GRAPH THEORY
BASIC TERMINOLOGY

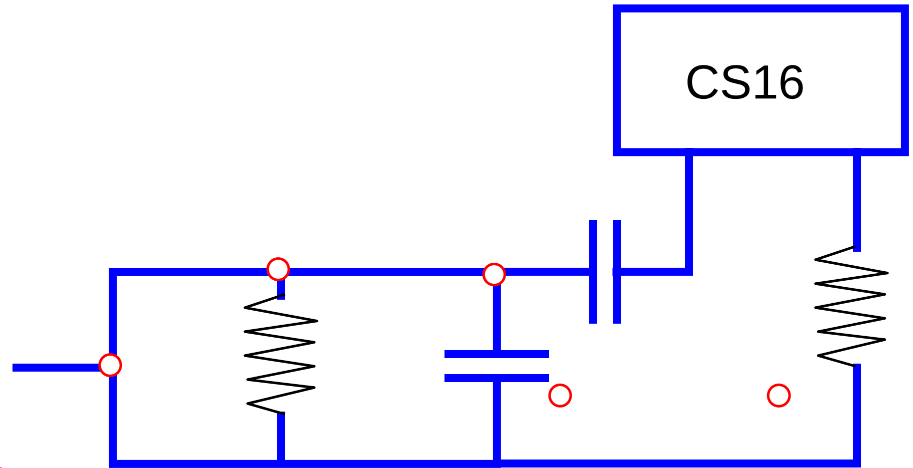
CS 441

Basic Graph Definitions

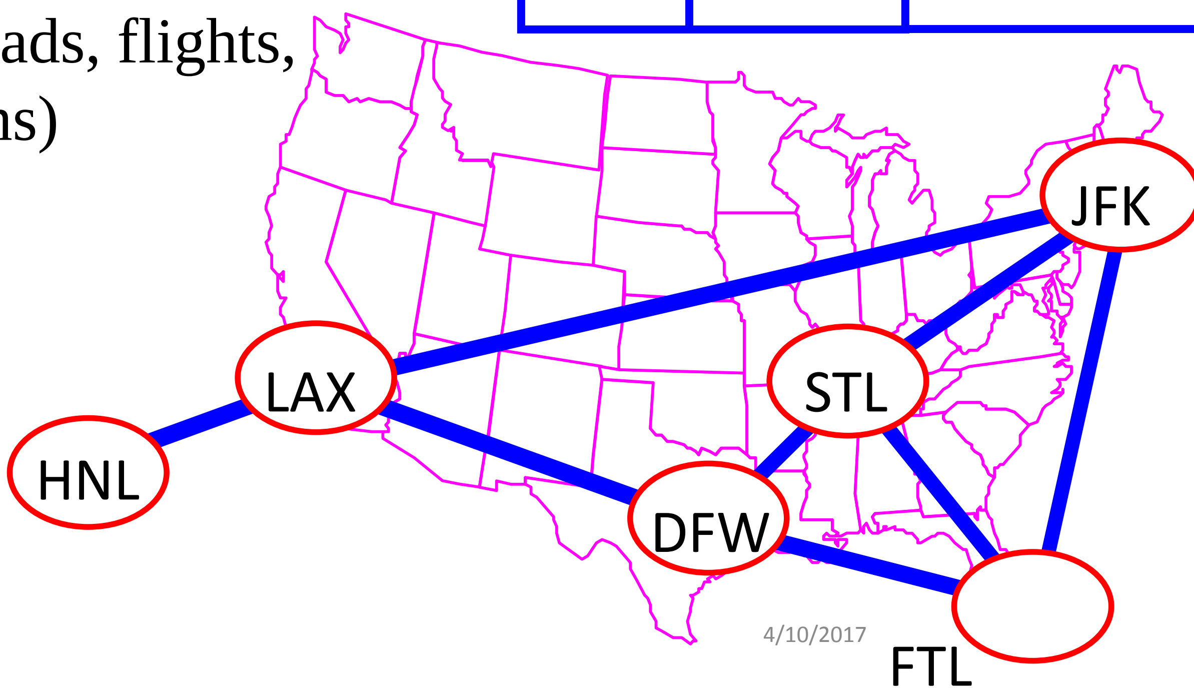
- A data structure that consists of a set of nodes (*vertices*) and a set of edges that relate the nodes to each other
- The set of edges describes relationships among the vertices. Some Examples,
  - Car navigation system
  - Efficient database
  - Build a bot to retrieve info off WWW
  - Representing computational models

Applications

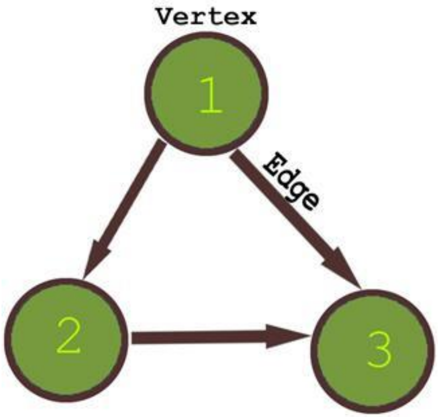
- electronic circuits



- networks (roads, flights, communications)

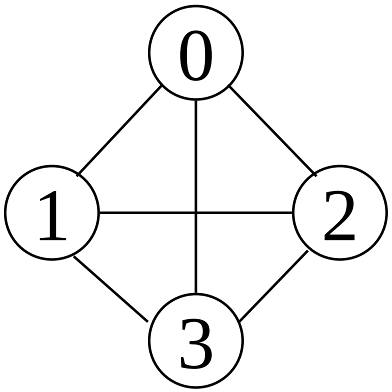


Formal Definition:

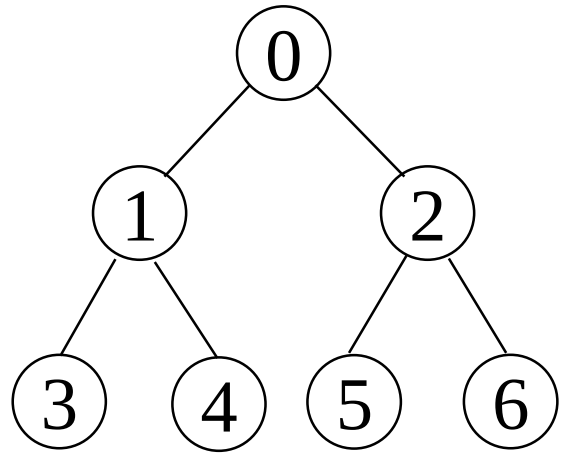


- A graph, *G=(V, E)*, consists of two sets:
  - a finite non empty set of ***vertices(V)***, and
  - a finite set ***(E)*** *of unordered pairs of distinct vertices called edges.*
  - *V(G)* and *E(G)* represent the sets of vertices and edges of *G*, respectively.
- Vertex: In graph theory, a **vertex** (plural **vertices**) or **node** or points is the fundamental unit out of which graphs are formed.
- Edge or Arcs or Links: Gives the relationship between the Two vertices.

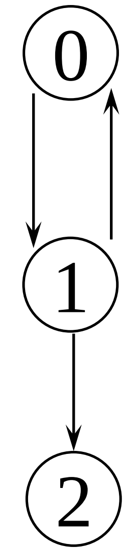
Examples for Graph



G_1



G_2



G_3

$V(G_1)=\{0,1,2,3\}$ $E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

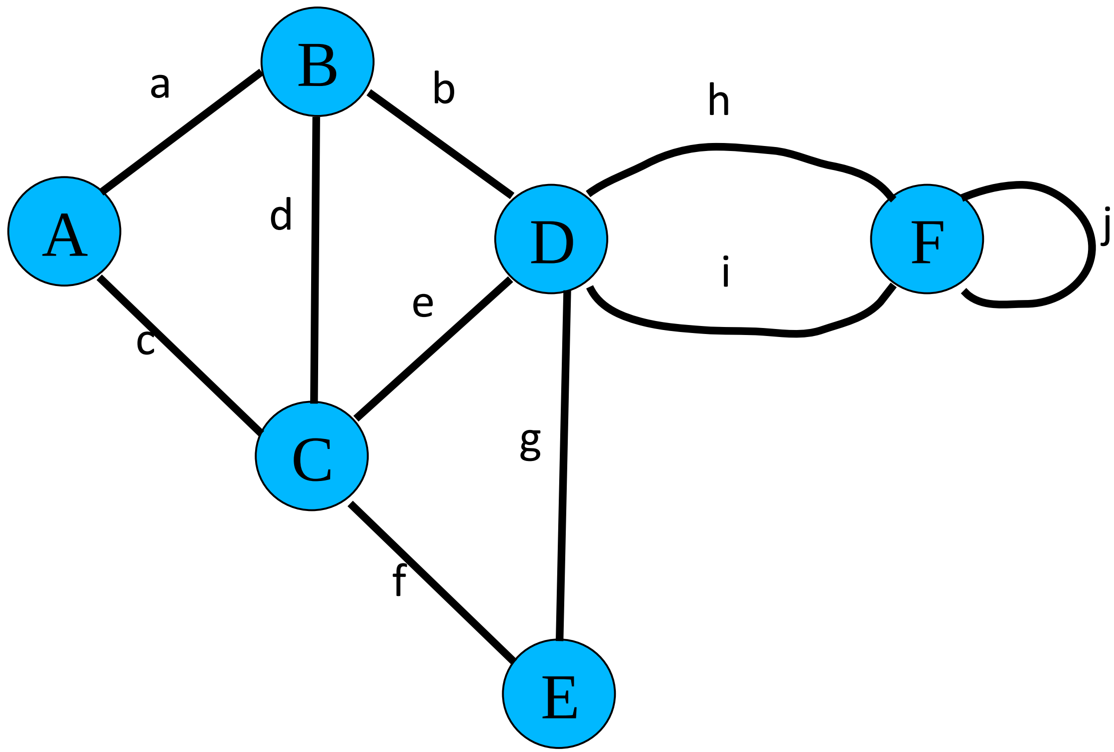
$V(G_2)=\{0,1,2,3,4,5,6\}$ $E(G_2)=\{(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)\}$

