6.4 Binomial Coefficients and Identities

As we remarked in Section 6.3, the number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a + b)^n$. We will discuss the **binomial theorem**, which gives a power of a binomial expression as a sum of terms involving binomial coefficients. We will prove this theorem using a combinatorial proof. We will also show how combinatorial proofs can be used to establish some of the many different identities that express relationships among binomial coefficients.

6.4.1 The Binomial Theorem

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The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial** expression is simply the sum of two terms, such as $x + y$. (The terms can be products of constants and variables, but that does not concern us here.)

Example 1 illustrates how the coefficients in a typical expansion can be found and prepares us for the statement of the binomial theorem.

EXAMPLE 1 The expansion of $(x + y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x + y)^3 = (x + y)(x + y)(x + y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3, x^2y, xy^2, and y^3 arise. To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1. To obtain a term of the form x^2y, an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2}$. Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums to obtain an x (and consequently take a y from each of the other two sums). This can be done in $\binom{3}{1}$ ways. Finally, the only way to obtain a y^3 term is to choose the y for each of the three sums in the product, and this can be done in exactly one way. Consequently, it follows that

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)(x + y) = (xx + xy + yx + yy)(x + y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

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We now state the binomial theorem.

THEOREM 1 **THE BINOMIAL THEOREM** Let x and y be variables, and let n be a nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Proof: We use a combinatorial proof. The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$. To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose $n - j$ xs from the n binomial factors (so that the other j terms in the product are ys). Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is equal to $\binom{n}{j}$. This proves the theorem. ◁

Some computational uses of the binomial theorem are illustrated in Examples 2–4.

EXAMPLE 2 What is the expansion of $(x + y)^4$?

Extra Examples

Solution: From the binomial theorem it follows that

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j}y^j\\ &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\end{aligned}$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$ ◀

EXAMPLE 3 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$ ◀

EXAMPLE 4 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: First, note that this expression equals $(2x + (-3y))^{25}$. By the binomial theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12}3^{13}.$$

Note that another way to find the solution is to first use the binomial theorem to see that

$$(u + v)^{25} = \sum_{j=0}^{25} \binom{25}{j} u^{25-j}v^j.$$

Setting $u = 2x$ and $v = -3y$ in this equation yields the same result. ◀

We can prove some useful identities using the binomial theorem, as Corollaries 1, 2, and 3 demonstrate.

COROLLARY 1 Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof: Using the binomial theorem with $x = 1$ and $y = 1$, we see that

$$2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

This is the desired result. ◁

There is also a nice combinatorial proof of Corollary 1, which we now present.

Proof: A set with n elements has a total of 2^n different subsets. Each subset has zero elements, one element, two elements, …, or n elements in it. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, …, and $\binom{n}{n}$ subsets with n elements. Therefore,

$$\sum_{k=0}^{n} \binom{n}{k}$$

counts the total number of subsets of a set with n elements. By equating the two formulas we have for the number of subsets of a set with n elements, we see that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

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COROLLARY 2 Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof: When we use the binomial theorem with $x = -1$ and $y = 1$, we see that

$$0 = 0^n = ((-1) + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

This proves the corollary. ◁

Remark: Corollary 2 implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

COROLLARY 3 Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Proof: We recognize that the left-hand side of this formula is the expansion of $(1 + 2)^n$ provided by the binomial theorem. Therefore, by the binomial theorem, we see that

$$(1 + 2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Hence

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

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6.4.2 Pascal's Identity and Triangle

The binomial coefficients satisfy many different identities. We introduce one of the most important of these now.

THEOREM 2 **PASCAL'S IDENTITY** Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: We will use a combinatorial proof. Suppose that T is a set containing $n + 1$ elements. Let a be an element in T, and let $S = T - \{a\}$. Note that there are $\binom{n+1}{k}$ subsets of T containing k

elements. However, a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a. Because there are $\binom{n}{k-1}$ subsets of $k-1$ elements of S, there are $\binom{n}{k-1}$ subsets of k elements of T that contain a. And there are $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S. Consequently,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

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Remark: It is also possible to prove this identity by algebraic manipulation from the formula for $\binom{n}{r}$ (see Exercise 23).

Remark: Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients. This recursive definition is useful in the

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\ldots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
. . .

(b)

FIGURE 1 Pascal's triangle.

computation of binomial coefficients because only addition, and not multiplication, of integers is needed to use this recursive definition.

Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in Figure 1.

The nth row in the triangle consists of the binomial coefficients

$$\binom{n}{k},\quad k = 0, 1, \ldots, n.$$

This triangle is known as **Pascal's triangle**, named after the French mathematician Blaise Pascal. Pascal's identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

Pascal's triangle has a long and ancient history, predating Pascal by many centuries. In the East, binomial coefficients and Pascal's identity were known in the second century B.C.E. by the Indian mathematician Pingala. Later, Indian mathematicians included commentaries relating to Pascal's triangle in their books written in the first half of the last millennium. The Persian

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6.4.3 Other Identities Involving Binomial Coefficients

We conclude this section with combinatorial proofs of two of the many identities enjoyed by the binomial coefficients.

