analytical solution is rather complicated. Generally, it is considered that numerical differentiation is basically an unstable process which means that small values of h can lead to greatly magnified errors in the final result. In fact, we may not always expect reasonable results even when the original data are known to be accurate. In actual practice this operation is avoided altogether if possible because it tends to enhance the effects of rounding errors present in the tabular values. This is particularly true when the $f(x_i)$ values are themselves subject to more error, as they would probably be if determined experimentally. If derivative values are computed in such cases, particularly when the results are to be used in subsequent calculations, it is usually better to consider curve fitting, using least-squares technique and differentiate the formula for the curve.

In this chapter, we shall derive some formulas for estimating derivatives. In spite of some inherent shortcomings, numerical differentiation is useful to derive formulas for solving integrals, ordinary and partial differential equations. Standard examples of numerical differentiation often use known functions so that the numerical approximation can be compared with the exact answer.

4.2 DERIVATION OF DIFFERENTIATION FORMULAS

In order to derive a differentiation formula, we differentiate a suitable interpolation formula with respect to p.

We shall write, $x = x_0 + ph$. Differentiating this w.r.t.p., we get,

$$\frac{dx}{dp} = h$$

or, $$\frac{dp}{dx} = \frac{1}{h} \qquad \ldots (4.1)$$

Also, $f_p = f(x) = f(x_0 + ph) \qquad \ldots (4.2)$

Differentiating (4.2) w.r.t.x., we get,