$V(G_3)=\{0,1,2\}$ $E(G_3)=\{<0,1>,<1,0>,<1,2>\}$

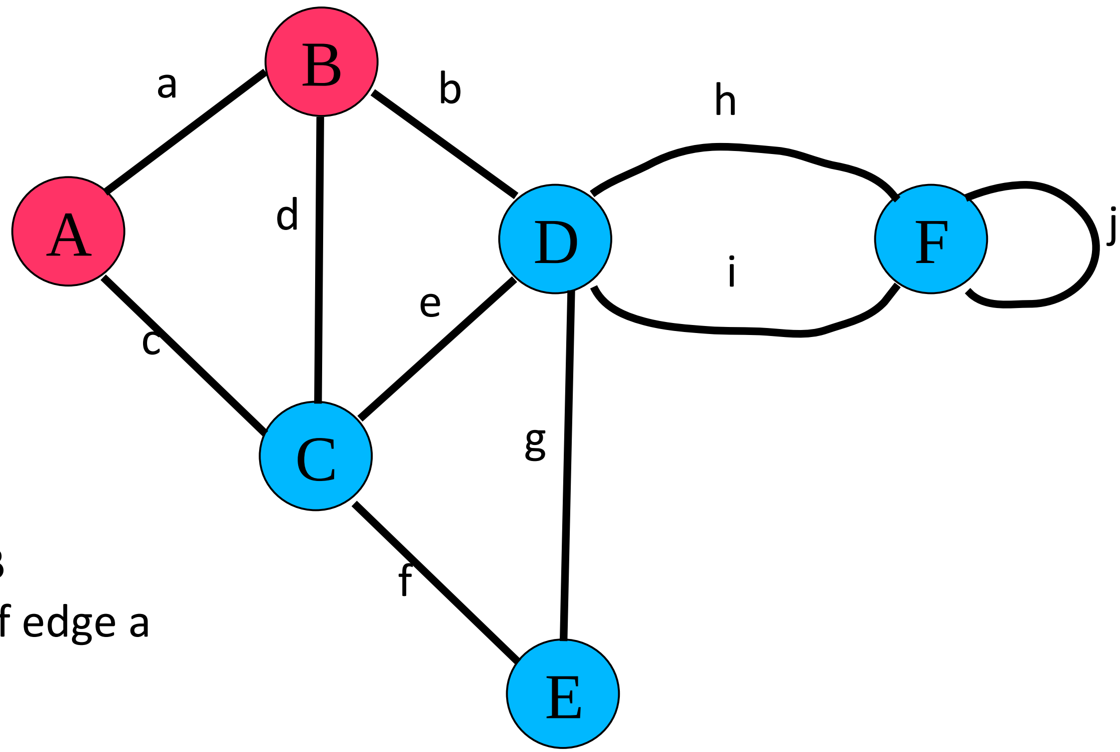
Graph Terminology

- Two vertices joined by an edge are called the end vertices or endpoints of the edge.
- If an edge is directed its first endpoint is called the origin and the other is called the destination.
- Two vertices are said to be adjacent if they are endpoints of the same edge.

Graph Terminology

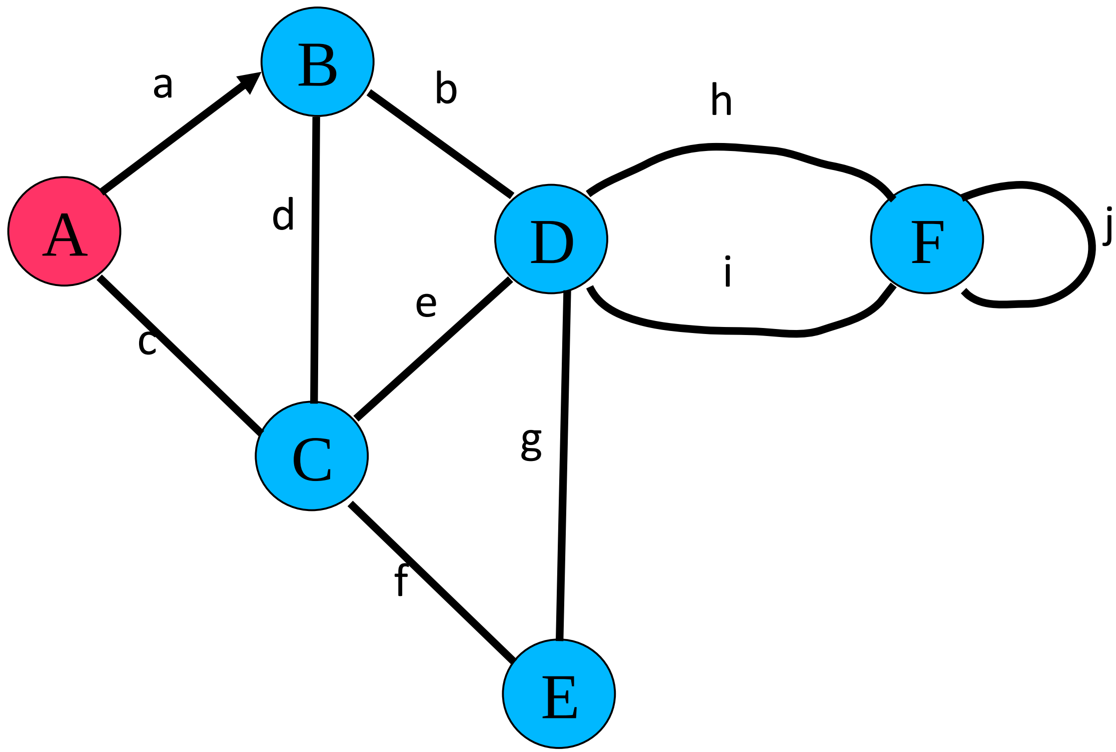


Graph Terminology



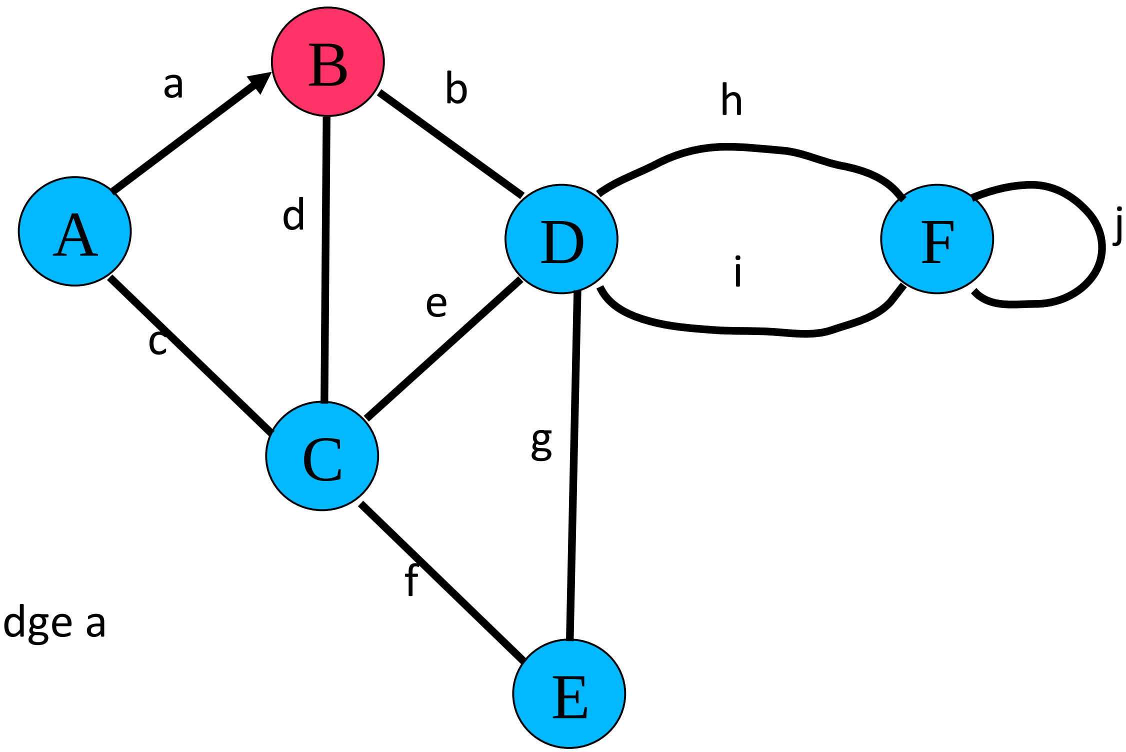
Vertices A and B
are endpoints of edge a