THEOREM 3 **VANDERMONDE'S IDENTITY** Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

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Remark: This identity was discovered by mathematician Alexandre-Théophile Vandermonde in the eighteenth century.

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$.

Another way to pick r elements from the union is to pick k elements from the second set and then $r-k$ elements from the first set, where k is an integer with $0 \le k \le r$. Because there are $\binom{n}{k}$ ways to choose k elements from the second set and $\binom{m}{r-k}$ ways to choose $r-k$ elements from the first set, the product rule tells us that this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Hence, the total number of ways to pick r elements from the union also equals $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$.

We have found two expressions for the number of ways to pick r elements from the union of a set with m items and a set with n items. Equating them gives us Vandermonde's identity. ◁

Corollary 4 follows from Vandermonde's identity.

ALEXANDRE-THÉOPHILE VANDERMONDE (1735–1796) Because Alexandre-Théophile Vandermonde was a sickly child, his physician father directed him to a career in music. However, he later developed an interest in mathematics. His complete mathematical work consists of four papers published in 1771–1772. These papers include fundamental contributions on the roots of equations, on the theory of determinants, and on the knight's tour problem (introduced in the exercises in Section 10.5). Vandermonde's interest in mathematics lasted for only 2 years. Afterward, he published papers on harmony, experiments with cold, and the manufacture of steel. He also became interested in politics, joining the cause of the French revolution and holding several different positions in government.

COROLLARY 4 If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof: We use Vandermonde's identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The last equality was obtained using the identity $\binom{n}{k} = \binom{n}{n-k}$. ◁

We can prove combinatorial identities by counting bit strings with different properties, as the proof of Theorem 4 will demonstrate.

THEOREM 4 Let n and r be nonnegative integers with $r \le n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We use a combinatorial proof. By Example 14 in Section 6.3, the left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones. This final one must occur at position $r+1, r+2, \ldots$, or $n+1$. Furthermore, if the last one is the kth bit there must be r ones among the first $k-1$ positions. Consequently, by Example 14 in Section 6.3, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r+1 \le k \le n+1$, we find that there are

$$\sum_{k=r+1}^{n+1}\binom{k-1}{r} = \sum_{j=r}^{n}\binom{j}{r}$$

bit strings of length n containing exactly $r+1$ ones. (Note that the last step follows from the change of variables $j = k-1$.) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof. ◁

Exercises

1. Find the expansion of $(x+y)^4$
   a) using combinatorial reasoning, as in Example 1.
   b) using the binomial theorem.
2. Find the expansion of $(x+y)^5$
   a) using combinatorial reasoning, as in Example 1.
   b) using the binomial theorem.
3. Find the expansion of $(x+y)^6$.
4. Find the coefficient of x^5y^8 in $(x+y)^{13}$.
5. How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?
6. What is the coefficient of x^7 in $(1+x)^{11}$?
7. What is the coefficient of x^9 in $(2-x)^{19}$?
8. What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?
9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

10. Use the binomial theorem to expand $(3x-y^2)^4$ into a sum of terms of the form cx^ay^b, where c is a real number and a and b are nonnegative integers.
11. Use the binomial theorem to expand $(3x^4-2y^3)^5$ into a sum of terms of the form cx^ay^b, where c is a real number and a and b are nonnegative integers.
12. Use the binomial theorem to find the coefficient of x^ay^b in the expansion of $(5x^2+2y^3)^6$, where
    a) $a=6, b=9$.
    b) $a=2, b=15$.
    c) $a=3, b=12$.
    d) $a=12, b=0$.
    e) $a=8, b=9$.
13. Use the binomial theorem to find the coefficient of x^ay^b in the expansion of $(2x^3-4y^2)^7$, where
    a) $a=9, b=8$.
    b) $a=8, b=9$.
    c) $a=0, b=14$.
    d) $a=12, b=6$.
    e) $a=18, b=2$.

*14. Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.

*15. Give a formula for the coefficient of x^k in the expansion of $(x^2-1/x)^{100}$, where k is an integer.

16. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

    1 10 45 120 210 252 210 120 45 10 1

    Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.
17. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \le k \le 9$?
18. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2\rfloor} = \binom{n}{\lceil n/2\rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.
19. Show that $\binom{n}{k} \le 2^n$ for all positive integers n and all integers k with $0 \le k \le n$.
25. Prove that if n and k are integers with $1 \le k \le n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,
    a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]
    b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3.
26. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \le n$ and $k \le r$,
    a) using a combinatorial argument.
    b) using an argument based on the formula for the number of r-combinations of a set with n elements.
27. Show that if n and k are positive integers, then

    $$\binom{n+1}{k} = (n+1)\binom{n}{k-1}\Big/ k.$$

    Use this identity to construct an inductive definition of the binomial coefficients.
28. Show that if p is a prime and k is an integer such that $1 \le k \le p-1$, then p divides $\binom{p}{k}$.
29. Let n be a positive integer. Show that

    $$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}\Big/2.$$

*30. Let n and k be integers with $1 \le k \le n$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}\Big/2 - \binom{2n}{n}.$$

*31. Prove the **hockeystick identity**

$$\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

a) using a combinatorial argument.
b) using Pascal's identity.