Graph Terminology



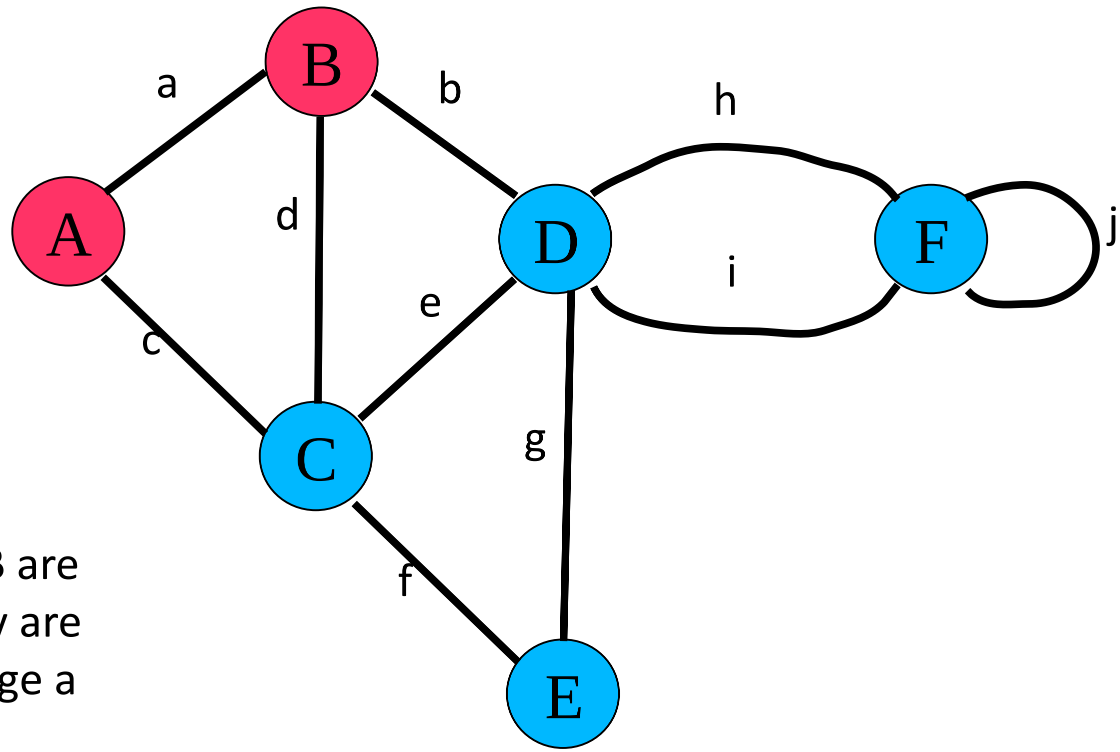
Vertex A is the origin of edge a

Graph Terminology



Vertex B is the destination of edge a

Graph Terminology

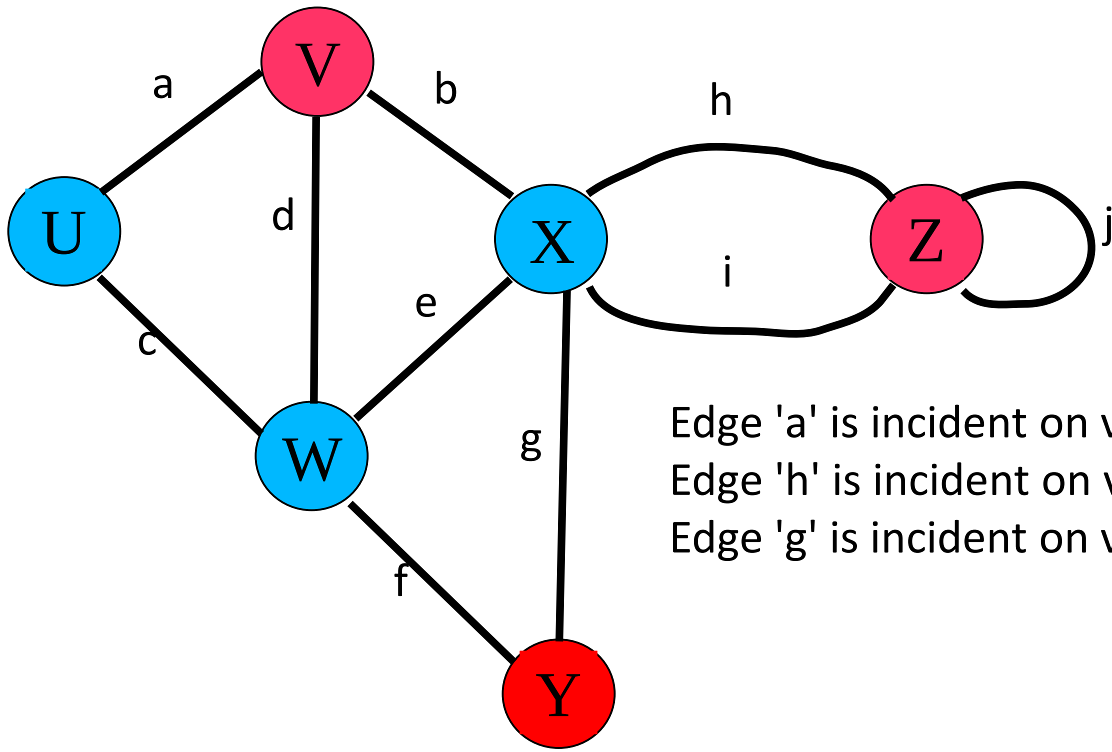


Vertices A and B are adjacent as they are endpoints of edge a

Graph Terminology

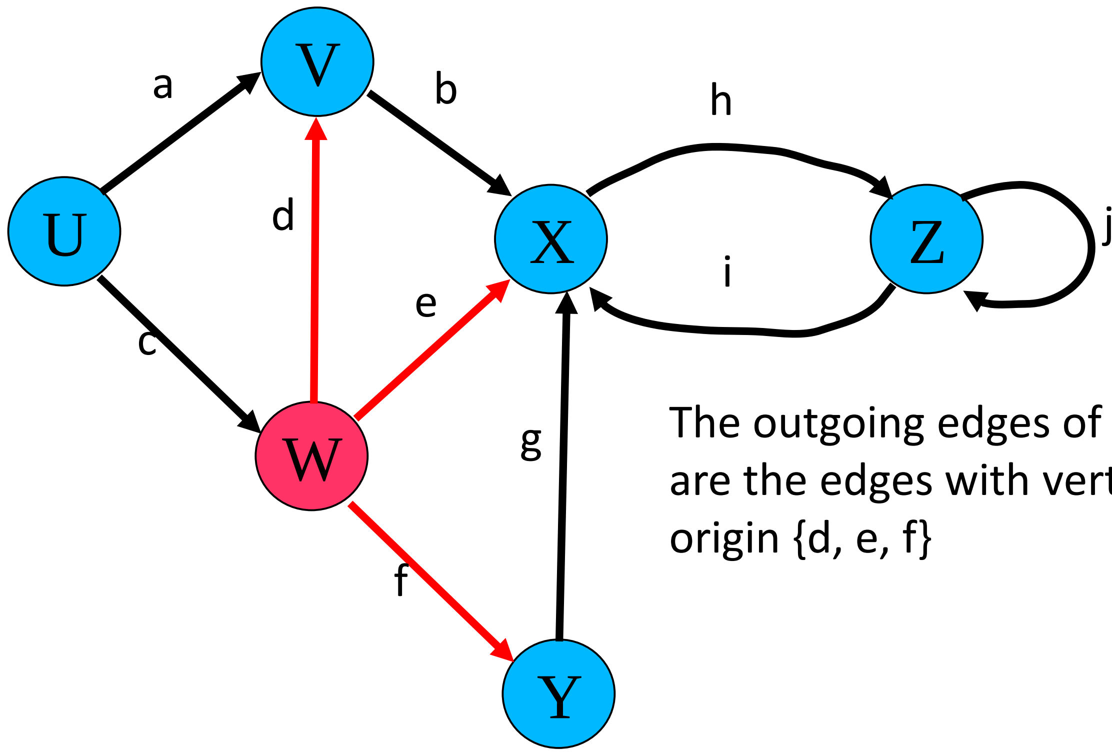
- An edge is said to be incident on a vertex if the vertex is one of the edges endpoints.
- The outgoing edges of a vertex are the directed edges whose origin is that vertex.
- The incoming edges of a vertex are the directed edges whose destination is that vertex.

Graph Terminology



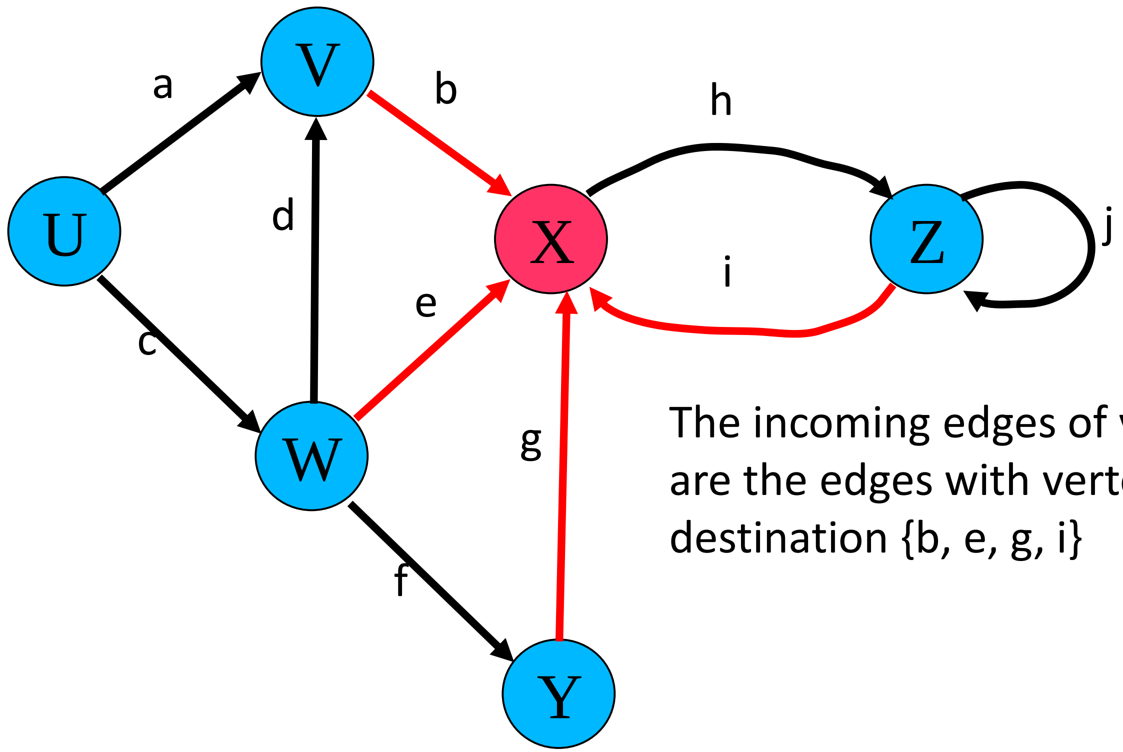
Edge 'a' is incident on vertex V
Edge 'h' is incident on vertex Z
Edge 'g' is incident on vertex Y

Graph Terminology



The outgoing edges of vertex W are the edges with vertex W as origin {d, e, f}

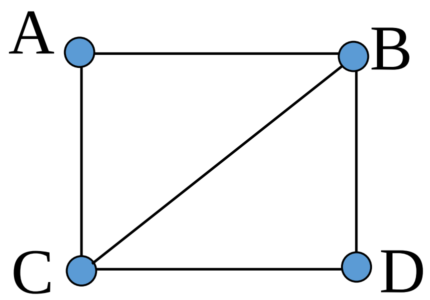
Graph Terminology



The incoming edges of vertex X are the edges with vertex X as destination {b, e, g, i}

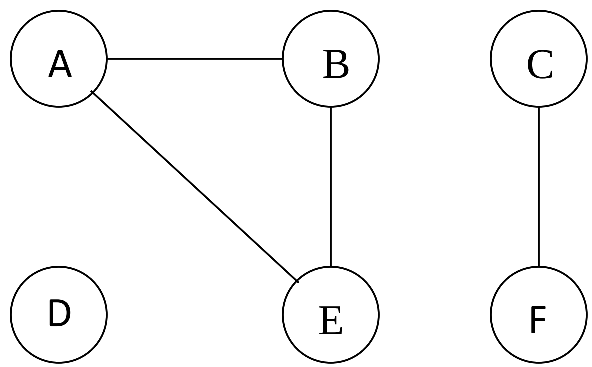
Adjacent, neighbors

- Two vertices are *adjacent* and are *neighbors* if they are the endpoints of an edge
- Example:
  - *A* and *B* are adjacent
  - *A* and *D* are not adjacent



Degree

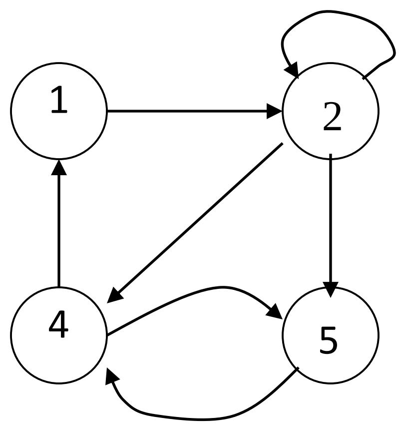
Degree: Number of edges incident on a node



The degree of B is 2.

Degree (Directed Graphs)

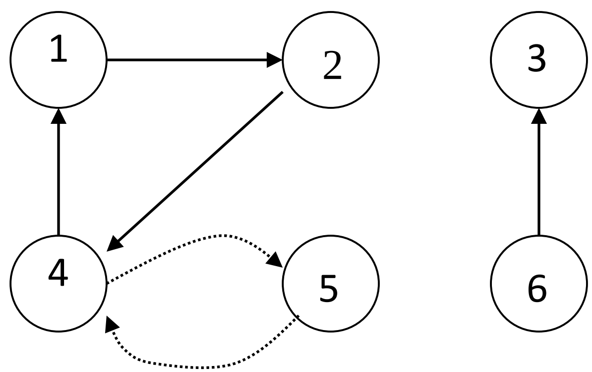
- In degree: Number of edges entering a node
- Out degree: Number of edges leaving a node
- Degree = Indegree + Outdegree



The in degree of 2 is 2 and
the out degree of 2 is 3.

Path

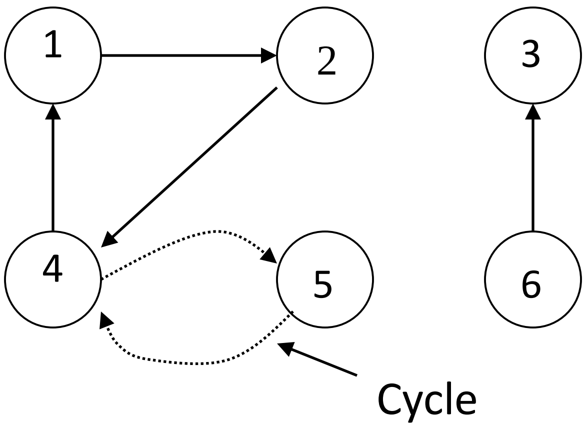
- A *path* is a sequence of vertices such that there is an edge from each vertex to its successor.
- A path is *simple* if each vertex is distinct.
- A *circuit* is a path in which the terminal vertex coincides with the initial vertex



Simple path: [1, 2, 4, 5]
Path: [1, 2, 4, 5, 4]
Circuit: [1, 2, 4, 5, 4, 1]

Cycle

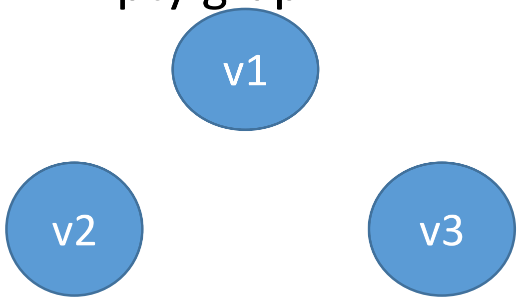
- A path from a vertex to itself is called a ***cycle***.
- A graph is called ***cyclic*** if it contains a cycle;
  - otherwise it is called ***acyclic***



Types of Graph

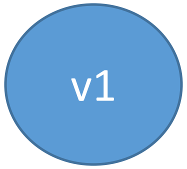
Null graph, Trivial Graph

- A graph G=(V,E) where E=0 is said to be Null or Empty graph



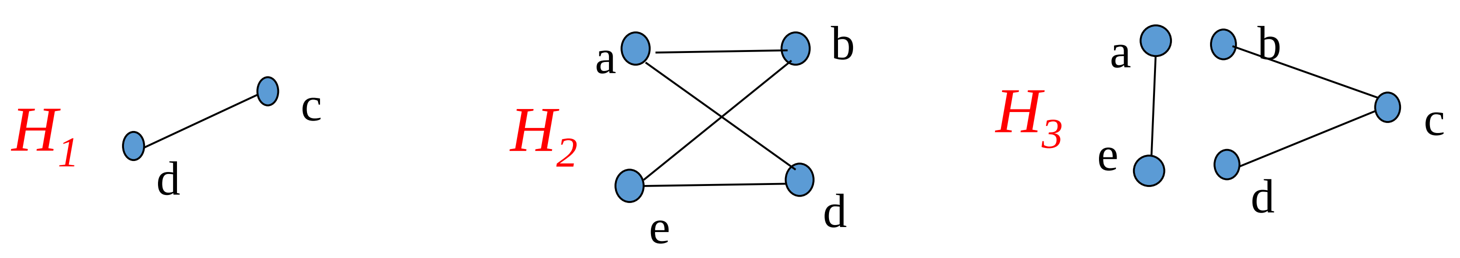
- A graph with One vertex

  and no edge is called as a trivial graph.



Connected and Disconnected

- ***Connected*** : There exists at least one path between two vertices
- ***Disconnected*** : Otherwise
- Example:
  - H_1 and H_2 are connected
  - H_3 is disconnected



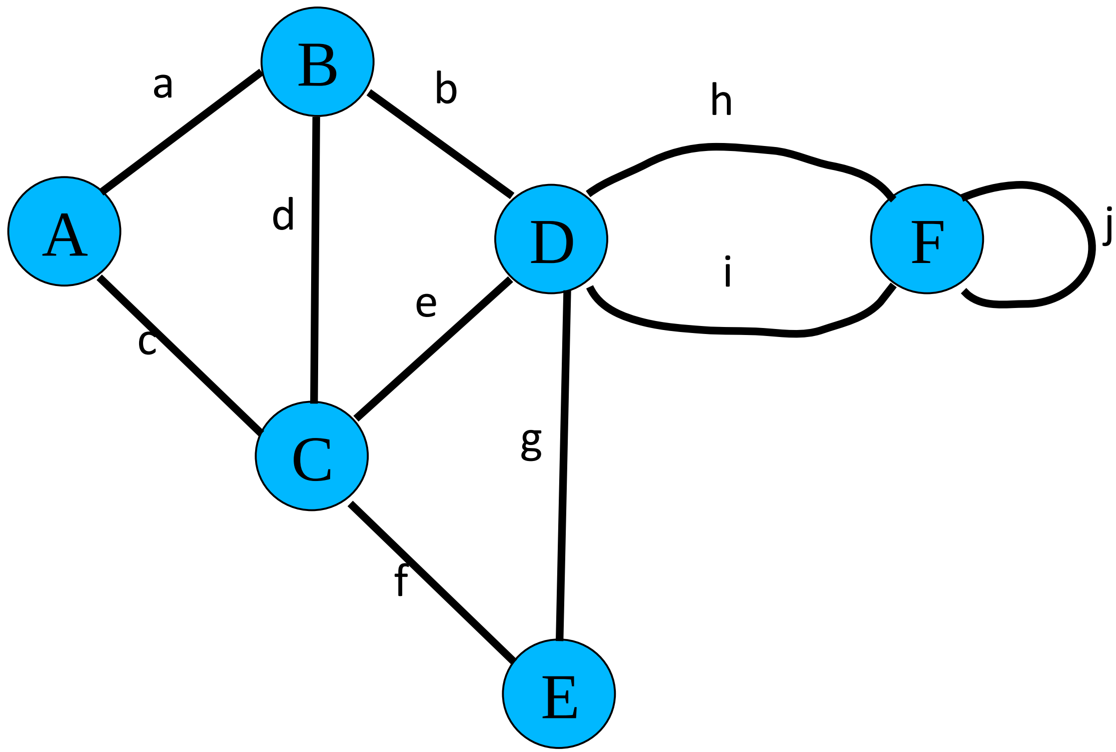
Undirected Graph

- In an undirected graph, there is no distinction between (u, v) and (v, u).
- An edge (u, v) is said to be directed from u to v if the pair (u, v) is ordered with u preceding v.

  E.g. A Flight Route
- An edge (u, v) is said to be undirected if the pair (u, v) is not ordered

  E.g. Road Map

Undirected Graph



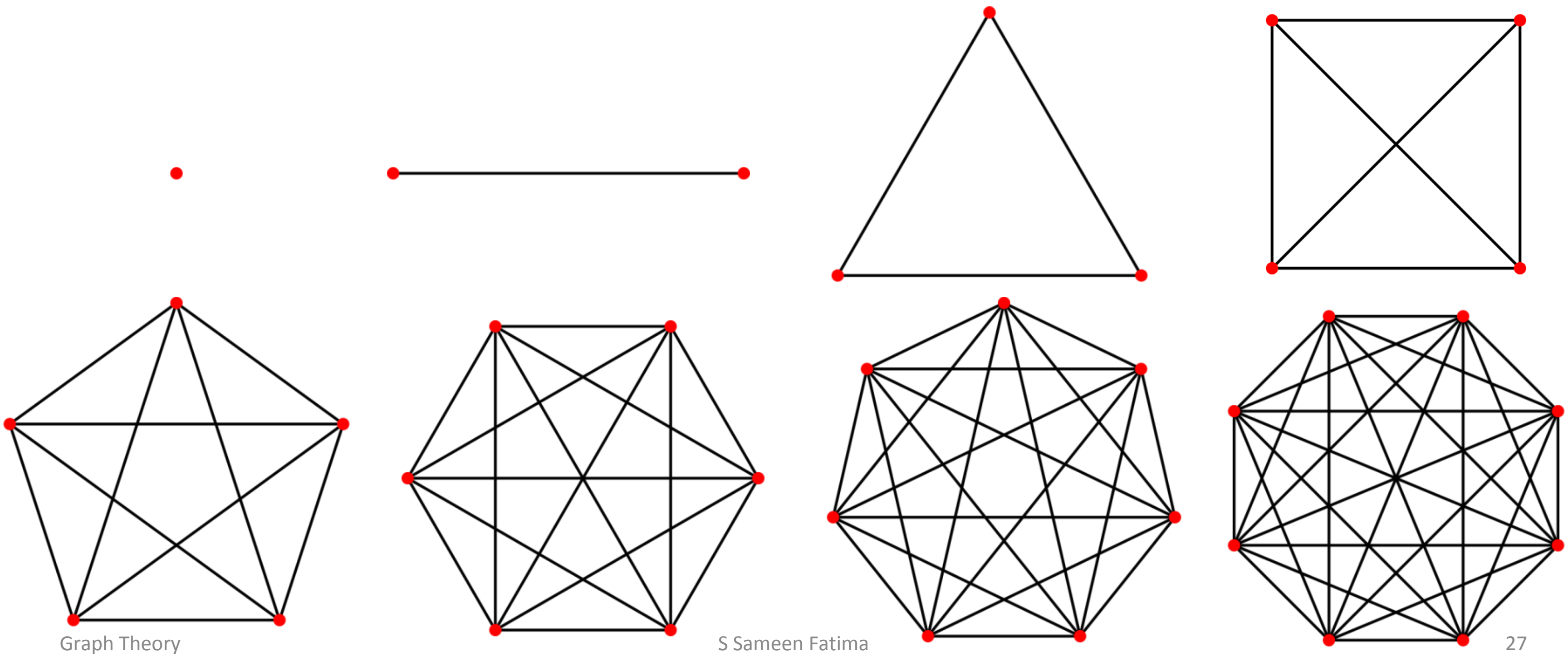
Here (u,v) and (v,u) both are possible.

Undirected Graph

- A graph whose definition makes reference to Unordered pairs of vertices as Edges is known as undirected graph.
- Thus an undirected edge (u,v) is equivalent to (v,u) where u and v are distinct vertices.
- In the case of undirected edge(u,v) in a graph, the vertices u,v are said to be adjacent or the edge(u,v) is said to be incident on vertices u,v.

Complete Graph

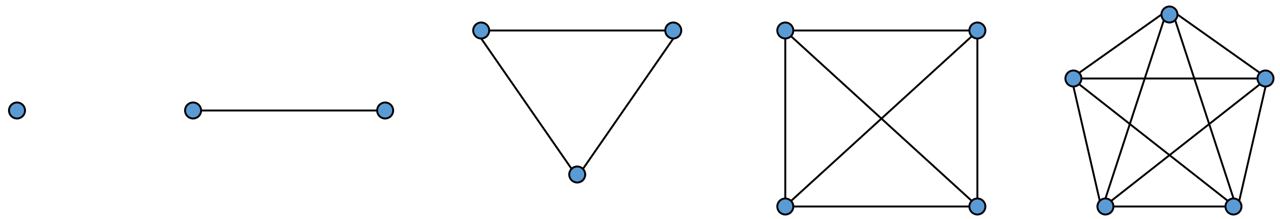
- Complete Graph: A simple graph in which every pair of vertices are adjacent
- If no of vertices = n, then there are n(n-1) edges



Complete Graph

- In a complete graph: Every node should be connected to all other nodes.
- The above means " Every node is adjacent to all other nodes in that graph".
- The degree of all the vertices must be same.

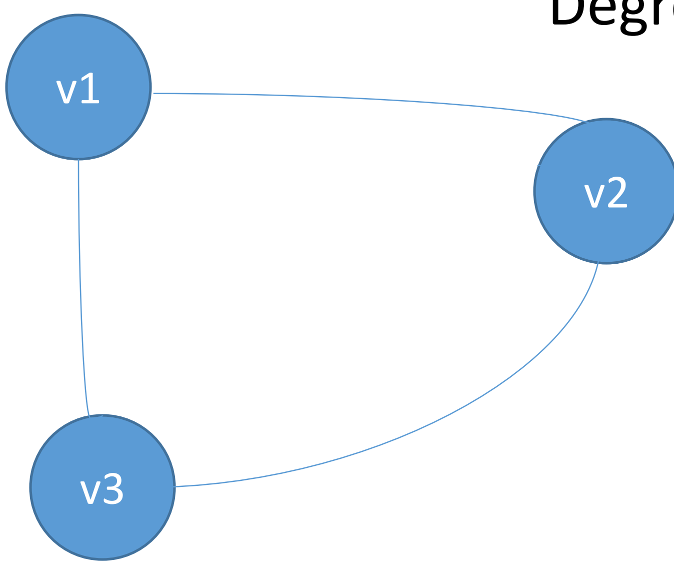
- K_n = Denotes a complete with n number of vertices.



K_1 K_2 K_3 K_4 K_5

Complete Undirected Graph

An undirected graph with ‘n’ number of vertices is said to be complete ,iff each vertex



Number of vertices=3

Degree of Each vertices

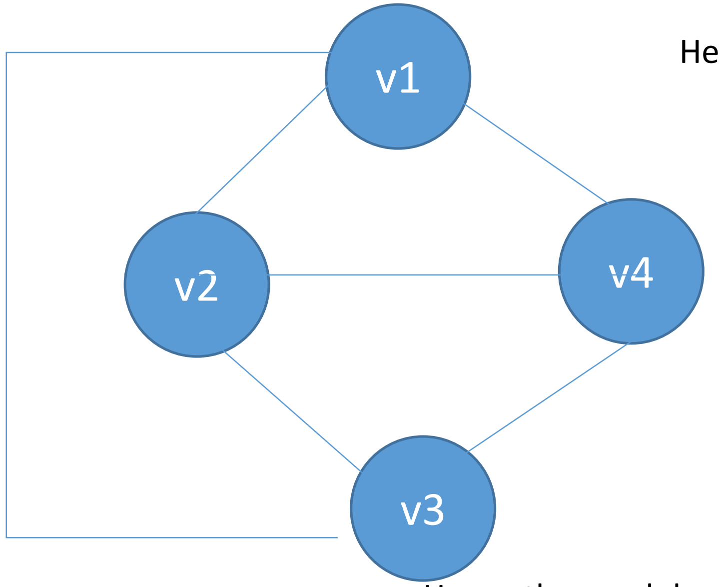
=(n-1)

=(3-1)

=2

Complete Undirected Graph

- An n vertex undirected graph with exactly (n.(n-1))/2 edges is said to be complete.



Here we have 4 number of vertices and hence
(4.(4-1))/2= (4.3)/2
=6

Hence the graph has 6 number of edges and it is a Complete Undirected graph.

Directed Graph

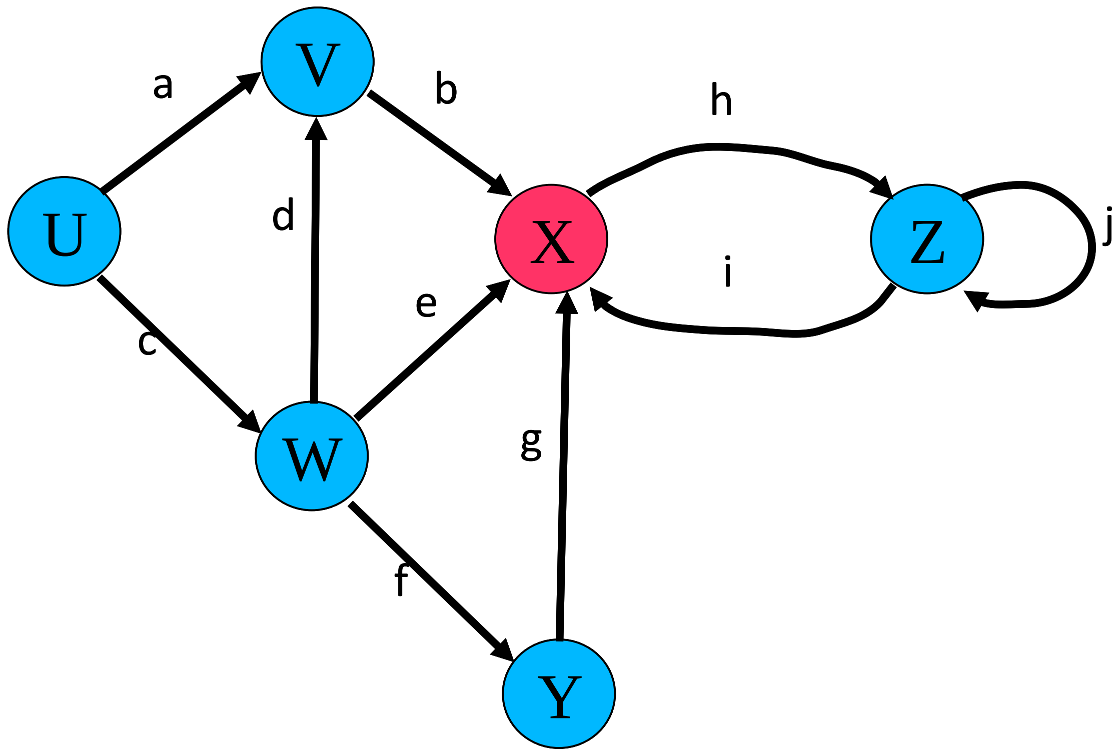
- A directed graph is one in which every edge (u, v) has a direction, so that (u, v) is different from (v, u)

There are two possible situations that can arise in a directed graph between vertices u and v.

- i) only one of (u, v) and (v, u) is present.

- ii) both (u, v) and (v, u) are present.

Directed Graph



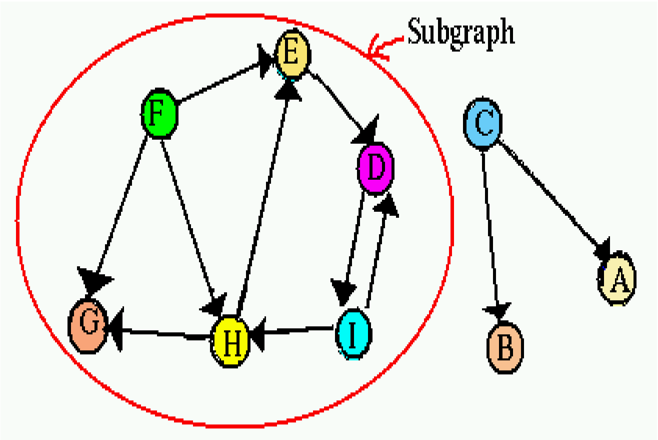
Here (u,v) is possible where as (v,u) is not possible

In a directed edge, u is said to be adjacent to v and v is said to be adjacent from u.

The edge <u,v> is incident to both u and v.

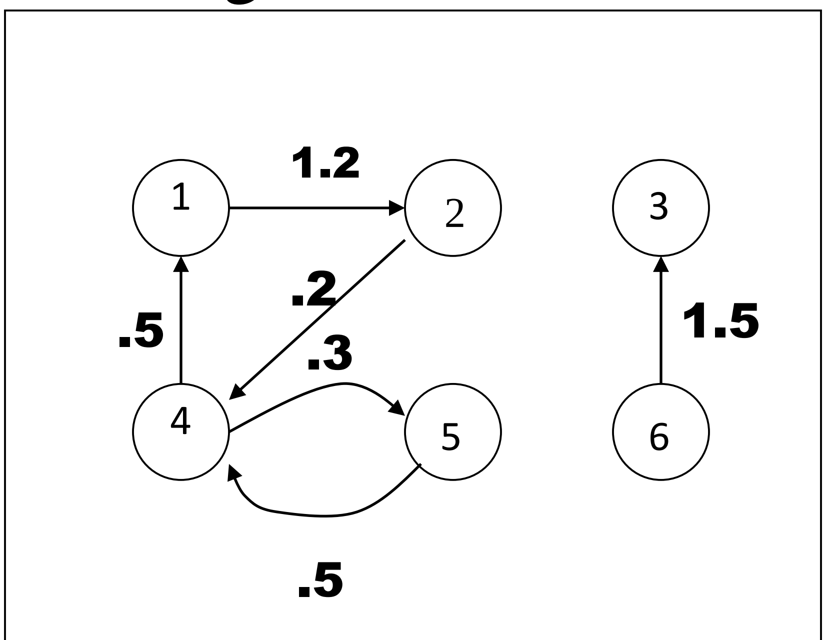
Directed Graph

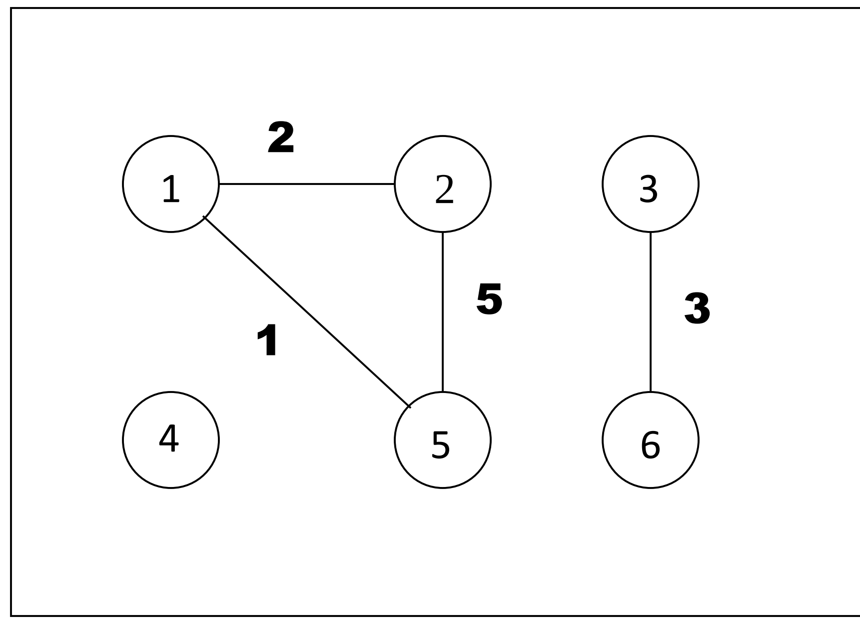
- Directed Graphs are also called as **Digraph.**
- Directed graph or the digraph make reference to edges which are directed (i.e) edges which are Ordered pairs of vertices.
- The edge(uv) is referred to as <u,v> which is distinct from <v,u> where u,v are distinct vertices.



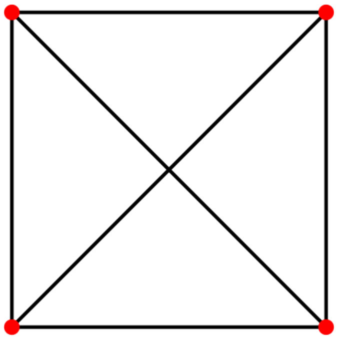
Weighted Graph

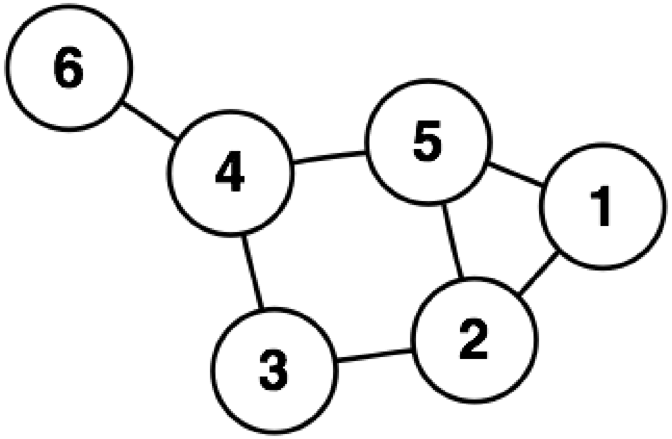
Weighted graph is a graph for which each edge has an associated ***weight***, usually given by a ***weight function*** $w: E \rightarrow \mathbf{R}$.





Planar Graph

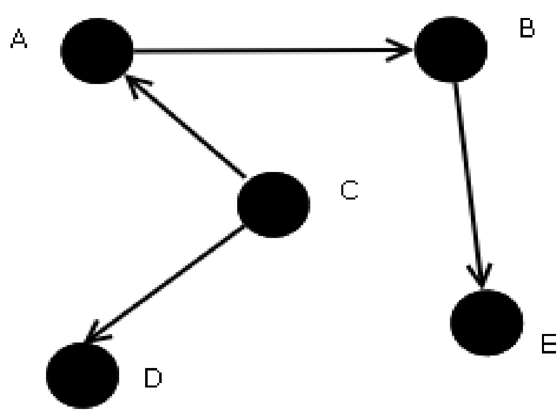




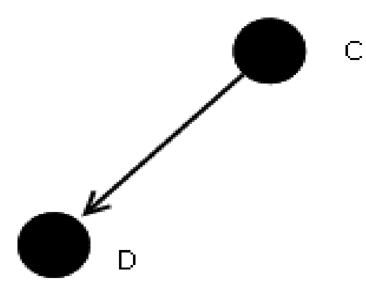
- Can be drawn on a plane such that no two edges intersect

Sub Graph

- A graph whose vertices and edges are subsets of another graph.
- A subgraph G'=(V',E') of a graph G = (V,E) such that V' ⊆ V and E' ⊆ E , Then G is a supergraph for G'.

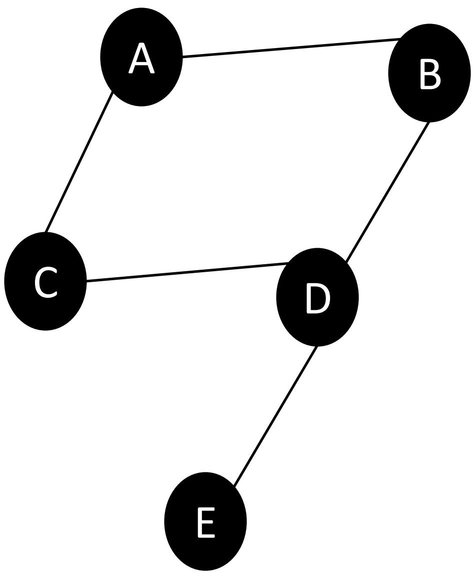


(G)

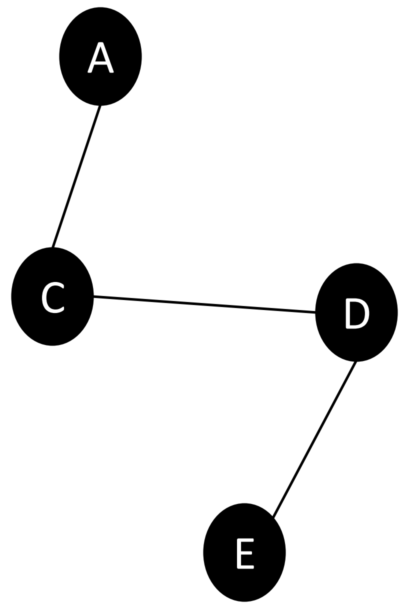


(G1)

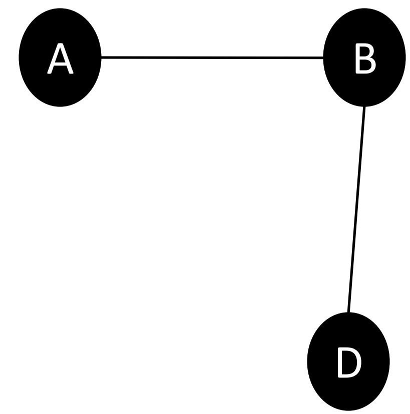
..Sub Graph



(G)



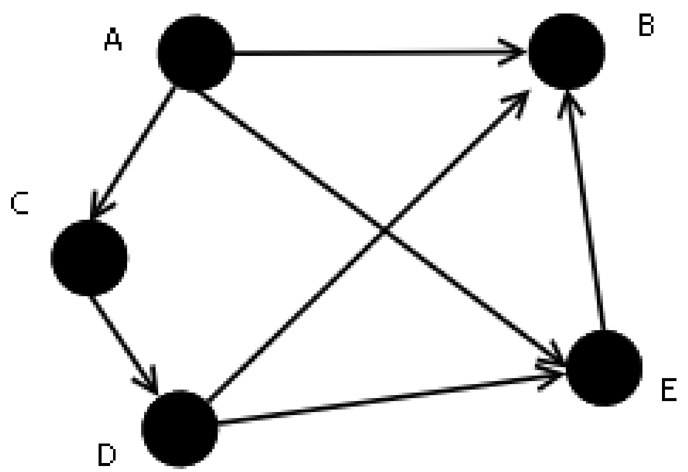
(G1)

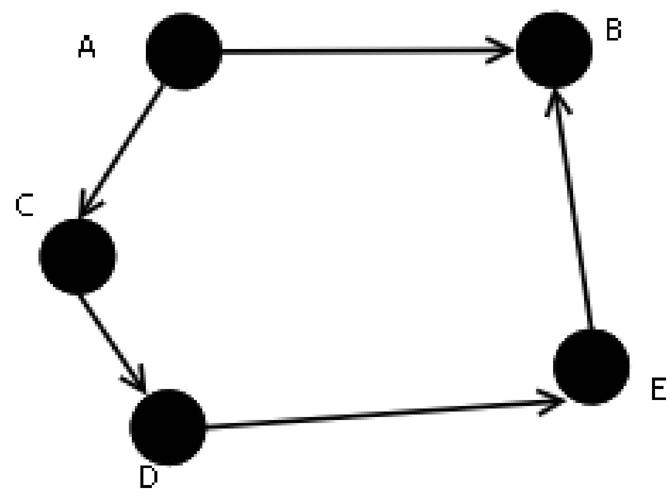


(G2)

Spanning Subgraph

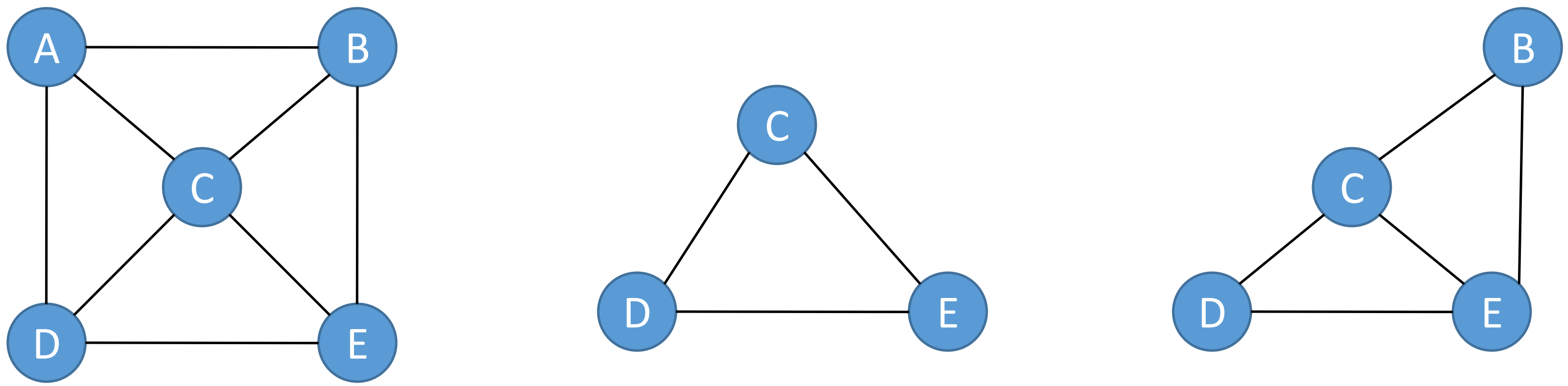
- A *spanning subgraph* is a subgraph that contains all the vertices of the original graph.





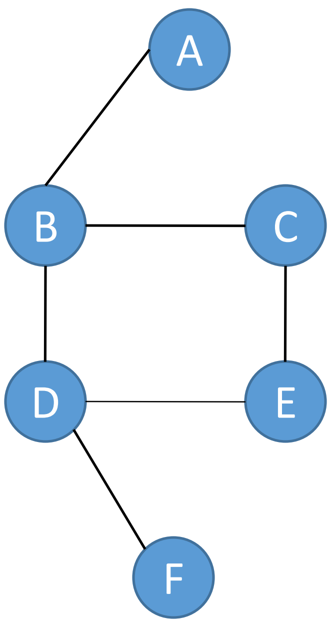
Induced-Subgraph

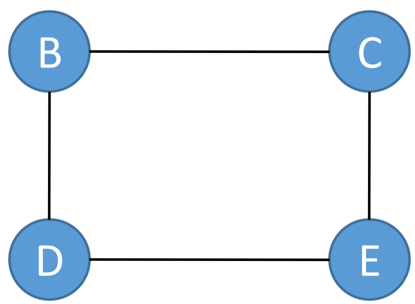
- Vertex-Induced Subgraph:
  - A *vertex-induced subgraph* is one that consists of some of the vertices of the original graph and all of the edges that connect them in the original.

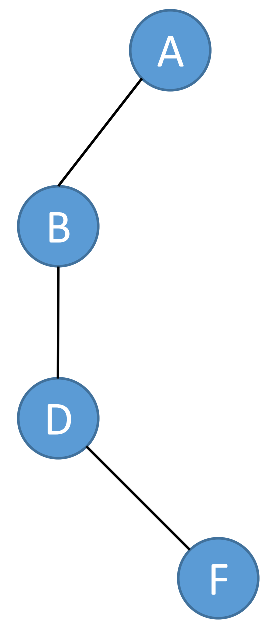


Induced-Subgraph

- Edge-Induced Subgraph:
  - An *edge-induced subgraph* consists of some of the edges of the original graph and the vertices that are at their endpoints.







Minimum Spanning Tree

Minimum Spanning Tree

- What is MST?
- Kruskal's Algorithm
- Prim's Algorithm

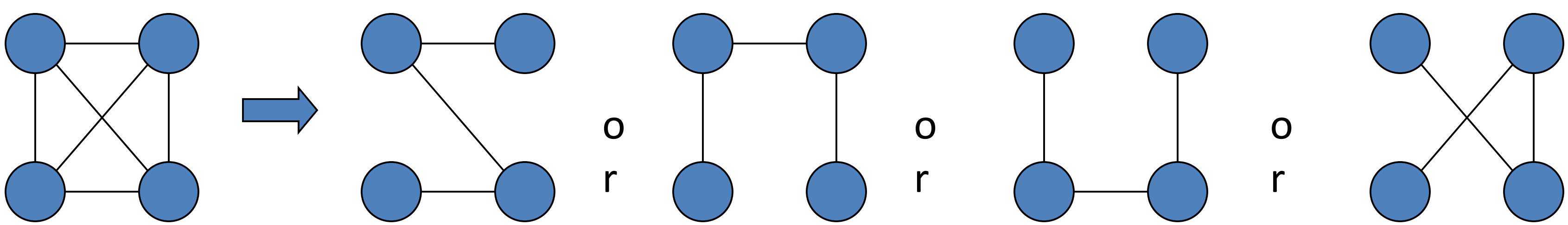
Spanning Trees

A spanning tree of a graph is just a subgraph that contains all the vertices and is a tree.

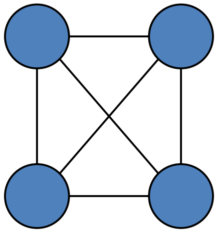
A graph may have many spanning trees.

Graph A

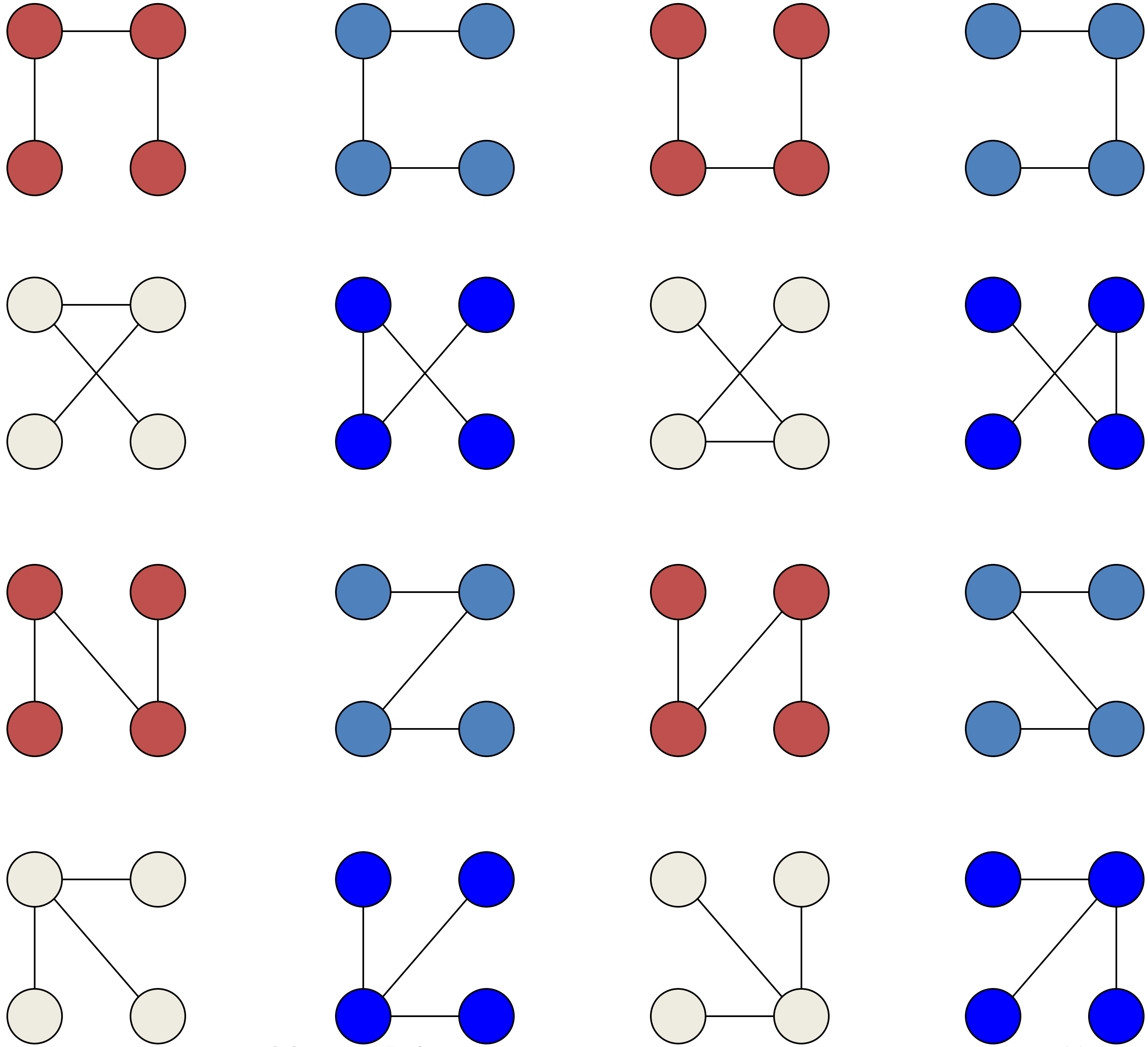
Some Spanning Trees from Graph A



Complete Graph



All 16 of its Spanning Trees

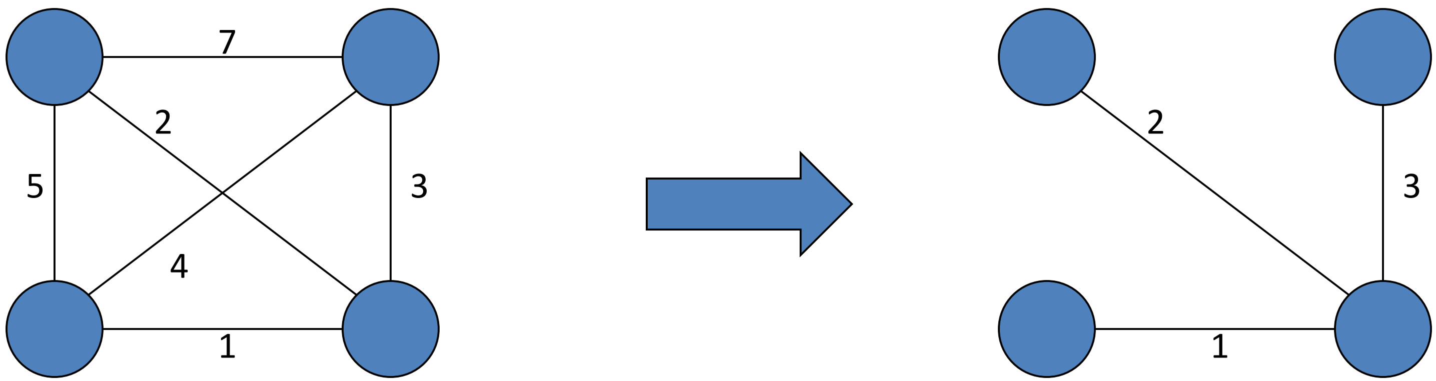


Minimum Spanning Trees

The Minimum Spanning Tree for a given graph is the Spanning Tree of minimum cost for that graph.

Complete Graph

Minimum Spanning Tree

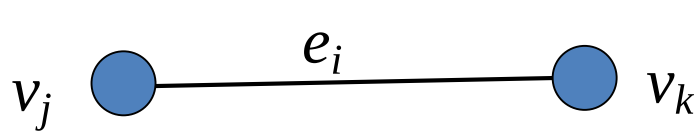


GRAPH REPRESENTATION

- Adjacency Matrix
- Incidence Matrix
- Adjacency List

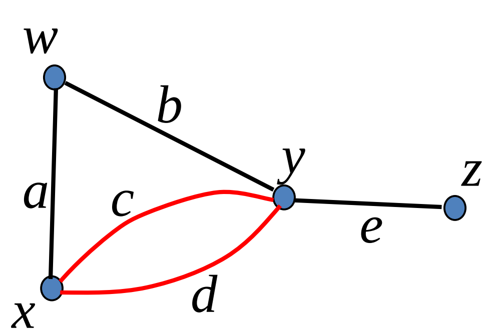
Adjacency, Incidence, and Degree

- Assume e_i is an edge whose endpoints are (v_j, v_k)
- The vertices v_j and v_k are said to be ***adjacent***
- The edge e_i is said to be ***incident upon*** v_j
- ***Degree*** of a vertex v_k is the number of edges incident upon v_k. It is denoted as $d(v_k)$



Adjacency Matrix

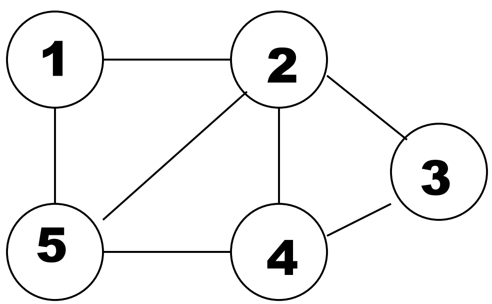
- Let $G = (V, E)$, $|V| = n$ and $|E|=m$
- The ***adjacency matrix*** of G written $A(G)$, is the $|V| \times |V|$ matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$.



$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Adjacency Matrix

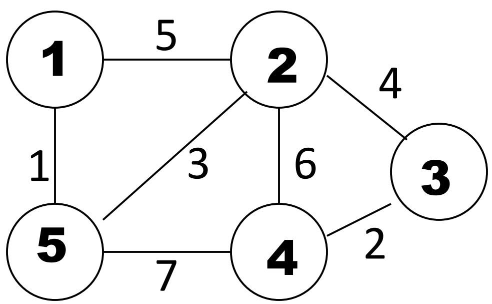
- Let $G = (V, E)$, $|V| = n$ and $|E|=m$
- The ***adjacency matrix*** of G written $A(G)$, is the $|V|$ x $|V|$ matrix in which entry $a_{i,j}$ is 1 if an edge exists otherwise it is 0



$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

Adjacency Matrix (Weighted Graph)

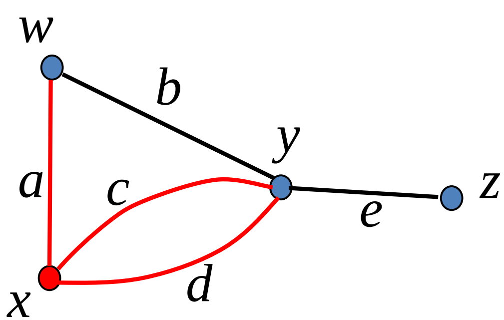
- Let $G = (V, E)$, $|V| = n$ and $|E|=m$
- The ***adjacency matrix*** of G written $A(G)$, is the $|V| \times |V|$ matrix in which entry $a_{i,j}$ is weight of the edge if it exists otherwise it is 0



$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 5 & 0 & 0 & 1 \\
2 & 5 & 0 & 4 & 6 & 3 \\
3 & 0 & 4 & 0 & 2 & 0 \\
4 & 0 & 6 & 2 & 0 & 7 \\
5 & 1 & 3 & 0 & 7 & 0 \\
\hline
\end{array}
$$

Incidence Matrix

- Let $G = (V, E)$, $|V| = n$ and $|E|=m$
- The ***incidence matrix*** $M(G)$ is the $|V|$ x $|E|$ matrix in which entry $m_{i,j}$ is 1 if v_i is an endpoint of e_i and otherwise is 0.

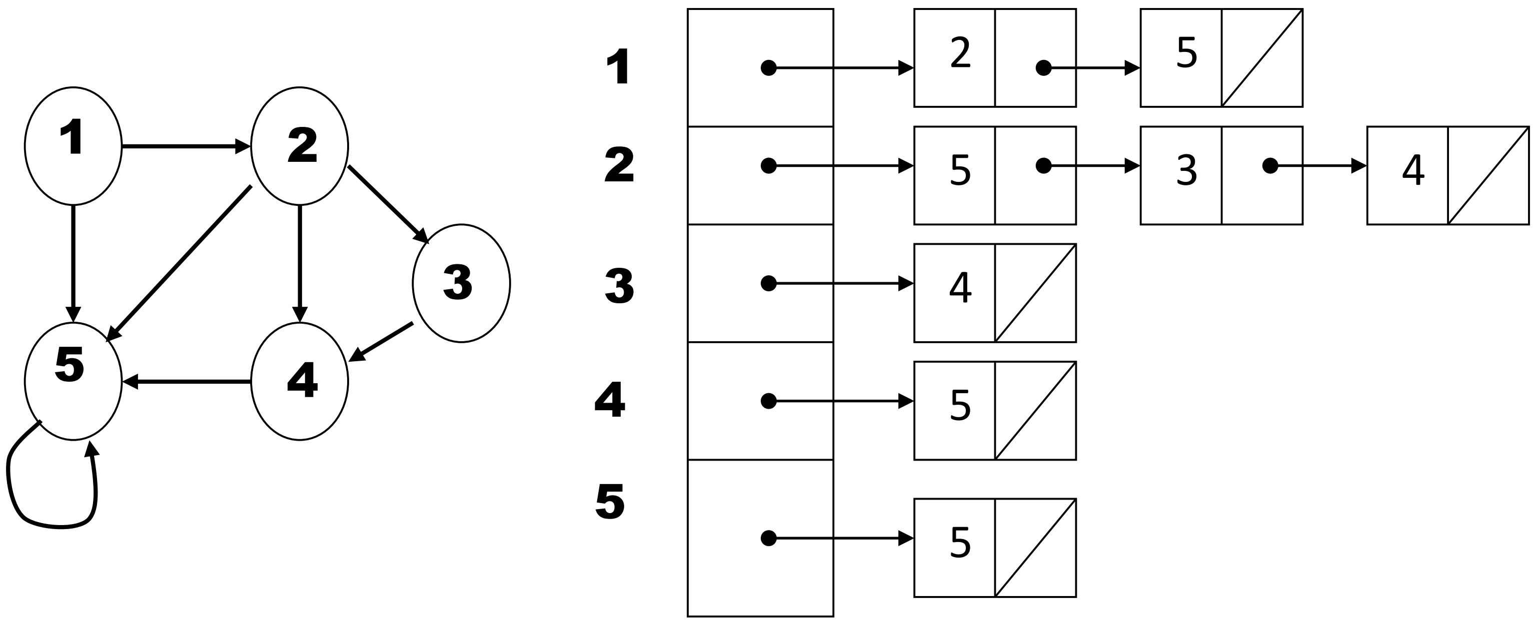


$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

Adjacency List Representation

- Adjacency-list representation
  - an array of $|V|$ elements, one for each vertex in V
  - For each $u \in V$, $ADJ[u]$ points to all its adjacent vertices.

Adjacency List Representation for a Digraph



Adjacency lists

- Advantage:
  - Saves space for sparse graphs. Most graphs are sparse.
  - Traverse all the edges that start at v, in θ(degree(v))
- Disadvantage:
  - Check for existence of an edge (v, u) in worst case time θ(degree(v))

Adjacency List

- Storage
  - For a directed graph the number of items are

$$\sum_{v \in V} (\text{out-degree } (v)) = |E|$$

So we need $\Theta(V + E)$

  - For undirected graph the number of items are

$$\sum_{v \in V} (\text{degree } (v)) = 2 \, |E|$$

Also $\Theta(V + E)$

- Easy to modify to handle weighted graphs. How?

Adjacency Matrix Representation

- Advantage:
  - Saves space for:
    - Dense graphs.
    - Small unweighted graphs using 1 bit per edge.
  - Check for existence of an edge in $\theta(1)$
- Disadvantage:
  - Traverse all the edges that start at v, in $\theta(|V|)$

Adjacency Matrix Representation

- Storage
  - $\Theta(\ |\ V\ |^2)$ (We usually just write, $\Theta(\ V^2)$)
  - For undirected graphs you can save storage (only $1/2(V^2)$) by noticing the adjacency matrix of an undirected graph is symmetric. How?
- Easy to handle weighted graphs. How?